Sequential Sweeps and High Dimensional Expansion

Vedat Levi Alev
Department of Mathematics, University of Haifa, Israel
Department of Computer Science, Ariel University, Israel

Ori Parzanchevski
Einstein Institute of Mathematics, Hebrew University of Jerusalem, Israel

Abstract

It is well known that the spectral gap of the down-up walk over an n-partite simplicial complex (also known as Glauber dynamics) cannot be better than $O(1/n)$ due to natural obstructions such as coboundaries. We study an alternative random walk over partite simplicial complexes known as the sequential sweep or the systematic scan Glauber dynamics: Whereas the down-up walk at each step selects a random coordinate and updates it based on the remaining coordinates, the sequential sweep goes through each of the coordinates one by one in a deterministic order and applies the same update operation. It is natural, thus, to compare n-steps of the down-up walk with a single step of the sequential sweep. Interestingly, while the spectral gap of the n-th power of the down-up walk is still bounded from above by a constant, under a strong enough local spectral assumption (in the sense of Gur, Lifschitz, Liu, STOC 2022) we can show that the spectral gap of this walk can be arbitrarily close to 1. We also study other isoperimetric inequalities for these walks, and show that under the assumptions of local entropy contraction (related to the considerations of Gur, Lifschitz, Liu), these walks satisfy an entropy contraction inequality. Concretely, we generalize a result of Lubetzky, Lubotzky, and Parzanchevski (Journal of the EMS) about the rapid mixing of sequential sweep in Ramanujan complexes to suitable high dimensional expanders.

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1 Introduction

Let Ω be a finite set of n-tuples, i.e. $\Omega \subset U_1 \times U_2 \times \cdots \times U_n$, where each U_i is a finite set. Given a tuple $\omega \in \Omega$, for each $i \in [n]$, the i-th update operation $\mathsf{update}_i(\omega)$ returns a uniformly random tuple from Ω that is conditioned to agree with ω everywhere except for the i-th coordinate, i.e.

$$\Pr_{\omega' \sim \Omega}[\mathsf{update}_i(\omega) = \omega'] = \Pr_{\omega' \sim \Omega}\left[\omega' \,\middle|\, \omega_j = \omega'_j \quad \text{for all} \quad j \in [n] \setminus i\right].$$

The main object of study in this paper will be the random walk called sequential sweep $\mathsf{P}_{\mathsf{seq}}$ on the collection Ω, otherwise known as the systematic scan Glauber dynamics[1] on Ω: A single step of this dynamics moves between tuples by applying the n update operations in sequential fashion, i.e. writing $\omega^{(t)}$ for the tuple visited by this dynamics at step t, $\omega^{(t+1)}$ is the result of the following random update operations,

$$\omega^{(t+1)} = (\mathsf{update}_n \circ \mathsf{update}_{n-1} \circ \cdots \circ \mathsf{update}_1)(\omega^{(t)}) = \mathsf{update}_n(\mathsf{update}_{n-1}(\cdots(\mathsf{update}_1(\omega^{(t)})))).$$

It is easy to see that the stationary distribution for this random walk is the uniform distribution on Ω.

It is instructive to compare $\mathsf{P}_{\mathsf{seq}}$ with the the down-up walk P_{gd} on Ω also known as the Glauber dynamics, which moves between tuples by first sampling a random coordinate $i \in [n]$ and then applying the operation $\mathsf{update}_i(\bullet)$. In other words, $\mathsf{P}_{\mathsf{seq}}$ can be thought as simulating n successive steps of P_{gd} in which we condition the coordinate that is updated at the i-th step to be i for all $i \in [n]$.

1.1 HDX, Spectral Independence, and ε-products

We recall the notion of local-spectral expansion in (partite) simplicial complexes introduced in [40, 23, 42, 51, 22, 4]. A (finite) simplicial complex of rank n is a collection of subsets of size at most n of a finite set U, i.e. $X \subset \binom{U}{\leq n}$, which is closed under containment. X is called partite, if we can partition U into n disjoint subsets $U_1, \ldots, U_n$ such that any set from X of size n contains a unique element from each U_i – the sets U_i are called the sides of the simplicial complex. In particular, each $\omega \in X$ with $|\omega| = n$ can be treated as a tuple $(\omega_1, \ldots, \omega_n)$ where $\omega_i \in U_i$. We will write $X^{(i)} = \{\omega \in X : |\omega| = i\}$. We will also write $\omega_S = (\omega_s)_{s \in S}$ for all $\omega \in X^{(n)}$.

The local structures in X, are described by its links. For a face $\alpha \in X$, we describe the weighted graph $G^{(\alpha)} = (V_\alpha, E_\alpha, w_\alpha)$ as follows. We have,

$$V_\alpha = \{u \in U : \alpha \cup u \in X\} \quad \text{and} \quad E_\alpha = \left\{uv \in \binom{U}{2} : \alpha \cup \{u, v\} \in X\right\}.$$

Every edge uv is assigned a weight based on the number of subsets in X of size n that contain $\alpha \cup \{u, v\}$, i.e.

$$w_\alpha(uv) = |\{\omega \in X : |\omega| = n, \omega \supset \alpha \cup \{u, v\}\}|.$$

Writing $G^{(\alpha)}$ for the random walk matrix of $G^{(\alpha)}$ also, we observe that $\lambda_2(G^{(\alpha)})$ measures an amortized notion of the correlation between the vertices in V_α. Formally, we say that the simplicial complex X is a $\vec{\gamma} = (\gamma_0, \ldots, \gamma_{n-2})$-local spectral expander if for each $\alpha \in X$ with $|\alpha| \leq n - 2$, we have $\lambda_2(G^{(\alpha)}) \leq \gamma_{|\alpha|}$. When $n = 2$, this notion is equivalent to the usual notion of a spectral expander, [37, 44].

For our purposes an alternative but related notion of high-dimensional expansion introduced in [34, 21] will be more useful to consider. Whereas, we have said that $\lambda_2(G^{(\alpha)})$ measures an amortized notion of the correlation between vertices in V_α, we will be more interested in measuring a worst case notion of the correlation between the vertices in V_α which come from pre-specified parts U_i and U_j. Formally, we are interested in the second eigenvalue of the following bipartite graph $G_{i,j}^{(\alpha)} = (V_{ij}^{(\alpha)}, E_{ij}^{(\alpha)}, w_{ij}^{(\alpha)})$ for each $\alpha \in X$ such

[1] More precisely, n steps of the systematic scan dynamics is the walk we call sequential sweep here.

that $|\alpha \cap U_i| = |\alpha \cap U_j| = 0$. In words, α can be thought as an assignment to a subset of coordinates $S \subset [n]$, such that $\{i,j\} \notin S$. Here,

$$V_{ij}^{(\alpha)} = \{u \in U_i \cup U_j : \alpha \cup u \in X\} \quad \text{and} \quad E_{ij}^{(\alpha)} = \{uv : u \in U_i, v \in U_j, \alpha \cup \{u,v\} \in X\}.$$

Similarly, we assign each edge uv a weight between the number of subsets in X of size n containing $\alpha \cup \{u,v\}$, i.e.

$$w_{ij}^{(\alpha)}(uv) = \left|\left\{\omega \in X^{(n)} : |\omega| = n, \omega_S = \alpha, \omega_i = u, \omega_j = v\right\}\right|,$$

where we have assumed that α to be a tuple with coordinates in $S \subset [n]$, $u \in U_i$, and $v \in V_j$. We will call the simplicial complex $\vec{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{n-2})$-product if for all α and i,j we have $\lambda_2(G_{i,j}^{(\alpha)}) \le \varepsilon_{|\alpha|}$. We observe, that the quantity $\lambda_2(G_{i,j}^{(\alpha)}) \le \varepsilon_{|\alpha|}$ quantifies the correlation between the i-th and j-th coordinate on an n-tuple drawn from X conditioned on agreeing with α. We say X is completely ε-product if it is $\vec{\varepsilon} = (\varepsilon_i)_{i=0}^{n-2}$-product where $\varepsilon_i \le \varepsilon$ for all $i = 0, \ldots, n-2$.

In [34] this definition was used in proving hypercontractive estimates for the noise operator over simplicial complexes and in [21], it was used in the context of agreement testing. We recall, that every strong enough local partite expander is also $\vec{\varepsilon}$-product [21, Corollary 7.6]:

▶ **Theorem 1.** *Let X be an n-partite simplicial complex. If X is a $\vec{\gamma}$-local spectral expander with $\gamma_{n-2} \le \frac{\varepsilon}{(n-2)\varepsilon+1}$, then X is also completely ε-product.*

And also, recall the following rudimentary remark

▶ Remark 2.[2] Let X be an n-partite simplicial complex. If X is a $\vec{\gamma}$-local spectral expander with $\gamma_i \le \frac{\varepsilon_i}{n-i-1}$, then X is also $(\varepsilon_0, \ldots, \varepsilon_{n-2})$-product.

More generally, we will consider the following bipartite graphs $G_{I,J}^{(\alpha)} = (V_{I,J}^{(\alpha)}, E_{I,J}^{(\alpha)}, w_{I,J}^{(\alpha)})$ given $\alpha \in X$, and disjoint sets I and J such that $|\alpha \cap U_k| = 0$ for all $k \in I \cup J$. Similarly, we are thinking of α as a partial assignment to a subset of the coordinates of an n-tuple which do not assign anything to the coordinates in I or J. Here,

$$V_{I,J} = \left\{\omega_I \in \prod_{i \in I} U_i : \alpha \cup \omega_I \in X\right\} \cup \left\{\omega_J \in \prod_{j \in J} U_j : \alpha \cup \omega_J \in X\right\},$$

and

$$E_{I,J} = \left\{\omega_I\omega_J : \omega_I \in \prod_{i \in I}, \omega_J \in \prod_{j \in I}, \alpha \cup \omega_I \cup \omega_J \in X\right\},$$

and assign a weight on each edge $\omega_I\omega_J \in E_{I,J}$ based on the number of subsets in X of size n that contain $\alpha \cup \{u,v\}$, i.e.

$$w_{I,J}^{(\alpha)}(\omega_I\omega_J) = \left|\left\{\bar{\omega} \in X^{(n)} : \bar{\omega}_S = \alpha, \bar{\omega}_I = \omega_I, \bar{\omega}_S = \omega_S\right\}\right|.$$

where we have assumed α to be a partial assignment to the coordinates in S. Similarly to before, $\lambda_2(G_{I,J}^{(\alpha)})$ is a measure of correlation between the coordinates on I and the coordinates in J.

In [21, 34], the following bound is shown,

▶ **Theorem 3.** *If X is an n-partite simplicial complex with parts $U_1, \ldots, U_n$ which is $\vec{\varepsilon}$-product (where $\varepsilon_i \le \varepsilon$ for all $i = 0, \ldots, n-2$), then for all α satisfying $|\alpha \cap U_k| = 0$ for $k \in I \cup J$, we have $\lambda_2(G_{I,J}^{(\alpha)})^2 \le |I||J| \cdot \varepsilon^2$.*

[2] We will prove this observation formally.

1.2 Results and Techniques

Let X be an n-partite simplicial complex. We write,

$$\varepsilon^{I\to J} := \varepsilon_X^{I\to J} = \max\Big\{\lambda_2(G_{I,J}^{(\alpha)}) : \alpha \text{ is an assignment to the coordinates in } [n]\setminus(I\cup J)\Big\}.$$

In particular, $\varepsilon^{I\to J}$ quantifies the maximum correlation between the coordinates in I and J from an n-tuple drawn from X given any assignment α to the coordinates in $(I\cup J)^c$.

We have,

▶ **Theorem 4** (Informal Version of Theorem 25). *Let X be an n-partite simplicial complex. We have,*

$$\sigma_2(\mathsf{P}_{\mathsf{seq}})^2 \le 1 - \prod_{j=2}^{n}(1-(\varepsilon^{[j-1]\to j})^2),$$

where $\sigma_2(\bullet)$ denotes the second largest singular value of $\bullet$.

In particular, writing $\mathsf{gap}(\bullet) = 1-\sigma_2(\bullet)$, we have

$$\mathsf{gap}(\mathsf{P}_{\mathsf{seq}}) \ge 1 - \sqrt{1-\prod_{j=2}^{n}(1-(\varepsilon^{[j-1]\to j})^2)}.$$

We also give an improved version of Theorem 3,

▶ **Theorem 5** (Simplified Version of Theorem 30). *Let (X,π) be an n-partite simplicial complex where π is $\vec{\varepsilon}$-product. For any assignment α to $S\subset[n]$ such that $S\cap[j]=\varnothing$, we have,*

$$\sigma_2(G_{\{j\},[j-1]}^{(\alpha)})^2 \le 1-\prod_{q=0}^{j-2}(1-\varepsilon_{|\alpha|+q}^2)$$

In particular, we have

$$(\varepsilon^{j\to[j-1]})^2 \le 1-\prod_{q=0}^{j-1}(1-\varepsilon_{n-j+q}^2)$$

In the full version of our paper, we study the expansion parameters $\varepsilon^{j\to[j-1]}$ in Ramanujan complexes of thickness p. We summarize our results for these complexes below in the following theorem.[3]

▶ **Theorem 6.** *Let X be a Ramanujan complex of thickness p and dimension $(d-1)$. We have,*

$$\left(\varepsilon^{[0,j-1]\to j}\right)^2 = \frac{1}{p+1}\big(1-\Theta(p^{-j})\big) \quad \textit{and} \quad \left(\varepsilon^{[0,\dots,d-2]\to d-1}\right)^2 = \frac{1}{p+1}\cdot\big(1-\Theta(p^{-j})\big) \tag{1}$$

for all $j=1,\dots,d-2$. Consequently,

$$\sigma_2\big(\mathsf{P}_{\mathsf{seq}}\big)^2 \le \frac{d+1}{p} - O\left(\frac{d^2}{p^2}\right)$$

where N is the number of vertices in X.

[3] Recall that a Ramanujan complex of dimension $d-1$ is a d-partite simplicial complex where the colors are named from 0 to $d-1$.

We note that the expansion parameters in Equation (1) are computed exactly and they do not suffer the co-dimension dependent blowups from Theorem 5, which would replace the RHS of the expansion parameters in Equation (1) with *roughly* $(j-1)/p$ and $(d-1)/(p+1)$, compare with Theorem 3. While our techniques fall short of recovering the optimal mixing results of [46], we hope our techniques will find further applications in the study of *spectral* expansion of Ramanujan complexes. We note that our spectral bound does not follow from the spectral techniques of [46] as this work concerns itself with spectral radii of non-backtracking operators, which in our context do not neatly translate into a bound on the singular values.

We note that Theorem 4 is proven using geometric considerations. For each $I \subset [n]$, we define the subspace $\mathcal{U}_I \subset \mathbb{R}^{X^{(n)}}$ as

$$\mathcal{U}_I = \operatorname{span}\Big\{\boldsymbol{f} \in \mathbb{R}^{X^{(n)}} : \boldsymbol{f}(\omega) = \boldsymbol{f}(\bar{\omega}) \ \text{ for all } \ \omega, \bar{\omega} \in X^{(n)} \text{ such that } \omega_{[n]\setminus I} = \bar{\omega}_{[n]\setminus I}\Big\}$$

and we will write $\mathsf{Q}_I = \operatorname{Proj}(\mathcal{U}_I)$ for the orthogonal projection to $\mathcal{U}_I$. It is well known that $\mathsf{P}_{\mathtt{seq}} = \mathsf{Q}_{\{1\}} \cdots \mathsf{Q}_{\{n\}}$, e.g. see [6]. We recall that the cosine between two subspaces $\mathcal{U}, \mathcal{V} \in \mathbb{R}^{X^{(n)}}$ is defined as,

$$\cos(\mathcal{U}, \mathcal{V}) = \max\big\{\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\pi : \boldsymbol{u} \in \mathcal{U} \cap (\mathcal{U}\cap\mathcal{V})^\perp, \boldsymbol{v} \in (\mathcal{U}\cap\mathcal{V})^\perp, \|\boldsymbol{u}\|_\pi = \|\boldsymbol{v}\|_\pi = 1\big\},$$

Similarly, $\sin^2(\mathcal{U}, \mathcal{V}) = 1 - \cos^2(\mathcal{U}, \mathcal{V})$. Now, we can follow in the footsteps of [6], who has proven estimates on the spectral gap of the sequential sweep in the Gaussian setting, by appealing to the following theorem:

▶ **Theorem 7** ([59]). *Let $\mathsf{Q}_1, \ldots, \mathsf{Q}_p \in \mathbb{R}^{\Omega\times\Omega}$ be projection operators to sub-spaces $\mathcal{U}_1, \ldots, \mathcal{U}_p$. Set $\mathcal{V}_j = \bigcap_{i=1}^j \mathcal{U}_i$. Writing $\mathsf{Q}_\star$ for the projection to $\mathcal{V}_p$, we have*

$$\|\mathsf{Q}_1 \cdots \mathsf{Q}_p \boldsymbol{f} - \mathsf{Q}_\star \boldsymbol{f}\|^2 \le \left(1 - \prod_{j=2}^p \sin^2(\mathcal{U}_j, \mathcal{V}_{j-1})\right) \cdot \|\boldsymbol{f} - \mathsf{Q}_\star \boldsymbol{f}\|^2.$$

Now, Theorem 4 follows from the following geometric interpretation of $\varepsilon^{I\to J}$:

▶ **Theorem 8** (Informal Version of Theorem 29 and Proposition 27). *Let X be an n-partite simplicial complex $I, J \subset [n]$ such that $I \cap J = \varnothing$. Under suitable assumptions,*

$$\cos(\mathcal{U}_I, \mathcal{U}_J) \le \varepsilon^{I\to J} \quad \text{and} \quad \bigcap_{i\in I} \mathcal{U}_i = \mathcal{U}_I.$$

We also show an analog of Theorem 4 for contraction in relative entropy: We recall that the KL-divergence between the distributions μ and π is defined as $\mathcal{D}(\mu \parallel \pi) = \mathbb{E}_{x\sim\pi}\left[\frac{\mu(x)}{\pi(x)} \log \frac{\mu(x)}{\pi(x)}\right]$. The entropy contraction factor $\mathsf{ec}(\mathsf{P})$ of the random walk operator $\mathsf{P} \in \mathbb{R}^{V\times V}$ with stationary measure π is defined as,

$$\mathsf{ec}(\mathsf{P}) = 1 - \kappa(\mathsf{P}) \ \text{ where } \ \kappa(\mathsf{P}) = \sup\left\{\frac{\mathcal{D}(\mu\mathsf{P} \parallel \pi)}{\mathcal{D}(\mu \parallel \pi)} : \mu \text{ is a probability distribution on } V\right\}.$$

We define the following entropy contraction parameters,

$$\eta^{I\to J} = 1 - \kappa^{I\to J} \ \text{ where } \ \kappa^{I\to J} = \sup\left\{\frac{\mathcal{D}(\mu \cdot G^{(\alpha)}_{I,J} \parallel \pi^{(\alpha)}_J)}{\mathcal{D}(\mu \parallel \pi^{(\alpha)}_I)} : \alpha \in X[(I\cup J)^c] \text{ and } \mu \in \triangle_{X_\alpha[I]}\right\}$$

where we have overloaded the notation $G^{(\alpha)}_{I,J}$ to mean the random walk from the part of $V^{(\alpha)}_{I,J}$ that describes the transition from assignments of I to assignments of J, and $\pi^{(\alpha)}_I$ and $\pi^{(\alpha)}_J$ are uniform distributions the assignments of I and assignments of J respectively.

In the full version of our paper, we prove

▶ **Theorem 9.** *Let X be an n-partite simplicial complex. Then,*

$$\mathsf{ec}(\mathsf{P}_{\mathsf{seq}}) \geq \prod_{j=1}^{n-1} \eta^{[j+1,n]\to j}. \tag{2}$$

In particular, writing π for the uniform distribution on $X^{(n)}$ we have

$$\mathcal{D}(\mu \mathsf{P}_{\mathsf{seq}} \parallel \pi) \leq (1 - \mathsf{ec}(\mathsf{P}_{\mathsf{seq}})) \cdot \mathcal{D}(\mu \parallel \pi).$$

The proof of Theorem 9 follows from an inductive argument. Our key observation is that conditional on the 1-st coordinate equalling any given value x, the distribution of $\mu\mathsf{P}_{\mathsf{seq}}$ on the remaining coordinates in $[2, \dots, n]$ can be thought as the distribution resulting from a sequential sweep in a smaller complex X_x with faces $(\omega_2, \dots, \omega_n)$ such that $(x, \omega_2, \dots, \omega_n)$ is a facet of X. The proof is then completed using the well-known Garland method [30] and the chain-rule for KL divergences. There is one interesting *twist*: usually the Garland method considers the localizations of functions, which need not be normalized. Applying the Garland method to distributions, requires understanding how the *localized/pinned* distributions interplay with the graphs $G^{(\alpha)}_{I,J}$. To the best of our knowledge, this is where our proofs differ from existing *local-to-global* arguments of [4, 17, 33, 5]. We hope our proof techniques will find more applications in the future.

We note that both $\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})$ and $\mathsf{gap}(\mathsf{P}_{\mathsf{seq}})$ are useful tools for bounding the mixing time $T(\mathsf{P}_{\mathsf{seq}})$ of the random walk $\mathsf{P}_{\mathsf{seq}}$, i.e. the number of steps one needs to take to come within small, say 0.01, distance of the uniform distribution. We have,

$$T(\mathsf{P}_{\mathsf{seq}}) \lesssim \left\lceil \frac{\log\left(|X^{(n)}|\right)}{\log\left(1/\sigma_2(\mathsf{P}_{\mathsf{seq}})\right)} \right\rceil \lesssim \left\lceil \frac{\log\left(|X^{(n)}|\right)}{\mathsf{gap}(\mathsf{P}_{\mathsf{seq}})} \right\rceil,$$

$$T(\mathsf{P}_{\mathsf{seq}}) \lesssim \left\lceil \frac{\log\log(|X^{(n)}|)}{\log(1/(1-\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})))} \right\rceil \lesssim \left\lceil \frac{\log\log\left(|X^{(n)}|\right)}{\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})} \right\rceil$$

When $\mathsf{gap}(\mathsf{P}_{\mathsf{seq}})$ and $\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})$ tend to 1, the first bound in each inequality is significantly sharper than the second. However, as they get closer and closer to 0, the latter inequalities are essentially lossless. In particular, when $\mathsf{gap}(\mathsf{P}_{\mathsf{seq}})$ and $\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})$ are of the same order, the bounds one obtains through $\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})$ are typically much sharper, though typically this constant is much more troublesome to bound.

▶ Remark 10. We note that our proof technique for Theorem 9 is general enough to prove contraction estimates for $\mathsf{P}_{\mathsf{seq}}$ with respect to general Φ-divergences, $\mathcal{D}_\Phi(\mu \parallel \pi) = \mathbb{E}_{\omega\sim\pi}\left[\Phi\left(\frac{\mu(\omega)}{\pi(\omega)}\right)\right]$ for convex Φ, assuming that $G^{(\alpha)}_{[j+1,n],\{j\}}$ contract Φ-divergences. In particular, we could give an alternative proof for Theorem 4 by setting $\Phi(t) = (t-1)^2$ and adapting the proof of Theorem 9.[4] We will refrain from taking this route and carrying this out explicitly, as we believe that proving Theorem 4 through Theorem 7 better highlights the geometric nature of this result. We refer to [16, 14, 56] for more information on Φ-divergences and the closely related Φ-entropies, and inequalities for Φ-divergence contraction called *generalized* data-processing inequalities.

[4] This would correspond to the so-called χ^2-divergence. The bound by which a random walk contracts the χ^2 divergence is well understood to correspond to the (square of the) second singular value.

1.3 Related Work

A similar proof strategy in proving a spectral gap bound for the sequential sweep in the Gaussian setting was employed by [6] – there he shows that the cosine of the angle between subspaces can be bounded in terms of the covariance matrix of the underlying Gaussian distribution, which is where our analysis diverges from his. We rely on the Garland method [30] to bound these values, which has been a staple in high-dimensional expansion literature, e.g. [51, 52, 42, 22, 4, 53, 54]. In particular, there is a non-trivial overlap between our techniques and the ones employed in [49, 50, 53, 54, 31]. Similar considerations about angles between subspaces have also been investigated in the context of Kazhdan constants and the property (T), e.g. [43, 27, 39].

Our work is close in spirit to [46] which study a non-stuttering version of the sequential sweep $\mathsf{P}_{\mathsf{seq}}$ in Ramanujan complexes, called the geodesic flow. The main result of [46] is that for these geodesic flows on Ramanujan complexes, the total-variation mixing time is as small as possible up to an additive logarithmic constant. Namely, a k-regular flow on n states achieves ϵ-mixing after only $\log_k(n) + O(\log\log(n))$ steps. In fact, this result holds for a larger class of walks on so-called *Ramanujan digraphs* [55]. While we cannot recover this result, our results hold in much greater generality without assumption of Ramanujanness, and also show that in suitable high-dimensional expanders the sequential sweep improves upon down-up walks.

As already mentioned a significant amount of work has been done in investigating the mixing properties of the down-up walk P_{gd} from considerations of high-dimensional expansion. See for example, [40, 23, 42, 22, 4] for works investigating local-spectral expansion (and variants) to get a bound on $\mathsf{gap}(\mathsf{P}_{\mathsf{gd}})$. Most of the concrete applications of these results rely on the spectral independence framework of [9, 28, 17] – we refer to the excellent lecture notes [60] and the references therein for more on this subject. Similarly, the works of [19, 33, 18, 7, 8] explore bounding $\mathsf{ec}(\mathsf{P}_{\mathsf{gd}})$ based on considerations of local entropy contraction. Our bound on $\mathsf{ec}(\mathsf{P}_{\mathsf{seq}})$ is proven using similar ideas. There has also been some work in connecting these concepts to classical Markov Chain techniques such as path coupling [1, 15], see e.g. [45, 12].

The random walks $G^{(\alpha)}_{I,J}$ that we have considered have also been studied in the works [21, 34], with applications in hypercontractivity and agreement testing. They are a natural generalization of swap-walks which have been studied in [3, 21]. Their study have found applications in approximation algorithms for constraint satisfaction problems [3], agreement testing [21], coding theory [2, 38], and hypercontractivity [10, 11, 34].

Although the sequential sweep $\mathsf{P}_{\mathsf{seq}}$ is a very natural random sampling algorithm considerably less is known about it and the results apply only in restricted and specialized settings, e.g. [20, 25, 35, 26]. These results show mixing by considering what is known as the Dobrushin matrix [24] whose entries give pessimistic estimates on the worst case dependencies between the coordinates. In [36, 57] the impact of the order of the vertices for $\mathsf{P}_{\mathsf{seq}}$ was studied and shown that the mixing time of the walk crucially depends on the order of the vertices. The relation between the relaxation time of $\mathsf{P}_{\mathsf{seq}}$ and P_{gd} (with a given order) was also subject of investigation in [32]. We refer the readers to the papers mentioned here and the references therein for more on the topic of scan order and mixing time.

The sequence of updated coordinates in $\mathsf{P}_{\mathsf{seq}}$ can be thought as the set of vertices visited by a (deterministic) walk on the directed cyle $C_n = ([n], E)$ initiated from the vertex 1. It is now a natural idea to replace the directed circle with an arbitrary graph $G = ([n], E)$ and picking the updated vertices by means of a random walk on $[n]$. [5] shows that if G is a two-sided spectral expander, the mixing time of this new walk is comparable to the mixing time of P_{gd} under mild assumptions. Their techniques fall short of establishing a mixing time for the sequential sweep in blackbox fashion since the directed cycle has no two-sided expansion. However, we observe that their result essentially translates an $O(n \log n)$ mixing time for $\mathsf{P}_{\mathtt{gd}}$ into an $O(\frac{n \log n}{1-\lambda(G)})$ mixing time for this new-walk. Observing that this mixing time is roughly the cover time[5] of the graph G, suggesting that a tighter analysis could perhaps establish an $O(1)$-mixing time[6] for our sweep[7].

Finally, we mention the work [29], which proves worst case comparison theorems between the sequential sweep $\mathsf{P}_{\mathtt{seq}}$ and the Glauber dynamics $\mathsf{P}_{\mathtt{gd}}$, which shows that $\mathtt{gap}(\mathsf{P}_{\mathtt{seq}}) = \Omega(n^{-1}) \cdot \mathtt{gap}(\mathsf{P}_{\mathtt{gd}})$. They further give examples of complexes in which this bound is tight. We note that our focus in this work are complexes where $\mathtt{gap}(\mathsf{P}_{\mathtt{seq}})$ is significantly better than $\mathtt{gap}(\mathsf{P}_{\mathtt{gd}})$, thus our result can be thought as complementary to their result.

2 Preliminaries

2.1 Linear Algebra

Vectors and Inner Products

Given $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$, and a measure π on V such that $\pi(x) > 0$ for all $x \in V$, we use the notations $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\pi$ and $\|\boldsymbol{f}\|_\pi$ to denote the inner-product and the norm with respect to the distribution π, i.e.

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\pi = \mathop{\mathbb{E}}_{x \sim \pi} \boldsymbol{f}(x)\boldsymbol{g}(x) = \sum_{x \in V} \pi(x) \cdot \boldsymbol{f}(x)\boldsymbol{g}(x) \quad \text{and} \quad \|\boldsymbol{f}\|_\pi^2 = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\pi. \tag{3}$$

Matrices and Eigenvalues

In this section, we will recall some results concerning eigenvalues and eigenvectors of matrices.

We will call a matrix $\mathsf{B} \in \mathbb{R}^{U \times V}$ row-stochastic if all entries of B are non-negative and all rows of B sum up to 1, i.e.

$$\text{for all } u \in U, v \in V \quad \mathsf{B}(u,v) \geq 0 \quad \text{and} \quad \mathsf{B}\mathbf{1} = \mathbf{1}.$$

The adjoint of the operator $\mathsf{B} \in \mathbb{R}^{U \times V}$, with respect to the inner-products defined by the measures π_U and π_V on U and V, is the operator $\mathsf{B}^* \in \mathbb{R}^{V \times U}$ satisfying

$$\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{g} \rangle_{\pi_U} = \langle \mathsf{B}^* \boldsymbol{f}, \boldsymbol{g} \rangle_{\pi_V} \quad \text{for all } \boldsymbol{f} \in \mathbb{R}^U, \boldsymbol{g} \in \mathbb{R}^V.$$

If $U = V$ and $\pi_U = \pi_V$, the operator B is called self-adjoint when $\mathsf{B}^* = \mathsf{B}$. If B is a row-stochastic matrix, we will call B^* the time-reversal of B with respect to π_U, π_V and say that B is reversible if $\mathsf{B} = \mathsf{B}^*$. It is well known that the operator $\mathsf{B}^* \in \mathbb{R}^{V \times U}$ is uniquely determined by the choice of $\mathsf{B} \in \mathbb{R}^{U \times V}$ and the inner-products defined by π_U and π_V (see e.g. [58, p. 318]),

► **Proposition 11.** *Let $\mathsf{B} \in \mathbb{R}^{U \times V}$ be arbitrary. We write B^* for the adjoint operator to B with respect to the inner-products defined by the distributions π_U and π_V. Then,*

$$\mathsf{B}^*(y,x) = \mathsf{B}(x,y) \cdot \frac{\pi_U(x)}{\pi_V(y)} \quad \textit{for all} \quad x \in U, y \in V.$$

[5] The number of steps following which the random is expected to have visited all the vertices after being initialized from an arbitrary vertex.
[6] i.e. $O(n)$ update steps
[7] where we observe that the cover time of the directed cycle is n, which would correspond to a single sweep

We also recall the following standard fact which is an immediate consequence of Proposition 11,

▶ **Proposition 12.** *If* $\mathsf{B} \in \mathbb{R}^{U,V}$ *is a row-stochastic matrix satisfying* $\pi_U \mathsf{B} = \pi_V$*, then the adjoint matrix* B^* *with respect to* π_U, π_V *is also row-stochastic and satisfies* $\pi_V \mathsf{B}^* = \pi_U$*.*

Throughout this paper, we will often deal with operators which are not self-adjoint, and therefore we will need to deal with singular values. Given an operator $\mathsf{B} \in \mathbb{R}^{V \times U}$ we will write $\sigma_i(\mathsf{B})$ for the i-th largest singular value of B, i.e. we have $\sigma_i(\mathsf{B}) = \sqrt{\lambda_i(\mathsf{B}^*\mathsf{B})}$.[8] This expression is well-defined since $\mathsf{B}^*\mathsf{B}$ is positive-semi definite and therefore $\lambda_i(\mathsf{B}^*\mathsf{B}) \geq 0$. We recall the following facts,

▶ **Proposition 13.** *Let* $\mathsf{B} \in \mathbb{R}^{U \times V}$ *be a row-stochastic matrix satisfying* $\pi_U \mathsf{B} = \pi_V$*. Then,*
1. $\sigma_1(\mathsf{B}) = 1$,
2. $\sigma_2(\mathsf{B}) =$
$\max\big\{\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{g}\rangle_{\pi_U} : \boldsymbol{f} \in \mathbb{R}^U, \boldsymbol{g} \in \mathbb{R}^V, \|\boldsymbol{f}\|_{\pi_U} = \|\boldsymbol{g}\|_{\pi_V} = 1, \langle \boldsymbol{f}, \mathbf{1}_U\rangle_{\pi_U} = \langle \boldsymbol{g}, \mathbf{1}_V\rangle_{\pi_V} = 0\big\}$.
3. $\sigma_2(\mathsf{B}) = \max\Big\{\frac{\|(\mathsf{B}-\mathbf{1}_U\pi_V)\boldsymbol{f}\|_{\pi_U}}{\|\boldsymbol{f}-\mathbf{1}_V\pi_V\boldsymbol{f}\|_{\pi_V}} : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f} - \mathbf{1}_V\pi_V\boldsymbol{f}\|_{\pi_V} \neq 0\Big\}$

A matrix $\mathsf{Q} \in \mathbb{R}^{V \times V}$ is called an orthogonal projection with respect to π if it is self adjoint with respect to π, i.e. $\mathsf{Q} = \mathsf{Q}^*$ and it satisfies the equation $\mathsf{Q}^2 = \mathsf{Q}$. We will say that Q is the orthogonal projection operator to a subspace $\mathcal{U} \subset \mathbb{R}^V$ if the image $\mathrm{im}(\mathsf{Q}) = \mathsf{Q}\mathbb{R}^V$ equals $\mathcal{U}$.

Finally, we will recall some results concerning the angles between subspaces and products of projection matrices. Let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^\Omega$ equipped with the inner-product π. The cosine of the angle between $\mathcal{U}$ and $\mathcal{V}$ is defined to be,

$$\cos(\mathcal{U}, \mathcal{V}) = \max\big\{\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\pi : \boldsymbol{u} \in \mathcal{U} \cap (\mathcal{U} \cap \mathcal{V})^\perp, \boldsymbol{v} \in (\mathcal{U} \cap \mathcal{V})^\perp, \|\boldsymbol{u}\|_\pi = \|\boldsymbol{v}\|_\pi = 1\big\}, \tag{4}$$

▶ **Theorem 14** ([59])**.** *Let* $\mathsf{Q}_1, \ldots, \mathsf{Q}_p \in \mathbb{R}^{\Omega \times \Omega}$ *be projection operators to sub-spaces* $\mathcal{U}_1, \ldots, \mathcal{U}_p$*. Set* $\mathcal{V}_j = \bigcap_{i=1}^j \mathcal{U}_i$*. Writing* $\mathsf{Q}_\star$ *for the projection to* $\mathcal{V}_p$*, we have*

$$\|\mathsf{Q}_1 \cdots \mathsf{Q}_p \boldsymbol{f} - \mathsf{Q}_\star \boldsymbol{f}\|_\pi^2 \leq \left(1 - \prod_{j=2}^p \sin^2(\mathcal{U}_j, \mathcal{V}_{j-1})\right) \cdot \|\boldsymbol{f} - \mathsf{Q}_\star \boldsymbol{f}\|_\pi^2$$

where $\sin^2(\mathcal{U}, \mathcal{V}) = 1 - \cos^2(\mathcal{U}, \mathcal{V})$*.*

2.2 Probability Distributions and Relative Entropy

Let Ω be a finite set. We will write $\triangle_\Omega$ for the probability simplex with vertices Ω, i.e.

$$\triangle_\Omega := \left\{\mu : \Omega \to \mathbb{R}_{\geq 0} : \sum_{\omega \in \Omega} \mu(\omega) = 1\right\}.$$

We will adopt the convention of treating elements of $\mu \in \triangle_\Omega$ as row-vectors. For $\mu \in \triangle_\Omega$, we will write $\mathrm{supp}(\mu) = \{\omega \in \Omega : \mu(\omega) > 0\}$ for the support of μ. Throughout the paper, Ω (or X) will always be a set of n-tuples for some $n \geq 1$, i.e. there exists a finite sequence of sets $\Omega_1, \ldots, \Omega_n$ such that $\Omega \subset \Omega_1 \times \cdots \times \Omega_n$. Given a set $S \subset [n]$, we will write $\Omega[S]$ for the projection of Ω to the coordinates in S, i.e.

[8] Crucially, the i-th singular value always depends on the choice of the measures which define the adjoint matrix. We will mostly deal with row-stochastic matrices that satisfy $\pi_U \mathsf{B} = \pi_V$ for some choice of distributions π_U, π_V. The adjoint matrix B^* is then defined with respect to these distributions.

$$\Omega[S] = \{(\omega_s)_{s\in S} : (\omega_1, \dots, \omega_n) \in \Omega\}.$$

The S-marginal distribution of $\mu \in \Omega$ is defined by,

$$\mu_S(\omega_S) = \sum_{\bar{\omega}\in\Omega[S^c]} \mu(\omega_S \oplus \bar{\omega}) \quad \text{for all} \quad \omega_S \in \Omega[S], \tag{5}$$

where $S^c = [n] \setminus S$.

Let $\omega_S \in \Omega_S$. We will write Ω_{ω_S} and $\mu^{(\omega_S)}$ for the ω_S-pinning of Ω and μ_S, where

$$\Omega_{\omega_S} = \{\bar{\omega} \in \Omega[S^c] : \omega_S \oplus \bar{\omega} \in \Omega\} \quad \text{and} \quad \mu^{(\omega_S)}(\bar{\omega}) = \frac{\mu(\omega_S \oplus \bar{\omega})}{\sum_{\tilde{\omega}\in\Omega[S^c]} \mu(\omega_S \oplus \tilde{\omega})}, \tag{6}$$

for all $\omega_S \in \Omega[S]$. The following relation between the S-marginal μ_S and ω_S-pinning is immediate from the definitions. Let, $S, T \subset [n]$ such that $S \cap T = \varnothing$,

$$\mu(\omega_S \oplus \omega_T) = \mu_S(\omega_S) \cdot \mu_T^{(\omega_S)}(\omega_T) \tag{7}$$

We note that the statement of Equation (7) is a simple consequence of the law of conditional probability (Bayes' rule).

The Kullback-Leibler divergence $\mathcal{D}(\mu \parallel \nu)$ between distributions $\mu, \nu \in \triangle_\Omega$ is defined as,

$$\mathcal{D}(\mu \parallel \nu) = \mathop{\mathbb{E}}_{x\sim\nu}\left[\frac{\mu(x)}{\nu(x)} \log \frac{\mu(x)}{\nu(x)}\right] \tag{KL divergence}$$

where we have adopted the convention $0/0 = 1$ and assumed $\mathrm{supp}(\mu) \subset \mathrm{supp}(\nu)$.

It is a well-known fact that $\mathcal{D}(\bullet \parallel \bullet)$ is monotone-decreasing under the application of a row-stochastic matrix,

▶ **Lemma 15** (Data Processing Inequality). *Let $\mathsf{M} \in \mathbb{R}^{\Omega\times\Omega'}$ be any row-stochastic operator. Then, for any pair of distributions $\mu, \nu \in \Delta_\Omega$ we have,*

$$\mathcal{D}(\mu\mathsf{M} \parallel \nu\mathsf{M}) \leq \mathcal{D}(\mu \parallel \nu).$$

We also recall the following consequence of the chain-rule for the KL divergence,

▶ **Lemma 16.** *Let $n \geq 0$ finite sets $\Omega_1, \dots, \Omega_n$ be given. Suppose the set $\Omega \subset \Omega_1 \times \cdots \times \Omega_n$ and the distribution $\mu \in \triangle_\Omega$ are arbitrary. Then, writing $S^c = [n] \setminus S$, we have*

$$\mathcal{D}(\mu \parallel \nu) = \mathcal{D}(\mu_S \parallel \nu_S) + \mathop{\mathbb{E}}_{\omega_S\sim\mu_S} \mathcal{D}(\mu_{S^c}^{(\omega_S)} \parallel \nu_{S^c}^{(\omega_S)}).$$

2.3 Random Walks and Mixing Times

The mixing time $T(\varepsilon, \mathsf{P})$ of the random walk operator $\mathsf{P} \in \mathbb{R}^{V\times V}$ is defined to be the least time step where the distribution of the random walk is ε-close to the stationary distribution π of P in the ℓ_1 distance, i.e. $\pi^{(t)}$ for the distribution of the random walk after t-steps, we have

$$T(\varepsilon, \mathsf{P}) = \min\Big\{t \in \mathbb{N}_{\geq 0} : \|\pi^{(t)} - \pi\|_{\ell_1} \leq \varepsilon\Big\}.$$

We will write $\mathsf{A}(x, \bullet)$ for the x-th row of the matrix A. Recalling that the distribution of the random walk P starting from x after t steps is given by $\pi^{(t)} = \mathsf{P}^t(x, \bullet)$, we formally define

$$T(\varepsilon, \mathsf{P}) = \min\big\{t \in \mathbb{N}_{\geq 0} : \|\mathsf{P}^t(x, \bullet) - \pi^\top\|_{\ell_1} \leq \varepsilon \text{ for all } x \in V\big\}. \tag{mixing time}$$

We define the spectral gap $\mathtt{gap}(\mathsf{P})$ of the random walk $\mathsf{P} \in \mathbb{R}^{\Omega\times\Omega}$ with stationary distribution π as,

$$\mathtt{gap}(\mathsf{P}) = 1 - \sigma_2(\mathsf{P}) \qquad \text{(spectral gap)}$$

The following mixing time bound is well-known, see for example [48, Proposition 1.12].

▶ **Theorem 17** (Spectral Mixing Time Bound). *Let $\mathsf{P} \in \mathbb{R}^{V\times V}$ be a random walk matrix P with stationary distribution π. One has,*

$$T(\varepsilon, \mathsf{P}) \le \left\lceil \frac{\log\left(\left(\varepsilon \cdot \sqrt{\min_{x\in V} \pi(x)}\right)^{-1}\right)}{\log(1/\sigma_2(\mathsf{P}))} \right\rceil \le \left\lceil \frac{\log\left(\left(\varepsilon \cdot \sqrt{\min_{x\in V} \pi(x)}\right)^{-1}\right)}{\mathtt{gap}(\mathsf{P})} \right\rceil$$

where $\mathtt{gap}(\mathsf{P})$ is the spectral gap of the operator P.

The entropy contraction factor $\mathtt{ec}(\mathsf{P})$ of the random walk operator P with stationary measure π is defined as,

$$\mathtt{ec}(\mathsf{P}) = 1 - \kappa(\mathsf{P}) \;\text{ where }\; \kappa(\mathsf{P}) = \sup\left\{\frac{\mathcal{D}(\mu\mathsf{P} \parallel \pi)}{\mathcal{D}(\mu \parallel \pi)} : \mu \in \triangle_\Omega\right\} \quad \text{(entropy contraction factor)}$$

Then, the following estimate can be used to bound the mixing time, see for example [13, Lemma 2.4],

▶ **Theorem 18.** *Let $\mathsf{P} \in \mathbb{R}^{V\times V}$ be a random walk operator P with stationary distribution π. One has,*

$$T(\varepsilon, \mathsf{P}) \le \frac{C}{\mathtt{ec}(\mathsf{P})} \cdot \log\log \frac{1}{\varepsilon \cdot \min_{x\in V} \pi(x)},$$

where $\mathtt{ec}(\mathsf{P})$ is the entropy contraction factor of P and C is a universal constant independent of (P, π).

2.4 (Partite) Simplicial Complexes

A simplicial complex is a downward closed collection of subsets of a finite set U. Formally, $X \subset 2^U$ and whenever $\beta \in X$ for all $\alpha \subset \beta$ we have $\alpha \in X$. The rank of a face α is $|\alpha|$. Given some j, we will adopt the notation $X^{(j)}$ to refer to the collection faces of X of rank j and the notation $X^{(\le j)}$ to refer to the collection of faces of X of rank at most j, i.e.

$$X^{(j)} := X \cap \binom{U}{j} \quad\text{and}\quad X^{(\le j)} := \bigcup_{i=0}^{j} X^{(i)}.$$

We say X is a simplicial complex of rank n if the largest rank of any face $\alpha \in X$ is n. We note that by definition $X^{(0)} = \{\varnothing\}$.

We say that a simplicial complex X of rank n is pure, if any face $\alpha \in X^{(j)}$ for any $j < n$ is contained in another face $\beta \in X^{(n)}$. Equivalently, in a pure simplicial complex the only inclusion maximal faces are those of maximal rank. In this article, we will only deal with pure simplicial complexes.

A rank-n pure simplicial complex X is called n-partite if we can partition $X^{(1)}$ into disjoint sets $X[1], \dots, X[n]$ such that

$$\text{for all } \beta \in X^{(n)} \text{ and for all } i = 1, \dots, n \;\text{ we have }\; |\beta \cap X[i]| = 1. \qquad (n\text{-partiteness})$$

We will call the sets $X[1], \ldots, X[n]$ the sides of the complex X. Equivalently, every element of a rank-n face $\beta \in X[n]$ comes from a distinct side $X[i]$. We observe that a bipartite graph is a 2-partite simplicial complex.

To keep our nomenclature simple, we will simply refer to a pure n-partite simplicial complex of rank n as an n-partite simplicial complex, i.e. we will not consider n-partite complexes which are not pure.

For a face $\alpha \in X$ we introduce the notation, $\mathrm{type}(\alpha) = \{i \in [n] : \alpha \cap X[i] \neq \varnothing\}$ for the type of the face α, i.e. the collection of sides of X that α intersects.

For any $i \in [n]$ and $\beta \in X^{(n)}$ we will write $\beta_i \in X^{(1)}$ for the unique element of β satisfying $\{\beta_i\} = \beta \cap X[i]$. We will refer to β_i as the i-th coordinate of β. We will also write $\beta_T = \{\beta_t : t \in T\}$ for all $T \subset [n]$. We extend this notation to arbitrary faces $\alpha \in X$ and $T \subset \mathrm{type}(\alpha)$. In keeping with the view that a face $\alpha \in X$ with $\mathrm{type}(\alpha) = \{t_1, \ldots, t_k\}$ can be represented as a tuple $(a_{t_1}, \cdots, a_{t_k})$, we will favour the notation $\alpha \oplus \alpha'$ to denote the union of two faces $\alpha, \alpha' \in X$ with $\mathrm{type}(\alpha) \cap \mathrm{type}(\alpha') = \varnothing$ over the usual notation $\alpha \cup \alpha'$.

We observe that for facets $\beta \in X^{(n)}$, i.e. faces of maximal rank, we have $\mathrm{type}(\beta) = [n]$. Given, $\alpha \in X$ we recall that the link X_α is defined as,

$$X_\alpha = \{(\beta \setminus \alpha) \in X : \beta \in X, \beta \supset \alpha\}.$$

The following observation is immediate,

▶ **Lemma 19.** *Let X be an n-partite simplicial complex with sides $X[1], \ldots, X[n]$ and $\alpha \in X^{(j)}$ for some $j \in [0, n]$. Then, the simplicial complex X_α is an $(n-j)$-partite simplicial complex with sides $X_\alpha[j] := X[j] \cap X_\alpha^{(1)}$ for $j \in [n] \setminus \mathrm{type}(\alpha)$.*

For $T \subset [n]$, we will also introduce the notation $X[T]$ to refer to all faces of X of type T, i.e. $X[T] = \{\alpha \in X : \mathrm{type}(\alpha) = T\}$.

2.5 Weighted Simplicial Complexes

A weighted simplicial complex (X, π) of rank n is a pure simplicial complex of rank n where $\pi := \pi_n$ is a probablity distribution on $X^{(n)}$ with full support, i.e. $\pi \in \triangle_{X^{(n)}}$ and $\mathrm{supp}(\pi) = X^{(n)}$. For $j \in [0, n-1]$, we inductively define the probability distributions $\pi_j : X^{(j)} \to \mathbb{R}$ as

$$\pi_j(\alpha) = \frac{1}{j+1} \sum_{\substack{\beta \in X[j+1], \\ \beta \subset \alpha}} \pi_{j+1}(\beta). \tag{8}$$

Following repeated applications of Equation (8), we obtain the following well-known result (see, e.g. [4, Proposition 2.3.1])

▶ **Proposition 20.** *Let (X, π) be a simplicial complex of rank n. For all $0 \leq j \leq k \leq n$ and $\alpha \in X^{(j)}$, one has*

$$\pi_j(\alpha) = \frac{1}{\binom{k}{j}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X^{(k)}}} \pi_k(\beta).$$

Similarly, given a face $\alpha \in X^{(j)}$, we define the distribution $\pi^{(\alpha)}$ on $X_\alpha^{(n-j)}$ by conditioning π on the containment of α. Consequently, we have

▶ **Proposition 21.** *Let (X, π) be a simplicial complex of rank n. Let $\alpha \in X^{(j)}$ and $\tau \in X_\alpha^{(\ell)}$ be faces for some $0 \leq \ell \leq j \leq n$. Then,*

$$\pi_l^{(\alpha)}(\tau) = \frac{\pi_{j+\ell}(\alpha \cup \tau)}{\binom{j+\ell}{\ell} \cdot \pi_j(\alpha)}.$$

When (X, π) is an n-partite weighted simplicial complex, for any $T \subset [n]$, we will define π_T as the marginal distribution of π on the coordinates in T according to Equation (5), i.e.

$$\pi_T(\omega_T) = \sum_{\bar{\omega} \in X[T^c]} \pi(\omega_T \oplus \bar{\omega}).$$

Similarly, given any $\omega_T \in X[T]$, we define the weighting $\pi^{(\omega_T)}$ on $(X_{\omega_T}, \pi^{(\omega_T)})$ according to Equation (6). Thus,

$$\pi^{(\omega_T)}(\bar{\omega}) = \Pr_{\omega' \sim \pi}[\omega'_{T^c} = \bar{\omega} \mid \omega'_T = \bar{\omega}] = \frac{\pi(\omega_T \oplus \bar{\omega})}{\pi_T(\omega_T)} \quad \text{for all} \quad \bar{\omega} \in X[T^c].$$

2.6 Update Operators and the Sequential Sweep

Let (X, π) be a pure, n-partite simplicial complex of rank n. We define the update operators $\mathsf{Q}_1, \ldots, \mathsf{Q}_n \in \mathbb{R}^{X^{(n)} \times X^{(n)}}$ as follows,

$$\mathsf{Q}_i(\omega, \omega') = \Pr_{\bar{\omega} \sim \pi}\left[\bar{\omega} = \omega' \mid \bar{\omega} \supset \omega_{[n] \setminus i}\right] = \pi_{\{i\}}^{(\omega_{[n]\setminus i})}(\omega'_i). \tag{9}$$

Thus, Q_i can be thought as describing the random walk which makes a transition from $\omega \in X^{(n)}$ to a random state $\omega' \in X^{(n)}$ by only re-sampling the i-th coordinate from the distribution π without changing the remaining coordinates, $\omega_{[n] \setminus i}$.

Given an ordering $\vec{s} = (s(1), \ldots, s(n))$ of $[n]$, we define the sequential sweep $\mathsf{P}_{\mathsf{seq}}^{(\vec{s})}$ as the random walk operator $\mathsf{P}_{\mathsf{seq}}^{(\vec{s})} := \mathsf{Q}_{s(1)} \cdots \mathsf{Q}_{s(n)}$, which updates the coordinates $s(1), \ldots, s(n)$ sequentially one after the other. When $\vec{s} = (1, \ldots, n)$, i.e. $\vec{s}$ is the canonical order of $[n]$, we will simply write $\mathsf{P}_{\mathsf{seq}} := \mathsf{P}_{\mathsf{seq}}^{(\vec{s})}$. We note that the weighting π of the simplicial complex is stationary for $\mathsf{P}_{\mathsf{seq}}$, since $\pi \mathsf{Q}_i = \pi$ for all $i = 1, \ldots, n$. Thus, $\pi \mathsf{P}_{\mathsf{seq}} = \pi$.

Notice that we have, $\mathsf{P}_{\mathsf{seq}}^* = (\mathsf{Q}_1 \cdots \mathsf{Q}_n)^* = \mathsf{Q}_n^* \cdots \mathsf{Q}_1^* = \mathsf{Q}_n \cdots \mathsf{Q}_1$. In particular the random-walk $\mathsf{P}_{\mathsf{seq}}$ is not necessarily reversible!

It has already been observed in [6, 53], that the update operators Q_i are orthogonal projections.

▶ **Proposition 22.** *For all $i = 1, \ldots, n$ the operator Q_i is an orthogonal projection to the subspace,*

$$\mathcal{U}_i = \operatorname{span}\{\boldsymbol{u}_\alpha : \alpha \in X[1, \cdots, i-1, i+1, \cdots, n]\},$$

where $\boldsymbol{u}_\alpha(\omega) = \mathbf{1}[\omega \supset \alpha]$.

2.7 Colored Random Walks on Partite Simplicial Complexes

It will also be important to recall the colored random walks, introduced in [21, 34]. Let $\alpha \in X$ be any given face of an n-partite weighted simplicial complex. Let $I, J \subset [n] \setminus \operatorname{type}(\alpha)$ be two disjoint sets, i.e. $I \cap J = \varnothing$. We define the (I, J) colored random walk $\mathsf{C}_\alpha^{I \to J} \in \mathbb{R}^{X_\alpha[I] \times X_\alpha[J]}$ from $X_\alpha[I]$ to $X_\alpha[J]$ by setting

$$\mathsf{C}_\alpha^{I \to J}(\tau_I, \tau_J) = \pi_J^{(\alpha \oplus \tau_I)}(\tau_J) = \Pr_{\omega \sim \pi}[\omega_J = \tau_J \mid \omega_I = \tau_I, \omega_{\operatorname{type}(\alpha)} = \alpha] \tag{10}$$

for all $\tau_I \in X_\alpha[I]$ and $\tau_J \in X_\alpha[J]$.

Let $I, J \subset [n]$ be two subsets satisfying $I \cap J = \varnothing$. Suppose for some $\alpha \in X[(I \cup J)^c]$ the marginal distributions $\pi_I^{(\alpha)}$ and $\pi_J^{(\alpha)}$ are independent, i.e. we have

$$\pi_{I\cup J}^{(\alpha)}(\tau_I \oplus \tau_J) = \pi_I^{(\alpha)}(\tau_I) \cdot \pi_J^{(\alpha)}(\tau_J) \quad \text{for all} \quad \tau_I \in X_\alpha[I], \tau_J \in X_\alpha[J].$$

It is easy to see, that in this case $\mathsf{C}_\alpha^{I\to J} = \mathbf{1}\pi_J^{(\alpha)}$. In particular, we have[9]

$$\sigma_2(\mathsf{C}_\alpha^{I\to J}) = \sigma_2(\mathbf{1}\pi_J^{(\alpha)}) = 0 \quad \text{and} \quad \mathcal{D}(\mu \mathsf{C}_\alpha^{I\to J} \parallel \nu \mathsf{C}_\alpha^{I\to J}) = \mathcal{D}(\pi_J^{(\alpha)} \parallel \pi_J^{(\alpha)}) = 0,$$

for all $\mu, \nu \in \triangle_{X_\alpha[I]}$. We will consider relaxations of both statements to consider the proximity of $\pi_I^{(\alpha)}$ and $\pi_J^{(\alpha)}$ to being product distributions. Thus, the (I, J) variance contraction parameter $\varepsilon^{I\to J}$ is defined as,

$$\varepsilon^{I\to J} := \varepsilon_{(X,\pi)}^{I\to J} = \max_{\alpha \in X[(I\cup J)^c]} \sigma_2\big(\mathsf{C}_\alpha^{I\to J}\big) \tag{11}$$

We also define, the related (I, J) entropy contraction parameter

$$\eta^{I\to J} = 1 - \kappa^{I\to J} \;\; ; \; \kappa^{I\to J} = \sup\left\{ \frac{\mathcal{D}(\mu \mathsf{C}_\alpha^{I\to J} \parallel \pi_J^{(\alpha)})}{\mathcal{D}(\mu \parallel \pi_I^{(\alpha)})} : \alpha \in X[(I \cup J)^c] \text{ and } \mu \in \triangle_{X_\alpha[I]} \right\} \tag{12}$$

$\vec{\varepsilon}$-product distributions and $\varepsilon^{I\to J}$

When dealing with the colored random walks $\mathsf{C}_\alpha^{I\to J}$ an alternative notion of high-dimensional expansion introduced in [34] proves more convenient: Let (X, π) be an n-partite simplicial complex with parts $X[1], \cdots, X[n]$. The distribution π is called ε-product if,

$$\sigma_2(\mathsf{C}_\varnothing^{i\to j}) \le \varepsilon \quad \text{for all} \quad i, j \in [n] \text{ where } i \neq j.$$

We extend this definition in the following manner: Let $\vec{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{n-2})$ and (X, π) be an arbitrary n-partite simplicial complex. We say the distribution π is $\vec{\varepsilon}$-product if for all $\alpha \in X^{(\le n-2)}$ we have,

$$\sigma_2(\mathsf{C}_\alpha^{i\to j}) \le \varepsilon_{|\alpha|} \quad \text{for all} \quad i, j \in [n] \setminus \alpha \text{ where } i \neq j.$$

Equivalently, π is $\vec{\varepsilon}$-product if and only if $\pi^{(\alpha)}$ is $\varepsilon_{|\alpha|}$-product for all $\alpha \in X^{(\le n-2)}$. We recall the following result of [21, Claim 7.10],

▶ **Proposition 23.** *Let (X, π) be an n-partite simplicial complex in which every link G_α is connected – equivalently, (X, π) is a one-sided $\vec{\gamma}$-local spectral expander where $\gamma_i < 1$ for all i. Then, the bipartite graph $G_{\{i\},\{j\}}$ described by $\mathsf{C}_\alpha^{i\to j}$ is connected. In particular, there exists a vector $\vec{\varepsilon} = (\varepsilon_i)_{i=0}^{n-2}$ such that $\varepsilon_i < 1$ for all i, for which the distribution π is $\vec{\varepsilon}$-product.*

They also prove the following down-trickling result, [21, Corollary 7.6]

▶ **Theorem 24.** *Let (X, π) be an n-partite simplicial complex all of whose links are connected. Suppose $\gamma_{n-2}(X, \pi) \le \frac{\varepsilon}{(n-2)\varepsilon+1}$. Then, for every $\alpha \in X^{(k)}$ and every $i, j \in [n] \setminus \mathrm{type}(\alpha)$, we have $\sigma_2(\mathsf{C}_\alpha^{i\to j}) \le \frac{\varepsilon}{k\varepsilon+1}$.*

[9] the first equality is the consequence of $\mathsf{C}_\alpha^{I\to J}$ being a rank one matrix

3 A Spectral Gap Bound for $\mathsf{P}_{\mathsf{seq}}$

Our main theorem in this section will be,

▶ **Theorem 25.** *Let (X,π) be an n-partite simplicial complex, $\vec{s} = (s(1),\ldots,s(n))$ be any ordering of $[n]$, and the parameters $\varepsilon^{I\to J}$ be defined as in Equation (11) for all $I, J \subset [n]$ with $I \cap J = \varnothing$. Then,*

$$\sigma(\mathsf{P}^{(\vec{s})}_{\mathsf{seq}})^2 \leq 1 - \prod_{j=2}^{n}\Big(1 - (\varepsilon^{s([j-1])\to s(j)})^2\Big),$$

where we have written $S(A) = \{s(a) : a \in S\}$. Equivalently,

$$\mathsf{gap}(\mathsf{P}^{(\vec{s})}_{\mathsf{seq}}) \geq 1 - \sqrt{1 - \prod_{j=2}^{n}(1 - (\varepsilon^{s([j-1])\to s(j)})^2)}.$$

In particular, if π is $\vec{\varepsilon}$-product, we have

$$\sigma(\mathsf{P}^{(\vec{s})}_{\mathsf{seq}})^2 \leq 1 - \prod_{j=2}^{n}\prod_{p=0}^{j-2}(1 - \varepsilon_p^2).$$

Equivalently,

$$\mathsf{gap}(\mathsf{P}^{(\vec{s})}_{\mathsf{seq}}) \geq 1 - \sqrt{1 - \prod_{j=2}^{n}\prod_{p=0}^{j-2}(1 - \varepsilon_p)^2}$$

In conjunction with Theorem 24, Theorem 25 immediately implies the following

▶ **Corollary 26.** *Let (X,π) be a link-connected n-partite simplicial complex that is a $\vec{\gamma}$-local spectral expander with $\gamma_i \leq \varepsilon/(n-1-i)$ (i.e. π $\vec{\varepsilon}$-product, with $\varepsilon_i \leq \varepsilon$), then*

$$\sigma_2(\mathsf{P}_{\mathsf{seq}}) \leq \frac{n\cdot\varepsilon}{\sqrt{2}} \quad \textit{and} \quad \mathsf{gap}(\mathsf{P}_{\mathsf{seq}}) \geq 1 - \frac{n\cdot\varepsilon}{\sqrt{2}}.$$

If we have $\gamma_{n-2} \leq \frac{\varepsilon}{2n}$ and all links of (X,π) are connected, then

$$\sigma_2(\mathsf{P}_{\mathsf{seq}}) \leq \frac{\varepsilon}{\sqrt{2}} \quad \textit{and} \quad \mathsf{gap}(\mathsf{P}_{\mathsf{seq}}) \geq 1 - \frac{\varepsilon}{\sqrt{2}}.$$

For the second statement we note that Theorem 24 implies that we are completely ε/n product assuming $\gamma_{n-2} \leq \varepsilon/2n$. In particular, the ε in the statement of Theorem 24 and the ε above differ by a factor of n. The above corollary shows that for constant n, having constant local spectral expansion guarentees that $\mathsf{gap}(\mathsf{P}_{\mathsf{seq}})$ is close to 1 – thus it allows us to conclude that $\mathsf{P}_{\mathsf{seq}}$ mixes rapidly in the complexes of [47, 41].

We will prove Theorem 25 using Theorem 14. To be able apply Theorem 14, we will need some preparatory work:

1. For all $T \subset [n]$, we will characterize the intersections $\bigcap_{t\in T}\mathcal{U}_t$, where $\mathcal{U}_t$ are as defined in Proposition 22, i.e. $\mathcal{U}_t = \operatorname{im}(\mathsf{Q}_t)$. This will be done in Proposition 27.
2. We will get a bound on $\sin^2(\mathcal{U}_p, \bigcap_{t\in T}\mathcal{U}_t)$ for all $p \in [n]$ and $T \subset [n]\setminus p$ in terms of the parameters $\varepsilon^{I\to J}$, defined in Equation (11). This will be done in Theorem 29.
3. We will get a bound on the parameters $\varepsilon^{I\to J}$ under the assumption that π is $\vec{\varepsilon}$-product. This will be done in Theorem 30.

We state these results now.

▶ **Proposition 27.** *For (X,π) a weighted and link-connected n-partite simplicial complex. Let $\mathcal{U}_1,\dots,\mathcal{U}_n$ be as defined in Proposition 22. For all $T \subset [n]$, we define*

$$\mathcal{U}_T := \operatorname{span}\{\boldsymbol{u}_\alpha : \alpha \in X[T^c]\}.$$

Then, $\mathcal{U}_T = \bigcap_{t\in T} \mathcal{U}_t$.

▶ Remark 28. We have,

$$\mathcal{U}_I = \operatorname{span}\Big\{\boldsymbol{f} \in \mathbb{R}^{X^{(n)}} : \boldsymbol{f}(\omega) = \boldsymbol{f}(\bar{\omega}) \text{ for all } \omega, \bar{\omega} \in X^{(n)} \text{ such that } \omega_{[n]\setminus I} = \bar{\omega}_{[n]\setminus I}\Big\}.$$

In particular, $\mathcal{U}_I$ is the space of vectors $\boldsymbol{f} \in \mathbb{R}^{X^{(n)}}$ whose entries are completely determined by the coordinates in $[n] \setminus I$.

We note that a special case of this result was already proven in [53, Proposition 3.6]. We will present a proof for Proposition 27, in Section 3.2.

Then, we show the geometric interpretation for the quantities $\varepsilon^{I\to J}$ as follows,

▶ **Theorem 29.** *Let (X,π) be an n-partite simplicial complex. Let $I, J \subset [n]$ be any pair of disjoint sets, i.e. $I \cap J = \varnothing$.*

Then, $\cos(\mathcal{U}_I,\mathcal{U}_J) \le \varepsilon^{I\to J}$, where the subspaces $\mathcal{U}_I,\mathcal{U}_J$ are as defined in Proposition 27 and the parameters $\varepsilon^{I\to J}$ are as defined in Equation (11).

We will prove Theorem 29 in Section 3.3.

Finally, we state our bound on the parameters $\varepsilon^{I\to J}$,

▶ **Theorem 30.** *Let (X,π) be an n-partite simplicial complex where π is $\vec{\varepsilon}$-product. We have,*

$$\sigma_2(\mathsf{C}_\alpha^{I\to J})^2 \le 1 - \prod_{p=0}^{|I|-1}\prod_{q=0}^{|J|-1}(1-\varepsilon^2_{|\alpha|+p+q}) \quad \textit{and thus} \quad (\varepsilon^{I\to J})^2 \le 1 - \prod_{p=0}^{|I|-1}\prod_{j=0}^{|J|-1}(1-\varepsilon^2_{n-|I|-|J|+p+q}) \tag{13}$$

▶ Remark 31. Notice that the RHS of Equation (13) is always strictly smaller than 1 provided that π is $\vec{\varepsilon}$-product with $\varepsilon_i < 1$ for all $i = 0,\dots,n-2$. By Proposition 23, this is always the case assuming all the links in (X,π) are connected.

We will present the proof for Theorem 30 in Section 3.1.

The proof of Theorem 25 is now immediate from Theorem 14.

3.1 Proof of Theorem 30

Proof of Theorem 30. If $|I| = |J| = 1$, then the RHS of Equation (13) equals $\varepsilon_{|\alpha|}$ and thus the statement follows since π is $\vec{\varepsilon}$-product. Suppose without loss of generality, that $\alpha = \varnothing$ and $|I| \ge 2$ where we have some fixed element $i \in I$. Write,

$$\sigma := \max_{x\in X[i]} \sigma_2\Big(\mathsf{C}_x^{(I\setminus i)\to J}\Big). \tag{14}$$

Let $\boldsymbol{F} \in X[J] \to \mathbb{R}$ satisfy $\|\boldsymbol{F}\|_{\pi_J} = 1$, $\langle \boldsymbol{F}, \mathbf{1}\rangle_{\pi_J} = 0$, and $\|\mathsf{C}_\varnothing^{I\to J}\boldsymbol{F}\|_{\pi_I} = \sigma_2(\mathsf{C}_\varnothing^{I\to J})$.

Then,

$$\begin{aligned}
\sigma_2\big(\mathsf{C}_{\varnothing}^{I\to J}\big)^2 = \|\mathsf{C}_{\varnothing}^{I\to J}\boldsymbol{F}\|_{\pi_I}^2 &= \mathop{\mathbb{E}}_{\alpha_I\sim\pi_I}\left(\mathop{\mathbb{E}}_{\alpha_J\sim\pi_J^{(\alpha_I)}}\boldsymbol{F}(\alpha_J)\right)^2,\\
&= \mathop{\mathbb{E}}_{x\sim\pi_{\{i\}}}\mathop{\mathbb{E}}_{\alpha'\sim\pi_{I\setminus i}^{(x)}}\left(\mathop{\mathbb{E}}_{\alpha_J\sim\pi_J^{(x\oplus\alpha')}}\boldsymbol{F}(\alpha_J)\right)^2,\\
&= \mathop{\mathbb{E}}_{x\sim\pi_{\{i\}}}\left\|\mathsf{C}_x^{(I\setminus i)\to J}\boldsymbol{F}_x\right\|_{\pi_{I\setminus i}^{(x)}}^2,\\
&\leq \mathop{\mathbb{E}}_{x\sim\pi_{\{i\}}}\left[\left\|\boldsymbol{F}_x^{\mathbf{1}}\right\|_{\pi_J^{(x)}}^2 + \sigma_2(\mathsf{C}_x^{(I\setminus i)\to J})^2\cdot\left\|\boldsymbol{F}_x^{\perp\mathbf{1}}\right\|_{\pi_J^{(x)}}^2\right],\\
&= \sigma^2 + (1-\sigma^2)\cdot\mathop{\mathbb{E}}_{x\sim\pi_{\{i\}}}\|\boldsymbol{F}_x^{\mathbf{1}}\|_{\pi_J^{(x)}}^2
\end{aligned}$$

where $\boldsymbol{F}_x^{\mathbf{1}} = \mathbf{1}\cdot\mathbb{E}_\alpha\,\boldsymbol{F}_x(\alpha_{J\setminus i})$. We note now for each $i \in X[i]$,

$$\left\|\boldsymbol{F}_x^{\mathbf{1}}\right\|_{\pi_J}^2 = \left(\mathop{\mathbb{E}}_{\alpha_J\sim\pi_J^{(x)}}\boldsymbol{F}(\alpha_J)\right)^2 = [\mathsf{C}_{\varnothing}^{i\to J}\boldsymbol{F}](x)^2.$$

Thus, using that $\langle\boldsymbol{F},\mathbf{1}\rangle_{\pi_J} = 0$,

$$\mathop{\mathbb{E}}_{x\sim\pi_{\{1\}}}\left\|\boldsymbol{F}_x^{\mathbf{1}}\right\|_{\pi_J}^2 = \|\mathsf{C}_{\varnothing}^{i\to J}\boldsymbol{F}\|_{\pi_{\{i\}}}^2 \leq \sigma_2\big(\mathsf{C}_{\varnothing}^{i\to J}\big)^2.$$

Thus,

$$\begin{aligned}
\sigma_2\big(\mathsf{C}_{\varnothing}^{I\to J}\big)^2 &\leq 1-(1-\sigma^2)(1-\sigma_2(\mathsf{C}_{\varnothing}^{i\to J})^2),\\
&= 1-\left(1-\sigma_2\Big(\mathsf{C}_{x^\star}^{I\setminus i\to J}\Big)^2\right)\left(1-\sigma_2\big(\mathsf{C}_{\varnothing}^{i\to J}\big)^2\right).
\end{aligned} \tag{15}$$

where $x^\star = \arg\max_{x\in X[i]}\sigma_2(\mathsf{C}_x^{I\setminus i\to J})$.

Now, we first assume $J = \{j\}$ and $I = \{i_1,\ldots,i_\ell\}$, applying Equation (15) recursively, we obtain

$$\sigma_2(\mathsf{C}_{\varnothing}^{I\to\{j\}}) \leq 1-(1-\sigma_2(\mathsf{C}_{\varnothing}^{i_1\to j})^2)\prod_{p=k}^{\ell}\left(1-\sigma_2\Big(\mathsf{C}_{x_{i_1}\oplus\cdots\oplus x_{i_{p-1}}}^{i_p\to\{j\}}\Big)^2\right),$$

where

$$x_{i_1} = x^\star,\quad \omega(p) = x_{i_1}\oplus\cdots x_{i_{p-1}},\quad\text{and}\quad x_p = \arg\max_{x\in X_{\omega(p)}[i_p]}\sigma_2\Big(\mathsf{C}_{x_{i_1}\oplus\cdots\oplus x_{i_{p-1}}\oplus x}^{I\setminus\{i_1,\ldots,i_p\}\to\{j\}}\Big),$$

In particular using the $\vec{\varepsilon}$-product assumption,

$$\sigma_2(\mathsf{C}_{\varnothing}^{I\to j})^2 \leq 1-\prod_{p=0}^{|I|-1}(1-\varepsilon_p^2).$$

Now, by using $\mathsf{C}_{\varnothing}^{J\to i} = \big(\mathsf{C}_{\varnothing}^{i\to J}\big)^*$ and that $\sigma_2(\mathsf{B}) = \sigma_2(\mathsf{B}^*)$ we have by the above

$$\sigma_2(\mathsf{C}_{\varnothing}^{i\to J})^2 \leq 1-\prod_{p=0}^{|J|-1}(1-\varepsilon_p^2). \tag{16}$$

Similarly, inserting Equation (16) inside Equation (15), we obtain

$$\sigma_2\big(\mathsf{C}_{\varnothing}^{I\to J}\big)^2 \le 1 - \prod_{p=0}^{|J|-1}(1-\varepsilon_p^2)\cdot\left(1-\sigma_2\left(\mathsf{C}_{x^\star}^{I\setminus i\to J}\right)^2\right). \tag{17}$$

Suppose $I=\{i_1,\dots,i_\ell\}$ write $x_{i_1}=x^\star$ and

$$\omega(q)=x_{i_1}\oplus\cdots x_{i_{q-1}},\quad\text{and}\quad x_q=\arg\max_{x\in X_{\omega(q)}[i_q]}\sigma_2\left(\mathsf{C}_{\omega(q)\oplus x}^{I\setminus\{i_1,\dots,i_p\}\to J}\right),$$

Applying Equation (17) repeatedly, we obtain

$$\sigma_2(\mathsf{C}_{\varnothing}^{I\to J})^2 \le 1-\left(\prod_{p=0}^{|J|-1}(1-\varepsilon_p^2)\right)\cdot\prod_{q=1}^{|I|-1}\left(1-\sigma_2\left(\mathsf{C}_{x_{i_1}\oplus\cdots\oplus x_{i_q}}^{i_{q+1}\to J}\right)^2\right). \tag{18}$$

By Equation (16), we have

$$\sigma_2\left(\mathsf{C}_{x_{i_1}\oplus\cdots\oplus x_{i_q}}^{i_{q+1}\to J}\right)^2 \le 1-\prod_{p=0}^{|J|-1}(1-\varepsilon_{q+p}^2)$$

And inserting this in Equation (18), we obtain

$$\begin{aligned}\sigma_2\big(\mathsf{C}_{\varnothing}^{I\to J}\big)^2 \;&\le\; 1-\left(\prod_{p=0}^{|J|-1}(1-\varepsilon_p^2)\right)\cdot\left(\prod_{p=0}^{|J|-1}(1-\varepsilon_{p+1}^2)\right)\cdots\left(\prod_{p=0}^{|J|-1}(1-\varepsilon_{p+|I|-1}^2)\right),\\ &=\; 1-\prod_{q=0}^{|I|-1}\prod_{p=0}^{|I|-1}(1-\varepsilon_{p+q}^2).\end{aligned}$$ ◀

3.2 Proof of Proposition 27

Proof of Proposition 27. We proceed by induction on the number of elements in T. For $|T|=1$, the statement is clear. Thus, we assume that

$$\mathcal{U}_T=\bigcap_{t\in T}\mathcal{U}_t \quad\text{for all}\quad T\quad\text{with}\quad |T|=\ell.$$

Let $t\in[n]\setminus T$, and consider $T\cup\{t\}$. First, fix any $\alpha\in X[[n]\setminus(T\cup t)]$. Then, note

$$\boldsymbol{u}_\alpha=\sum_{\substack{\beta\in X[[n]\setminus T],\\ \beta\supset\alpha}}\boldsymbol{u}_\beta=\sum_{\substack{\beta'\in X[1,\dots,t-1,t+1,\dots,n],\\ \beta'\supset\alpha}}\boldsymbol{u}_{\beta'}.$$

Thus, indeed: $\operatorname{span}\{\boldsymbol{u}_\alpha:\alpha\in X[[n]\setminus(T\cup t)]\}=\mathcal{U}_{T\cup t}\subset\mathcal{U}_T\cap\mathcal{U}_t$.

To show the reverse inclusion, we recall that a function $\boldsymbol{f}\in\mathbb{R}^{X^{(n)}}$ is in $\mathcal{U}_A$ if and only if we have

$$\boldsymbol{f}(\omega_A\oplus\omega_{[n]\setminus A})=\boldsymbol{f}(\tilde\omega_A\oplus\omega_{[n]\setminus A})$$

for all $\omega_A,\tilde\omega_A\in X[A]$ and $\omega_{[n]\setminus A}\in X[[n]\setminus A]$ such that $\omega_A\oplus\omega_{[n]\setminus A},\widetilde{\omega}_A\oplus\omega_{[n]\setminus A}\in X^{(n)}$.

i.e. the value of $\boldsymbol{f}$ is determined only by the coordinates in $[n]\setminus A$ (qv. Remark 28).

Suppose $\boldsymbol{f} \in \mathcal{U}_T \cap \mathcal{U}_t$. We want to prove,

$$\boldsymbol{f}(\omega_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)}) = \boldsymbol{f}(\tilde{\omega}_t \oplus \tilde{\omega}_T \oplus \omega_{[n]\setminus(T\cup t)}), \tag{19}$$

for all $\omega_t, \tilde{\omega}_t \in X[t]$, $\omega_T, \tilde{\omega}_T \in X[T]$ and $\omega_{[n]\setminus(T\cup t)} \in X[(T\cup t)^c]$ such that $\omega_t \oplus \omega_T \oplus \omega_{(T\cup t)^c}, \widetilde{\omega}_t \oplus \widetilde{\omega}_T \oplus \omega_{(T\cup t)^c} \in X^{(n)}$.

Since $\boldsymbol{f} \in \mathcal{U}_t$, we have,

$$\boldsymbol{f}(\omega_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)}) = \boldsymbol{f}(\tilde{\omega}_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)})$$

for all $\omega_t, \tilde{\omega}_t \in X[t]$, $\omega_T \in X[T]$, and $\omega_{[n]\setminus(T\cup t)} \in X[[n] \setminus (T\cup t)]$ such that $\omega_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)}, \tilde{\omega}_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)} \in X^{(n)}$. Similarly, since $\boldsymbol{f} \in \mathcal{U}_T$, we have,

$$\boldsymbol{f}(\omega_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)}) = \boldsymbol{f}(\omega_t \oplus \tilde{\omega}_T \oplus \omega_{[n]\setminus(T\cup t)})$$

for all $\omega_t \in X[t]$, $\omega_T, \tilde{\omega}_T \in X[T]$, and $\omega_{[n]\setminus(T\cup t)} \in X[[n] \setminus (T\cup t)]$ such that $\omega_t \oplus \omega_T \oplus \omega_{[n]\setminus(T\cup t)}, \omega_t \oplus \tilde{\omega}_T \oplus \omega_{[n]\setminus(T\cup t)} \in X^{(n)}$. Fixing $\omega_{[n]\setminus(T\cup t)}$, this means that Equation (19) is satisfied by any pair $(\omega_t \oplus \omega_T, \widetilde{\omega}_t \oplus \widetilde{\omega}_T)$ which is connected by an edge in the line graph[10] $L(G_{t,T}^{(\omega_{[n]\setminus(T\cup t)})})$, where $G_{t,T}^{(\omega_{[n]\setminus(T\cup t)})}$ is the graph underlying $\mathsf{C}^{t\to T}_{\omega_{[n]\setminus(T\cup t)}}$. By Remark 31, since the links of (X,π) are connected, so is the graph $G_{t,T}^{(\omega_{[n]\setminus(T\cup t)})}$ for any $\omega_{[n]\setminus(T\cup t)} \in X[[n]\setminus(T\cup t)]$. In particular, the graph $L(G_{t,T}^{(\omega_{[n]\setminus(T\cup t)})})$ is connected, meaning any pair of $\omega_t \oplus \omega_T, \tilde{\omega}_t \oplus \tilde{\omega}_T \in X_{\omega_{[n]\setminus(T\cup t)}}[T\cup t]$ are connected by a path and therefore satisfy Equation (19). This allows us to conclude $\boldsymbol{f} \in \mathcal{U}_{T\cup t}$. ◀

3.3 Proof of Theorem 29

Proof. By Proposition 27 we have $\mathcal{U}_I \cap \mathcal{U}_J = \mathcal{U}_{I\cup J}$ and by Equation (4) we have,

$$\cos(\mathcal{U}_I, \mathcal{U}_J) = \max\{\langle \boldsymbol{h}, \boldsymbol{g}\rangle_\pi : \|\boldsymbol{h}\|_\pi = \|\boldsymbol{g}\|_\pi = 1 \;\text{ and }\; \boldsymbol{g} \in \mathcal{U}_I \cap \mathcal{U}_{I\cup J}^{\perp}, \boldsymbol{h} \in \mathcal{U}_J \cap \mathcal{U}_{I\cup J}^{\perp}\}. \tag{20}$$

Now, let $\boldsymbol{h}$ and $\boldsymbol{g}$ be the optimizers of the formula in Equation (20). For convenience we introduce the functions $\boldsymbol{G} : X[[n] \setminus I] \to \mathbb{R}$ and $\boldsymbol{H} : X[[n] \setminus J] \to \mathbb{R}$ where,

$$\begin{aligned} \boldsymbol{h} &:= \sum_{\alpha \in X[[n]\setminus J]} \boldsymbol{H}(\alpha) \cdot \boldsymbol{u}_\alpha, \\ \boldsymbol{g} &:= \sum_{\bar{\alpha} \in X[[n]\setminus I]} \boldsymbol{G}(\bar{\alpha}) \cdot \boldsymbol{u}_{\bar{\alpha}}. \end{aligned}$$

where the vectors $\boldsymbol{u}_\bullet$ are as defined in Proposition 22, i.e. $\boldsymbol{u}_\tau(\omega) = \mathbf{1}[\omega \supset \tau]$ for all $\tau \in X$. We observe that both $\boldsymbol{H}$ and $\boldsymbol{G}$ have the same norm as $\boldsymbol{h}$ and $\boldsymbol{g}$ respectively.

▷ Claim 32. We have,

$$\|\boldsymbol{H}\|_{\pi_{[n]\setminus J}} = \|\boldsymbol{G}\|_{\pi_{[n]\setminus I}} = 1.$$

Proof. Since the proofs are identical, we only prove the statement for $\boldsymbol{H}$. By the orthogonality of the collection $\boldsymbol{u}_\bullet$ we have,

$$\|\boldsymbol{h}\|_\pi^2 \;=\; \sum_{\alpha \in X[[n]\setminus J]} \boldsymbol{H}^2(\alpha) \cdot \|\boldsymbol{u}_\alpha\|_\pi^2.$$

[10] The line graph $L(G)$ of a simple graph G is the graph $L(G) = (E, E')$ where $ef \in E'$ if $|e \cap f| = 1$, i.e. there is an edge $ef \in E'$ for any pair of distinct edges sharing an end point.

Notice, for any $\alpha \in X[[n] \setminus J]$,

$$\|\boldsymbol{u}_\alpha\|_\pi^2 = \sum_{\substack{\beta \in X^{(n)},\\ \beta \supset \alpha}} \pi(\beta) = \pi_{[n]\setminus J}(\alpha),$$

where we have used the definition of $\pi_{[n]\setminus J}$, i.e. Equation (5) for this equality.

In particular,

$$\|\boldsymbol{h}\|_\pi^2 = \sum_{\alpha \in X[[n]\setminus J]} \boldsymbol{H}(\alpha)^2 \cdot \pi_{[n]\setminus J}(\alpha) = \|\boldsymbol{H}\|_{\pi_{[n]\setminus J}}.$$

Since $\boldsymbol{h}$ is picked as an optimizer to Equation (20), we have $\|\boldsymbol{H}\|_{\pi_{[n]\setminus J}} = 1$. ◁

Then, note

$$\begin{aligned} \langle \boldsymbol{h}, \boldsymbol{g} \rangle_\pi &= \sum_{\alpha \in X[[n]\setminus J]} \sum_{\bar{\alpha} \in X[[n]\setminus I]} \boldsymbol{H}(\alpha)\boldsymbol{G}(\bar{\alpha}) \cdot \langle \boldsymbol{u}_\alpha, \boldsymbol{u}_{\bar{\alpha}} \rangle_\pi, \\ &= \sum_{\alpha \in X[[n]\setminus J]} \sum_{\bar{\alpha} \in X[[n]\setminus I]} \boldsymbol{H}(\alpha)\boldsymbol{G}(\bar{\alpha}) \cdot \pi_n(\alpha \wedge \bar{\alpha}). \end{aligned}$$

where we define $\alpha \wedge \bar{\alpha}$ as $\perp$[11] if there exists some $\ell \in [n] \setminus (I \cup J)$ such that $\alpha_\ell \neq (\bar{\alpha})_\ell$ and otherwise we have,

$$(\alpha \wedge \bar{\alpha})_i = \begin{cases} \alpha_i & \text{if } i \in I, \\ \bar{\alpha}_i & \text{if } i \in J, \\ \alpha_i = (\bar{\alpha})_i & \text{if } i \in [n] \setminus (I \cup J) \end{cases}$$

In particular, since the $(\alpha, \bar{\alpha})$-term is only non-$\perp$ when α and $\bar{\alpha}$ agree on the coordinates $[n]\setminus(I\cup J)$, we can localize this sum at the intersections $\tilde{\alpha} \in X[[n]\setminus(I\cup J)]$. In particular, writing $\boldsymbol{H}_{\tilde{\alpha}}(\alpha_I) = \boldsymbol{H}(\tilde{\alpha} \oplus \alpha_I)$ and $\boldsymbol{G}_{\tilde{\alpha}}(\alpha_J) = \boldsymbol{G}(\tilde{\alpha} \oplus \alpha_J)$ for $\alpha_I \in X_{\tilde{\alpha}}[I]$ and $\alpha_J \in X_{\tilde{\alpha}}[J]$, we have

$$\begin{aligned} \langle \boldsymbol{h}, \boldsymbol{g} \rangle_\pi &= \sum_{\alpha \in X[[n]\setminus J]} \sum_{\bar{\alpha} \in X[\tilde{\alpha} \in X[[n]\setminus I]]} \boldsymbol{H}(\alpha)\boldsymbol{G}(\bar{\alpha}) \cdot \pi_n(\alpha \wedge \bar{\alpha}), \\ &= \sum_{\tilde{\alpha} \in X[[n]\setminus(I\cup J)]} \left(\sum_{\alpha_I \in X_{\tilde{\alpha}}[I]} \sum_{\alpha_J \in X_{\tilde{\alpha}}[J]} \boldsymbol{H}_{\tilde{\alpha}}(\alpha_I)\boldsymbol{G}_{\tilde{\alpha}}(\alpha_J)\pi(\alpha_I \oplus \alpha_J \oplus \tilde{\alpha}) \right), \\ &=^{\blacklozenge} \sum_{\tilde{\alpha} \in X[[n]\setminus(I\cup J)]} \pi_{[n]\setminus(I\cup J)}(\tilde{\alpha}) \cdot \left(\sum_{\alpha_J \in X_{\tilde{\alpha}}[J]} \sum_{\alpha_I \in X_{\tilde{\alpha}}[I]} \boldsymbol{H}_{\tilde{\alpha}}(\alpha_I)\boldsymbol{G}_{\tilde{\alpha}}(\alpha_J) \cdot \pi_{I\cup J}^{(\tilde{\alpha})}(\alpha_I \oplus \alpha_J). \right), \\ &= \mathop{\mathbb{E}}_{\tilde{\alpha} \sim \pi_{[n]\setminus(I\cup J)}} \left\langle \boldsymbol{H}_{\tilde{\alpha}}, \mathsf{C}_{\tilde{\alpha}}^{J\to I} \boldsymbol{G}_{\tilde{\alpha}} \right\rangle_{\pi_I^{\tilde{\alpha}}} \end{aligned} \tag{21}$$

where we have used Equation (7) to get the equality marked by (♦). To conclude, we observe that both $\boldsymbol{G}_{\tilde{\alpha}}$ and $\boldsymbol{H}_{\tilde{\alpha}}$ are perpendicular to the constant functions.

▷ Claim 33. For all $\tilde{\alpha} \in X[[n] \setminus (I \cup J)]$ we have,

$$\langle \boldsymbol{H}_{\tilde{\alpha}}, \boldsymbol{1} \rangle_{\pi_I^{(\tilde{\alpha})}} = \langle \boldsymbol{G}_{\tilde{\alpha}}, \boldsymbol{1} \rangle_{\pi_J^{(\tilde{\alpha})}} = 0.$$

▷ Claim 34. For any $S, T \subset [n]$ with $S \cap T = \varnothing$ and $\boldsymbol{F} : X[S \cup T] \to \mathbb{R}$, we have

$$\|\boldsymbol{F}\|_{S\cup T}^2 = \mathop{\mathbb{E}}_{\alpha \sim \pi_T} \|\boldsymbol{F}_\alpha\|_{\pi_S^{(\alpha)}}^2.$$

[11] in particular, $\pi(\alpha \wedge \bar{\alpha}) = 0$

Thus, we can continue upper-bounding Equation (21) as follows,

$$\begin{aligned}\langle \boldsymbol{h}, \boldsymbol{g}\rangle_\pi &\leq \mathop{\mathbb{E}}_{\tilde{\alpha}\sim\pi_{[n]\setminus(I\cup J)}}\left[\sigma_2\big(\mathsf{C}_{\tilde{\alpha}}^{J\to I}\big)\cdot \|\boldsymbol{H}_{\tilde{\alpha}}\|_{\pi_I^{(\tilde{\alpha})}}\cdot\|\boldsymbol{G}_{\tilde{\alpha}}\|_{\pi_J^{(\tilde{\alpha})}}\right],\\ &\leq \mathop{\mathbb{E}}_{\tilde{\alpha}\sim\pi_{[n]\setminus(I\cup J)}}\left[\sigma_2\big(\mathsf{C}_{\tilde{\alpha}}^{J\to I}\big)\cdot\frac{\|\boldsymbol{H}_{\tilde{\alpha}}\|^2_{\pi_I^{(\tilde{\alpha})}}+\|\boldsymbol{G}_{\tilde{\alpha}}\|^2_{\pi_I^{(\tilde{\alpha})}}}{2}\right],\\ &\leq \varepsilon^{J\to I}.\end{aligned}$$

We now present the proofs of Claim 33 and Claim 34.

Proof of Claim 33. Let $\tilde{\alpha}\in X[[n]\setminus(I\cup J)]$ be fixed. Since the proofs will be identical we will only prove the statement for $\boldsymbol{H}_{\tilde{\alpha}}$. Since $\boldsymbol{h}$ is picked as an optimizer to Equation (20), we have $\langle \boldsymbol{h}, \boldsymbol{u}_{\tilde{\alpha}}\rangle_\pi = 0$. We also have,

$$\langle \boldsymbol{u}_{\tilde{\alpha}}, \boldsymbol{h}\rangle_\pi = \sum_{\alpha\in X[[n]\setminus J]} \boldsymbol{H}(\alpha)\cdot\langle \boldsymbol{u}_{\tilde{\alpha}}, \boldsymbol{u}_\alpha\rangle_\pi = \sum_{\alpha\in X[[n]\setminus J]} \boldsymbol{H}(\alpha)\cdot\langle \boldsymbol{1}, \boldsymbol{u}_\alpha\rangle_\pi\cdot \boldsymbol{1}[\alpha\supset\tilde{\alpha}],$$

where we have used that $\mathrm{supp}(\boldsymbol{u}_\alpha)\subset\mathrm{supp}(\boldsymbol{u}_{\tilde{\alpha}})$ whenever $\tilde{\alpha}\subset\alpha$. Thus, using $\langle \boldsymbol{1}, \boldsymbol{u}_\alpha\rangle_\pi = \pi_{[n]\setminus J}(\alpha)$ we can simplify the above into

$$0 = \sum_{\substack{\alpha\in X[[n]\setminus J],\\ \alpha\supset\tilde{\alpha}}} \boldsymbol{H}(\alpha)\cdot\pi_{[n]\setminus J}(\alpha).$$

Every $\alpha\in X[[n]\setminus J]$ satisfying $\alpha\supset\tilde{\alpha}$ can be written as $\alpha=\tilde{\alpha}\oplus\alpha_I$ for a unique choice of $\alpha_I\in X_{\tilde{\alpha}}[I]$. Thus, using $\boldsymbol{H}(\tilde{\alpha}\oplus\alpha_I)=\boldsymbol{H}_{\tilde{\alpha}}(\alpha_I)$, we obtain

$$0 = \sum_{\alpha_I\in X_{\tilde{\alpha}}[I]} \boldsymbol{H}_{\tilde{\alpha}}(\alpha_I)\cdot\pi_{[n]\setminus J}(\tilde{\alpha}\oplus\alpha_I) = \pi_{[n]\setminus(I\cup J)}(\tilde{\alpha})\cdot\sum_{\alpha_I\in X_{\tilde{\alpha}}[I]}\boldsymbol{H}_{\tilde{\alpha}}(\alpha_I)\cdot\pi_I^{(\tilde{\alpha})}(\alpha_I),$$

where the last inequality is through Equation (7). The theorem then follows by noting that the term on the RHS is a multiple of $\langle \boldsymbol{H}_{\tilde{\alpha}}, \boldsymbol{1}\rangle_{\pi_I^{(\tilde{\alpha})}}$. ◁

Proof Claim 34. We have,

$$\begin{aligned}\|\boldsymbol{F}\|^2_{\pi_{S\cup T}} &= \sum_{\alpha\in X[S\cup T]} \boldsymbol{F}(\alpha)^2\cdot\pi_{S\cup T}(\alpha),\\ &= \sum_{\alpha\in X[S\cup T]} \pi_S(\alpha_S)\pi_T^{(\alpha_S)}(\alpha_T)\boldsymbol{F}_{\alpha_S}(\alpha_T)^2,\\ &= \mathop{\mathbb{E}}_{\alpha_S\sim\pi_S}\|\boldsymbol{F}_{\alpha_S}\|^2_{\pi_T^{(\alpha_S)}}\end{aligned}$$

where we have used Equation (7) to obtain the second equality. ◁

◀

References

1 David Aldous. Random walks on finite groups and rapidly mixing Markov chains. In *Séminaire de Probabilités XVII 1981/82*, pages 243–297. Springer, 1983.

2 Vedat Levi Alev, Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, and Madhur Tulsiani. List decoding of lifted codes. In *SODA*, 2020.

3 Vedat Levi Alev, Fernando Granha Jeronimo, and Madhur Tulsiani. Approximating constraint satisfaction problems on high dimensional expanders. In *FOCS*, 2019.

4 Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *STOC*, pages 1198–1211, 2020. doi:10.1145/3357713.3384317.

5 Vedat Levi Alev and Shravas Rao. Expanderizing higher order random walks, 2024. doi:10.48550/arXiv.2405.08927.

6 Yali Amit. On rates of convergence of stochastic relaxation for gaussian and non-gaussian distributions. *Journal of Multivariate Analysis*, 38(1):82–99, 1991.

7 Nima Anari, Vishesh Jain, Frederic Koehler, Huy Tuan Pham, and Thuy-Duong Vuong. Entropic independence ii: optimal sampling and concentration via restricted modified log-sobolev inequalities. *arXiv preprint*, 2021. arXiv:2111.03247.

8 Nima Anari, Vishesh Jain, Frederic Koehler, Huy Tuan Pham, and Thuy-Duong Vuong. Entropic independence: optimal mixing of down-up random walks. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1418–1430, 2022. doi:10.1145/3519935.3520048.

9 Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model. *CoRR*, abs/2001.00303, 2020. arXiv:2001.00303.

10 Mitali Bafna, Max Hopkins, Tali Kaufman, and Shachar Lovett. High dimensional expanders: Eigenstripping, pseudorandomness, and unique games. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1069–1128. SIAM, 2022. doi:10.1137/1.9781611977073.47.

11 Mitali Bafna, Max Hopkins, Tali Kaufman, and Shachar Lovett. Hypercontractivity on high dimensional expanders. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 185–194, 2022. doi:10.1145/3519935.3520040.

12 Antonio Blanca, Pietro Caputo, Zongchen Chen, Daniel Parisi, Daniel Štefankovič, and Eric Vigoda. On mixing of markov chains: Coupling, spectral independence, and entropy factorization. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 3670–3692. SIAM, 2022. doi:10.1137/1.9781611977073.145.

13 Antonio Blanca, Pietro Caputo, Daniel Parisi, Alistair Sinclair, and Eric Vigoda. Entropy decay in the swendsen–wang dynamics on zd. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1551–1564, 2021. doi:10.1145/3406325.3451095.

14 Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities - A Nonasymptotic Theory of Independence*. Oxford University Press, 2013. doi:10.1093/ACPROF:OSO/9780199535255.001.0001.

15 Russ Bubley and Martin E. Dyer. Path coupling: A technique for proving rapid mixing in Markov chains. In *FOCS*, pages 223–231, 1997. doi:10.1109/SFCS.1997.646111.

16 Djalil Chafaï. Entropies, convexity, and functional inequalities, on ϕ-entropies and ϕ-sobolev inequalities. *Journal of Mathematics of Kyoto University*, 44(2):325–363, 2004.

17 Zongchen Chen, Andreas Galanis, Daniel Stefankovic, and Eric Vigoda. Rapid mixing for colorings via spectral independence. *CoRR*, abs/2007.08058, 2020. arXiv:2007.08058.

18 Zongchen Chen, Kuikui Liu, and Eric Vigoda. Rapid mixing of Glauber dynamics up to uniqueness via contraction. *arXiv preprint*, 2020. arXiv:2004.09083.

19 Mary Cryan, Heng Guo, and Giorgos Mousa. Modified log-Sobolev inequalities for strongly log-concave distributions. *CoRR*, abs/1903.06081, 2019. arXiv:1903.06081.

20 Persi Diaconis and Arun Ram. Analysis of systematic scan metropolis algorithms using iwahori-hecke algebra techniques. *Michigan Mathematical Journal*, 48(1):157–190, 2000.

21 Yotam Dikstein and Irit Dinur. Agreement testing theorems on layered set systems. In *Proceedings of the 60th annual IEEE symposium on foundations of computer science, FOCS 2019, Baltimore, MD, USA, November 10–12, 2019*, pages 1495–1524. Los Alamitos, CA: IEEE Computer Society, 2019. doi:10.1109/FOCS.2019.00088.

22 Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. Boolean function analysis on high-dimensional expanders. In *APPROX/RANDOM*, pages 38:1–38:20, 2018. doi:10.4230/LIPIcs.APPROX-RANDOM.2018.38.

23 Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders. In *FOCS*, pages 974–985, 2017. doi:10.1109/FOCS.2017.94.

24 Roland L Dobrushin. Prescribing a system of random variables by conditional distributions. *Theory of Probability & Its Applications*, 15(3):458–486, 1970.

25 Martin Dyer, Leslie Ann Goldberg, and Mark Jerrum. Systematic scan for sampling colorings. *The Annals of Applied Probability*, 16(1):185–230, 2006.

26 Martin Dyer, Leslie Ann Goldberg, and Mark Jerrum. Dobrushin conditions and systematic scan. *Combinatorics, Probability and Computing*, 17(6):761–779, 2008. doi:10.1017/S0963548308009437.

27 Jan Dymara and Tadeusz Januszkiewicz. New kazhdan groups. *Geometriae Dedicata*, 80:311–317, 2000.

28 Weiming Feng, Heng Guo, Yitong Yin, and Chihao Zhang. Rapid mixing from spectral independence beyond the boolean domain. *arXiv preprint*, 2020. arXiv:2007.08091.

29 Jason Gaitonde and Elchanan Mossel. Comparison theorems for the mixing times of systematic and random scan dynamics. *arXiv preprint*, 2024. doi:10.48550/arXiv.2410.11136.

30 Howard Garland. p-adic curvature and the cohomology of discrete subgroups of p-adic groups. *Annals of Mathematics*, pages 375–423, 1973.

31 Zohar Grinbaum-Reizis and Izhar Oppenheim. Curvature criterion for vanishing of group cohomology. *Groups, Geometry, and Dynamics*, 16(3):843–862, 2022.

32 Heng Guo, Kaan Kara, and Ce Zhang. Layerwise systematic scan: Deep boltzmann machines and beyond. In *International Conference on Artificial Intelligence and Statistics*, pages 178–187. PMLR, 2018. URL: http://proceedings.mlr.press/v84/guo18a.html.

33 Heng Guo and Giorgos Mousa. Local-to-global contraction in simplicial complexes, 2021. arXiv:2012.14317.

34 Tom Gur, Noam Lifshitz, and Siqi Liu. Hypercontractivity on high dimensional expanders. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 176–184, 2022. doi:10.1145/3519935.3520004.

35 Thomas P Hayes. A simple condition implying rapid mixing of single-site dynamics on spin systems. In *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 39–46. IEEE, 2006. doi:10.1109/FOCS.2006.6.

36 Bryan D He, Christopher M De Sa, Ioannis Mitliagkas, and Christopher R é. Scan order in gibbs sampling: Models in which it matters and bounds on how much. *Advances in neural information processing systems*, 29, 2016.

37 Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

38 Fernando Granha Jeronimo, Shashank Srivastava, and Madhur Tulsiani. Near-linear time decoding of ta-shma's codes via splittable regularity. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1527–1536, 2021. doi:10.1145/3406325.3451126.

39 Martin Kassabov. Subspace arrangements and property t. *Groups, Geometry, and Dynamics*, 5(2):445–477, 2011.

40 Tali Kaufman and David Mass. High dimensional random walks and colorful expansion. In *ITCS*, pages 4:1–4:27, 2017. doi:10.4230/LIPIcs.ITCS.2017.4.

41 Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In *STOC*, pages 773–786, 2018. doi:10.1145/3188745.3188782.

42 Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap. In *APPROX/RANDOM*, pages 47:1–47:17, 2018. doi:10.4230/LIPIcs.APPROX-RANDOM.2018.47.

43 David A Kazhdan. On the connection of the dual space of a group with the structure of its closed subgroups. *Funkcional. Anal. i Prilozen*, 1(1):71–74, 1967.

44 David A Levin, Yuval Peres, and Elizabeth L Wilmer. Markov chains and mixing times. *American Mathematical Soc., Providence*, 2009.

45 Kuikui Liu. From coupling to spectral independence and blackbox comparison with the down-up walk. *arXiv preprint*, 2021. arXiv:2103.11609.

46 Eyal Lubetzky, Alexander Lubotzky, and Ori Parzanchevski. Random walks on ramanujan complexes and digraphs. *Journal of the European Mathematical Society*, 22(11):3441–3466, 2020.

47 Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of Ramanujan complexes of type. *Eur. J. Comb.*, 26(6):965–993, 2005. doi:10.1016/j.ejc.2004.06.007.

48 Ravi Montenegro and Prasad Tetali. Mathematical aspects of mixing times in Markov chains. *Foundations and Trends in Theoretical Computer Science*, 1(3), 2005. doi:10.1561/0400000003.

49 Izhar Oppenheim. Averaged projections, angles between groups and strengthening of banach property (t). *Mathematische Annalen*, 367:623–666, 2017.

50 Izhar Oppenheim. Angle criteria for uniform convergence of averaged projections and cyclic or random products of projections. *Israel Journal of Mathematics*, 223:343–362, 2018.

51 Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part I: Descent of spectral gaps. *Discrete & Computational Geometry*, 59(2):293–330, 2018. doi:10.1007/s00454-017-9948-x.

52 Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part ii: Mixing and geometrical overlapping, 2018. arXiv:1803.01320.

53 Izhar Oppenheim. Banach zuk's criterion for partite complexes with application to random groups. *arXiv preprint*, 2021. arXiv:2112.02929.

54 Izhar Oppenheim. Garland's method with banach coefficients. *Analysis & PDE*, 16(3):861–890, 2023.

55 Ori Parzanchevski. Ramanujan graphs and digraphs. *Anal. Geom. Graphs Manifolds*, 461:344, 2020.

56 Maxim Raginsky. Strong data processing inequalities and ϕ-sobolev inequalities for discrete channels. *IEEE Transactions on Information Theory*, 62(6):3355–3389, 2016. doi:10.1109/TIT.2016.2549542.

57 Gareth O Roberts and Jeffrey S Rosenthal. Surprising convergence properties of some simple gibbs samplers under various scans. *International Journal of Statistics and Probability*, 5(1):51–60, 2015.

58 Laurent Saloff-Coste. Lectures on finite Markov chains. In *Lectures on probability theory and statistics*, pages 301–413. Springer, 1997.

59 Kennan T. Smith, Donald C. Solmon, and Sheldon L. Wagner. Practical and mathematical aspects of the problem of reconstructing objects from radiographs. *Bull. Am. Math. Soc.*, 83:1227–1270, 1977. doi:10.1090/S0002-9904-1977-14406-6.

60 Daniel Stefankovic and Eric Vigoda. Lecture notes on spectral independence and bases of a matroid: Local-to-global and trickle-down from a markov chain perspective, 2023. arXiv:2307.13826.