

Bichromatic Geometric Spanners

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Abstract

For an edge-weighted graph $G = (V, E)$ and a stretch parameter $t \geq 1$, a t -spanner is a subgraph $H \subseteq G$ such that the shortest path distances in G and H satisfy $\delta_H(u, v) \leq t \delta_G(u, v)$ for all $u, v \in V$. In metric spanners, V is a finite metric space, and G is the complete graph with edge weights corresponding to the distances between the endpoints. When G is the complete graph on n points in the plane, $O(n)$ -size t -spanners are possible for any $t > 1$: For every $\varepsilon > 0$, there is an $(1 + \varepsilon)$ -spanner with $O(n/\varepsilon)$ edges (the stretch can be arbitrarily close to 1).

When $G = K(R, B)$ is the complete bipartite graph on n bichromatic points in the plane, in general, no spanner construction can achieve stretch $t < 3$ with $o(n^2)$ edges. Bose et al. (SICOMP 2009) constructed a $(3 + \varepsilon)$ -spanner with $O(n \log n)$ edges for any constant $\varepsilon > 0$. Our main result is a new construction for a $(3 + \varepsilon)$ -spanner with $O(\sqrt{1/\varepsilon} \cdot n)$ edges. Eliminating the $O(\log n)$ factor resolves a problem left open for more than 17 years, and raises a new research problem about optimizing the dependence on ε . We also study spanners for $G = K(R, B)$ on n bichromatic points on the real line: In this case, we show that the MST of $K(R, B)$ is a 7-spanner, and we construct a 3-spanner with at most $2n - 3$ edges.

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1 Introduction

A t -spanner for an edge-weighted graph $G = (S, E)$ is a subgraph $H = (S, E')$ that contains, for every edge $ab \in E$, an ab -path of weight at most $t \cdot w(ab)$. Let S be a set of n bichromatic points in Euclidean plane, with coloring $c : S \rightarrow \{\text{red}, \text{blue}\}$. Let R and B , resp., denote the set of red and blue points in S , and let $K(R, B)$ denote the complete bipartite graph on the partite sets R and B . The weight of an edge ab is the Euclidean length $|ab|$ of the line segment ab .

For a set S of n (uncolored) points in the plane and any $\varepsilon > 0$, there exists a $(1 + \varepsilon)$ -spanner with $O(n/\varepsilon)$ edges [18, 23]. Bose et al. [15] noted that for a complete bipartite graph $K(R, B)$, in general, one cannot find a t -spanner with $o(n^2)$ edges and stretch $t < 3$. Specifically, for every $\varepsilon > 0$, one can arrange $\lceil n/2 \rceil$ red and $\lfloor n/2 \rfloor$ blue points in two clusters of diameter $\frac{\varepsilon}{3}$ at unit distance apart: If a subgraph $H \subseteq K(R, B)$ misses any bichromatic



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edge ab , then every ab -path in H has length at least $(3 - \varepsilon)|ab|$. On the positive side, Bose et al. [15] constructed, for every bichromatic set $S = R \cup B \subset \mathbb{R}^2$ and constant $\varepsilon > 0$, an $(5 + \varepsilon)$ -spanner with $O(n)$ edges and a $(3 + \varepsilon)$ -spanner with $O(n \log n)$ edges. They did not analyze the dependency on the stretch parameter ε . The main problem left open in their work is whether a bichromatic $(3 + \varepsilon)$ -spanner with $O(n)$ edges always exists. Our main result settles this problem in the affirmative.

► **Theorem 1.** *For every $\varepsilon > 0$ and every set of n bichromatic points in the plane, there exists a $(3 + \varepsilon)$ -spanner with $O(\sqrt{1/\varepsilon} \cdot n)$ edges, which can be constructed in $O(\sqrt{1/\varepsilon} \cdot n \log n)$ time.*

While we settle the problem posed by Bose et al. [15], our result raises a new problem to determine the optimal tradeoff between $\varepsilon > 0$ and the size of a bichromatic $(3 + \varepsilon)$ -spanner in the plane. Theorem 1 generalizes to Euclidean d -space for constant dimension $d \geq 2$, and yields a $(3 + \varepsilon)$ -spanner with $O(n/\varepsilon^{(d-1)/2})$ edges. Moreover, in the related version of the paper, Theorem 1 has been generalized to every set of n multichromatic points in the plane. We describe the spanner construction and analyze it in detail in the plane, and then briefly sketch the generalization to higher dimensions.

We also study spanners for $K(R, B)$ in the real line. The lower bound construction by Bose et al. [15] can be realized already in the line: It shows that, in general, a spanner construction cannot achieve stretch $t < 3$ with $o(n^2)$ edges. We construct a 3-spanner with at most $2n - 3$ edges (Proposition 9). We also show that the bichromatic Euclidean MST (i.e., the MST of the complete bipartite graph $K(R, B)$ with Euclidean edge weights) is a 7-spanner; and the stretch bound 7 is the best possible (Proposition 10). For comparison, recall that for n (uncolored) points on the line, the Euclidean MST is an x -monotone path, which is a 1-spanner. However, for n (uncolored) points in the plane, the Euclidean MST has unbounded stretch.

Comparison to Previous Work and Technical Highlights. As mentioned above, Bose et al. [15] constructed a $(3 + \varepsilon)$ -spanner with $O(n \log n)$ edges for a bichromatic set $S = R \cup B$ of n points and constant $\varepsilon > 0$. Their spanner construction uses a hierarchical space partition, but a crucial step of their approach has a $\Omega(n \log n)$ barrier. Our approach is significantly different and perhaps conceptually simpler. We briefly review the spanner construction in [15] and point out key differences from our construction.

For a bichromatic set $S = R \cup B$ of n points in the plane, Bose et al. [15] use the classical *well-separated pair decomposition* (WSPD, for short) by Callahan and Kosaraju [17]. A WSPD with separation parameter $\varrho > 0$ is a collection of unordered pairs $\mathcal{P} = \{\{S_i, T_i\} : i = 1, \dots, m\}$ such that every point pair $\{s, t\} \in \binom{S}{2}$ appears there is an i such that $|\{s, t\} \cap S_i| = 1$ and $|\{s, t\} \cap T_i| = 1$; and $\text{dist}(R_i, B_i) \geq \varrho \cdot \max\{\text{diam}(R_i), \text{diam}(B_i)\}$ for all i . A WSPD of size $m = O(n)$ can be constructed using split trees [17, 26] or compressed quadtrees [20]. For an uncolored set of n points in $\mathbb{R}^d$, WSPDs with separation parameter $\varrho = \Theta(1/\varepsilon)$ were previously used for constructing $(1 + \varepsilon)$ -spanners of size $O(n/\varepsilon^d)$ [17, 19, 21]; and this bound is the best possible. Given a WSPD, each set S_i (resp., T_i) could be either monochromatic or bichromatic. Bose et al. [15] choose an arbitrary *representative* from each monochromatic set S_i and T_i ; and two representatives of different colors from bichromatic sets. Their spanner consists of specific edges chosen for each pair $\{S_i, T_i\}$ of the WSPD. If S_i and T_i are bichromatic, they add a single edge between a pair of representative from S_i and T_i , respectively. However, if a set S_i is monochromatic, they need a path from all points of S_i to its representative: They achieve this by connecting all points in S_i to the

representative of a closest cluster $S_j = \text{cl}(S_i)$ that contains a point of the other color (hence every point in S_i will have a 2-hop path to the representative of S_i). They prove that this graph is $(5 + \varepsilon)$ -spanner of size $O(n)$. They further show that *if* the WSPD has the additional property that $\min\{|S_i|, |T_i|\} = 1$ for all i , then their construction yields a $(3 + \varepsilon)$ -spanner. However, a WSPD with the additional property that size $m = O(n \log n)$, which is the best possible [17, 26], and dominates the size of their $(3 + \varepsilon)$ -spanner.

Our spanner design is crucially different in handling well-separated pairs with monochromatic clusters. If both S_i and T_i are bichromatic, we also add just two edges between representative. If S_i is monochromatic, we connect all points in S_i to the representative of T_i of the opposite color (and similarly, if T_i is monochromatic, we connect all points in T_i to a representative of S_i). Overall, for each well-separated pair (S_i, T_i) , a star, a bistar, or just a pair of edges between S_i and T_i (see Cases 1–3 in Section 2.2). In particular, we do not use a detour via any closest cluster $S_j = \text{cl}(S_i)$. This approach would already give a bichromatic $(3 + \varepsilon)$ -spanner of size $O(n)$ for any constant $\varepsilon > 0$.

Instead of a quadtree-based WSPD, however, we use a custom-made space partition scheme with rectangular cells of aspect ratio $\Theta(\sqrt{1/\varepsilon})$ to improve the dependence on $\varepsilon > 0$. In a standard quadtree-based WSPD with separation parameter $\varrho = \Theta(1/\varepsilon)$, the number of pairs is $m = O(\varrho^2 n) = O(n/\varepsilon^2)$ [20, Theorem 4.1.3]. Even though the separation parameter must be $\varrho = \Theta(1/\varepsilon)$ for a $(1 + \varepsilon)$ -spanner on uncolored point sets, our stretch analysis for a $(3 + \varepsilon)$ -spanner also works with a smaller separation parameter $\varrho = \Theta(\sqrt{1/\varepsilon})$. This would reduce the size of our spanner to $O(n/\varepsilon)$.

Furthermore, we have noticed that our inductive stretch analysis goes through even if we replace the square cells of a quadtree with rectangular cells with a larger aspect ratio provided that any two edges between the well-separated cells make an angle $O(\sqrt{\varepsilon})$. This relaxes the separation condition to $\text{dist}(S_i, T_i) \geq \Theta(\sqrt{1/\varepsilon}) \cdot \max\{\text{width}(S_i), \text{width}(T_i)\}$, and further reduces the size of our $(3 + \varepsilon)$ -spanner to $O(n/\sqrt{\varepsilon})$. We present this construction (without standard quadtrees and WSPDs) in Section 2. Our system of tilings with exponentially increasing scales and $\Theta(\sqrt{\varepsilon})$ directions is presented in Section 2.1, and our spanner is constructed based on pairs of well-separated tiles in Section 2.2. The stretch and sparsity analyses are presented in Section 2.3 and Section 2.4, respectively. We analyze the running time of our spanner construction in Section 2.5, and briefly outline generalizations to dimensions $d \geq 3$ in Section 2.6.

Previous work on bichromatic points. Bose et al. [16] studied a related problem: Given a finite set S in the plane, partition S into two nonempty color classes, $S = R \cup B$, so as to minimize the spanning ratio of the resulting complete bipartite graph $G = K(R, B)$, where the weight of every edge equals its Euclidean length. This problem is fundamentally different: In our problem, the color classes and $K(R, B)$ are fixed, and we need to find a small spanner in $K(R, B)$.

Bereg et al. [8] recently considered Euclidean spanners for the graph $G(R, B)$ of all red-blue and blue-blue edges (excluding red-red edges) on a set $S = R \cup B$ of n bichromatic points in $\mathbb{R}^d$. They constructed a 1.998-spanner of size at most $13n$ for $G(R, B)$ in the plane, and showed that the stretch of any spanner for $G(R, B)$ with $O(n)$ edges is at least $1 + \eta$ for some absolute constant $\eta > 0$. They also constructed a 1-spanner of size at most $2n$ for points in the real line.

Bichromatic variants for network optimization problems on n points in the plane, restricted to $K(R, B)$, have been studied extensively. In general, these problems are known to be more challenging than their original uncolored versions: For example, the Euclidean MST is always

a noncrossing straight-line graph; however, the bichromatic Euclidean MST may have crossing edges [3, 12]. Researchers studied bichromatic MSTs [11], bichromatic noncrossing MSTs [4, 14], bichromatic bottleneck MSTs [2], and bichromatic bottleneck noncrossing matchings [13]. Bichromatic TSP [22, 27], Euclidean TSP [6] and bichromatic Voronoi diagrams [5, 25] were also studied. Many of these problems are already nontrivial for bichromatic points on a line [1, 7].

2 Bichromatic $(3 + \varepsilon)$ -Spanners in the Plane

2.1 Tilings in the Plane

We begin by establishing some preliminary definitions. For a given $\varepsilon \in (0, 1)$, let $\delta = \sqrt{\varepsilon/3} = \Theta(\sqrt{\varepsilon})$. We construct a tiling of the plane with rectangular tiles of aspect ratio $1/\delta$.

► **Definition 2.** For every $i, j \in \mathbb{Z}$, we define the tile

$$T(i, j) = [i - 1, i] \times [(j - 1)\delta, j\delta].$$

The set of all tiles forms a tiling $T = \{T(i, j) : i, j \in \mathbb{Z}\}$.

In our spanner construction, we use several affine copies of the tiling T , each copy is obtained by transforming all tiles in T by the same affine transformation (rotation, scaling, and translation). Recall that every tile $T(i, j) \in T$ is an axis-aligned rectangle of width 1. Let $T' = \{T(i, j) + (0, \frac{\delta}{2}) : i, j \in \mathbb{Z}\}$ denote the tiling obtained by T by translation, with vector $(0, \frac{\delta}{2})$. For every angle $\theta \in [0, 2\pi)$ and every $\lambda > 0$, let $T_{\theta, \lambda}$ (resp., $T'_{\theta, \lambda}$) be the tiling obtained from T (resp., T') by a counterclockwise rotation through θ about the origin, followed by a dilation with factor λ .

However, we restrict ourselves to a discrete family of tilings: We use only $O(\sqrt{1/\varepsilon})$ angles θ , and a discrete set of exponentially increasing scaling factors, that we define now. Let $\varphi = \arctan(\delta/22)$ and $k = \lceil \pi/\varphi \rceil$. Note that $\varphi \in \Theta(\sqrt{\varepsilon})$ and $k \in \Theta(\sqrt{1/\varepsilon})$. Our discrete set of k angles is $\Phi = \{2r\pi/k : r = 0, 1, \dots, k-1\}$. Let $\gamma = 3^{1/11}$, and note that $3^{1/11} < 10/9$. Our discrete set of scaling factors is $\Lambda = \{\gamma^p : p \in \mathbb{Z}\}$. The family of all tilings with rotation angle in Φ and scaling factor in Λ is

$$\mathcal{T} = \bigcup_{\theta \in \Phi} \bigcup_{\lambda \in \Lambda} (T_{\theta, \lambda} \cup T'_{\theta, \lambda}).$$

The following lemma provides an important property of the family of tilings $\mathcal{T}$, which also plays a crucial role in the stretch analysis of our construction.

► **Lemma 3.** For every pair of distinct points $a, b \in \mathbb{R}^2$, there exist $\theta^* \in \Phi$, $\lambda^* \in \Lambda$ and $i^*, j^* \in \mathbb{Z}$ such that at least one of the following holds:

1. tiles $T_{\theta^*, \lambda^*}(i^*, j^*)$ and $T_{\theta^*, \lambda^*}(i^* + 10, j^*)$ each contain one of a and b ,
2. tiles $T'_{\theta^*, \lambda^*}(i^*, j^*)$ and $T'_{\theta^*, \lambda^*}(i^* + 10, j^*)$ each contain one of a and b .

Proof. Let ψ be the counterclockwise angle between the positive x -axis and vector $\vec{ab}$. We choose $\theta^* \in \Phi$ such that θ^* is the closest angle to ψ ; i.e., θ^* is the solution to the minimization problem

$$\theta^* = \arg \min_{\theta \in \Phi} |\theta - \psi|.$$

We may assume (by suitable rotation and reflections, if necessary) that $\theta^* = 0$, $\varphi \geq 0$, and $x(a) < x(b)$, where $x(a)$ and $x(b)$, resp., denote the x -coordinate of a and b . Next, we show that there exists a scale $\lambda^* \in \Lambda$ that satisfies

$$9\lambda^* \leq x(b) - x(a) < 10\lambda^*. \quad (1)$$

Let $\lambda^* = \gamma^{p^*}$, where

$$p^* = \left\lfloor \frac{\ln \frac{x(b)-x(a)}{9}}{\ln \gamma} \right\rfloor.$$

This gives $\gamma^{p^*} \leq \frac{x(b)-x(a)}{9} \leq \gamma^{p^*+1}$; and combined with $\gamma = 3^{1/11} < 10/9$, it implies Equation (1).

Since T_{θ^*, λ^*} is a tiling of $\mathbb{R}^2$, there exist $i^*, j^* \in \mathbb{Z}$ such that $a \in T_{\theta^*, \lambda^*}(i^*, j^*)$. Now Equation (1) implies that $x(a) + 9\lambda^* \leq x(b) \leq x(a) + 10\lambda^*$, and so $b \in T_{\theta^*, \lambda^*}(i^* + 10, j^{**})$ for some $j^{**} \in \mathbb{Z}$. Let $y(a)$ and $y(b)$, resp., denote the y -coordinate of a and b . If $y(a) = y(b)$, then we must have $j^* = j^{**}$. Suppose $y(a) \neq y(b)$, we may assume w.l.o.g. that $y(a) < y(b)$ and $\lambda^* = 1$. Under these assumptions, we have $T_{\theta^*, \lambda^*}(i^*, j^*) = T(i^*, j^*)$, which is an axis-aligned rectangle of width 1 and height δ . By our choice of φ , we can bound $y(b)$ from above, and conclude that

$$y(b) - y(a) = (x(b) - x(a)) \tan(\psi) \leq 11 \tan(\varphi) = \frac{\delta}{2}. \quad (2)$$

A horizontal median of the tile $T_{\theta^*, \lambda^*}(i^*, j^*)$ subdivides $T_{\theta^*, \lambda^*}(i^*, j^*)$ into two congruent rectangles, that we call the *upper* and *lower half* of $T_{\theta^*, \lambda^*}(i^*, j^*)$. Then one of the following two cases occurs

- **Case 1:** a is in the lower half of $T_{\theta^*, \lambda^*}(i^*, j^*)$. In this case, we have $0 \leq y(a) \leq \delta/2$, and Equation (2) gives $0 \leq y(b) \leq \delta/2$. Thus $a \in T_{\theta^*, \lambda^*}(i^*, j^*)$ and $b \in T_{\theta^*, \lambda^*}(i^* + 10, j^*)$.
- **Case 2:** a is in the upper half of $T_{\theta^*, \lambda^*}(i^*, j^*)$. In this case, we have $\delta/2 \leq y(a) \leq \delta$, and Equation (2) gives $\delta/2 \leq y(b) \leq \frac{3}{2}\delta$. Consequently, $a \in T'_{\theta^*, \lambda^*}(i^*, j^*)$ and $b \in T'_{\theta^*, \lambda^*}(i^* + 10, j^*)$.

This concludes the proof of Lemma 3. ◀

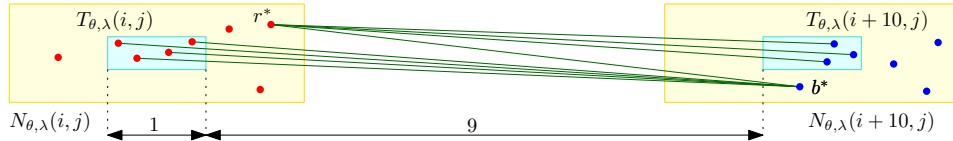
2.2 Spanner Construction

We are given a set $S = R \cup B$ of n bichromatic points in the plane and a parameter $\varepsilon > 0$. Let $\mathcal{T}$ be the family of tilings defined in Section 2.1. For every nonempty tile $T_{\theta, \lambda}(i, j)$, we denote by $N_{\theta, \lambda}(i, j)$ its *neighborhood* obtained by taking the union of $T_{\theta, \lambda}(i, j)$ and its eight surrounding tiles at the same scale. We say that $N_{\theta, \lambda}(i, j)$ is *bichromatic* if it contains at least one point of each color. Similarly, $N_{\theta, \lambda}(i, j)$ is *monochromatic* if all points in $N_{\theta, \lambda}(i, j)$ have the same color. It is important to note that $N_{\theta, \lambda}(i, j)$ can be bichromatic even if all points in $T_{\theta, \lambda}(i, j)$ are of the same color. For our construction, we begin by initializing a graph $G = (S, E)$ to be the empty graph.

For a set $Q \subseteq S$, we say that point q is *leftmost* in $Q \cap N_{\theta, \lambda}(i, j)$ if, after a clockwise rotation of $Q \cap N_{\theta, \lambda}(i, j)$ through θ , its x -coordinate is minimal. Likewise, the point q is *rightmost* in $Q \cap N_{\theta, \lambda}(i, j)$ if, after the same rotation, its x -coordinate is maximal. Note that several leftmost (resp., rightmost) points may exist (which have the same x -coordinate after suitable rotation). We choose a rightmost red point r^* in $N_{\theta, \lambda}(i, j)$ and a leftmost blue point b^* in $N_{\theta, \lambda}(i + 10, j)$. Similarly, we choose a rightmost blue point b^{**} in $N_{\theta, \lambda}(i, j)$ and a leftmost red point r^{**} in $N_{\theta, \lambda}(i + 10, j)$.

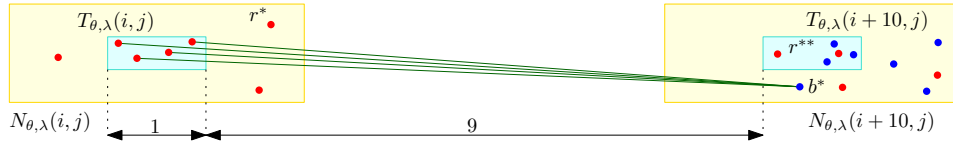
Consider every $\theta \in \Phi$, $\lambda \in \Lambda$, and $i, j \in \mathbb{Z}$ such that both $T_{\theta,\lambda}(i, j)$ and $T_{\theta,\lambda}(i + 10, j)$ are nonempty (the translated tiles are handled similarly if both $T'_{\theta,\lambda}(i, j)$ and $T'_{\theta,\lambda}(i + 10, j)$ are nonempty). We distinguish between four cases based on whether the neighborhoods $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are monochromatic or bichromatic:

- **Case 0:** Both $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are monochromatic, and all points in $N_{\theta,\lambda}(i, j) \cup N_{\theta,\lambda}(i + 10, j)$ have the same color. In this case, we do not add any edges to G .
- **Case 1:** Both $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are monochromatic, but they contain different colors. Assume w.l.o.g. that $N_{\theta,\lambda}(i, j) \cap S \subset R$ and $N_{\theta,\lambda}(i + 10, j) \cap S \subset B$ (the other case is symmetric); see Figure 1. In this case, we add edges to G between r^* and every point in $T_{\theta,\lambda}(i + 10, j) \cap B$, between b^* and every point in $T_{\theta,\lambda}(i, j) \cap R$, as well as the edge r^*b^* . In particular, the edges added to G form a *bistar* centered at r^* and b^* .



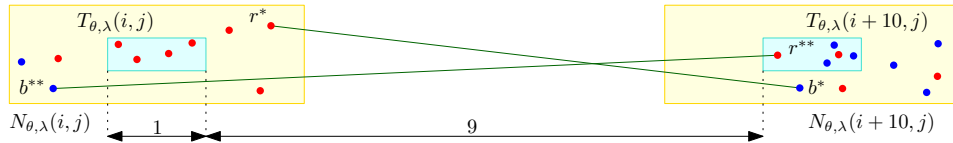
■ **Figure 1** Case 1: Both $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are monochromatic.

- **Case 2:** Exactly one of $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ is monochromatic. Assume w.l.o.g. that $N_{\theta,\lambda}(i, j)$ is monochromatic and $N_{\theta,\lambda}(i + 10, j)$ is bichromatic; see Figure 2 (the other case is symmetric). If $N_{\theta,\lambda}(i, j) \cap S \subset R$, we add an edge to G between b^* and every point in $T_{\theta,\lambda}(i, j) \cap R$. Otherwise, (when $N_{\theta,\lambda}(i, j) \cap S \subset B$), we add an edge to G between r^{**} and every point in $T_{\theta,\lambda}(i, j) \cap B$. In this case, the edges added to G form a *star*.



■ **Figure 2** Case 2: $N_{\theta,\lambda}(i, j)$ is monochromatic and $N_{\theta,\lambda}(i + 10, j)$ is bichromatic.

- **Case 3:** Both $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are bichromatic; see Figure 3. In this case, we add two edges to G : the edges r^*b^* and $r^{**}b^{**}$.



■ **Figure 3** Case 3: Both $N_{\theta,\lambda}(i, j)$ and $N_{\theta,\lambda}(i + 10, j)$ are bichromatic.

This completes the construction of the bipartite graph $G = (R \cup B, E)$. We show that G is a $(3 + \varepsilon)$ -spanner of $K(R, B)$ in Section 2.3, and that it contains $O(n/\sqrt{\varepsilon})$ edges in Section 2.4. We describe how to compute G in $O(\sqrt{1/\varepsilon} \cdot n \log n)$ time in Section 2.5.

2.3 Stretch Analysis

In this section, we prove that graph G is a $(3 + \varepsilon)$ -spanner of $K(R, B)$. We begin with a preliminary observation. The Euclidean length of an edge is given by the Pythagorean theorem for its x - and y -extents. However, if the y -extent is much smaller than the x -extent,

we can bound the Euclidean length from above by a polynomial as follows: For all $x \geq 1$ and $y > 0$, we have

$$\sqrt{x^2 + y^2} \leq x + y^2/2. \quad (3)$$

We can now prove the main result of this subsection.

► **Lemma 4.** *For every finite bichromatic set $S = R \cup B$ in the plane, the graph $G \subseteq K(R, B)$ constructed in Section 2.2 is $(3 + \varepsilon)$ -spanner for $K(R, B)$.*

Proof. We proceed by induction on the length of the bichromatic edges in $K(R, B)$. Specifically, let $D = \{|rb| : (r, b) \in R \times B\}$ be the set of distances between red-blue pairs in S ; and let $0 = d_0 < d_1 < \dots < d_m$ be elements of $\{0\} \cup D$ in increasing order.

We prove, by induction on $\ell = 0, 1, \dots, m$, that for all $(r, b) \in R \times B$ with $|rb| \leq d_\ell$, the graph G contains an rb -path of length at most $(3 + \varepsilon)|rb|$. In the base case, where $\ell = 0$ and $d_\ell = 0$, the claim vacuously holds, as there is no red-blue pair (r, b) with $|rb| = 0$.

In the induction step, let $d_\ell \in D$ and assume that for all $(r', b') \in R \times B$, if $|r'b'| < d_\ell$, then G contains an $r'b'$ -path of length at most $(3 + \varepsilon)|r'b'|$. Let $(r, b) \in R \times B$ with $|rb| = d_\ell$. By Lemma 3, there exist $\theta \in \Phi$, $\lambda \in \Lambda$ and $i, j \in \mathbb{Z}$ such that $T_{\theta, \lambda}(i, j)$ and $T_{\theta, \lambda}(i + 10, j)$ each contain one of r and b ; or $T'_{\theta, \lambda}(i, j)$ and $T'_{\theta, \lambda}(i + 10, j)$ each contain one of r and b . To simplify our argument, we may assume w.l.o.g. that $\theta = 0$ and $\lambda = 1$, $r \in T_{\theta, \lambda}(i, j)$ and $b \in T_{\theta, \lambda}(i + 10, j)$. Under these assumptions, $N_{\theta, \lambda}(i, j) = N(i, j)$, which is an axis-aligned rectangle of width 3 and height 3δ . We distinguish between three cases:

- **Case 1:** Both $N(i, j)$ and $N(i + 10, j)$ are monochromatic; see Figure 1. By construction (Case 1), G contains the rb -path $P = (r, b^*, r^*, b)$. Since the pair r^* is the rightmost red point in $N(i, j)$ and b^* is the leftmost blue point in $N(i + 10, j)$, then we have $x(r) \leq x(r^*) < x(b^*) \leq x(b)$, where $x(r)$, $x(r^*)$, $x(b^*)$ and $x(b)$, resp., denote the x -coordinate of r , r^* , b^* and b . Then, the length of b^*r^* can be bounded above by $|b^*r^*| \leq \sqrt{(x(b^*) - x(r^*))^2 + (3\delta)^2} \leq \sqrt{(x(b) - x(r))^2 + (3\delta)^2}$. Using Equation (3) and $9 \leq |rb|$, it follows that

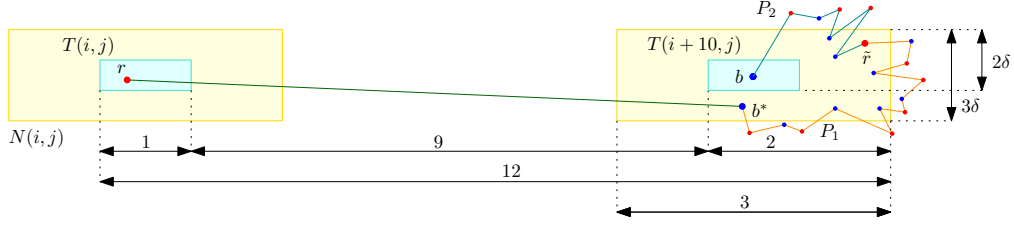
$$\begin{aligned} |b^*r^*| &\leq \sqrt{(x(b) - x(r))^2 + (3\delta)^2} \\ &\leq x(b) - x(r) + 9\delta^2/2 \\ &= x(b) - x(r) + 3\varepsilon/2 \\ &\leq (1 + \varepsilon/3)|rb|. \end{aligned}$$

By a similar process, we obtain $|rb^*| \leq (1 + \varepsilon/3)|rb|$ and $|r^*b| \leq (1 + \varepsilon/3)|rb|$. Thus, the length of the rb -path P is at most $(3 + \varepsilon)|rb|$ since $|P| = |rb^*| + |b^*r^*| + |r^*b| \leq 3(1 + \varepsilon/3)|rb| = (3 + \varepsilon)|rb|$. (Note that we did not use the induction hypothesis in Case 1.)

- **Case 2:** One of $N(i, j)$ and $N(i + 10, j)$ is monochromatic and the other is bichromatic. Assume w.l.o.g. that $N(i, j)$ is monochromatic and $N(i + 10, j)$ is bichromatic; see Figure 4 (the other case is symmetric). Let $\tilde{r} \in R \cap N(i + 10, j)$. Since $b^*, \tilde{r} \in N(i + 10, j)$, we have $|b^*\tilde{r}| \leq \text{diam}(N(i + 10, j)) = 3 \text{diam}(T(i, j))$. Furthermore, since $b \in T(i + 10, j)$, then the edge $\tilde{r}b$ lies in an axis-aligned rectangle spanned by $T(i + 10, j)$ and an adjacent tile, which implies $|\tilde{r}b| \leq 2 \text{diam}(T(i, j))$. Note that $\text{diam}(T(i, j)) = \sqrt{1 + \delta^2}$ by the Pythagorean Theorem. Equation (3) yields $\text{diam}(T(i, j)) = \sqrt{1 + \delta^2} \leq 1 + \varepsilon/6$. Since $9 \leq |rb|$, it now follows that $|b^*\tilde{r}| \leq 3 \text{diam}(T(i, j)) \leq 3(1 + \varepsilon/6) < 9 \leq |rb|$. Similarly, we have $|\tilde{r}b| < |rb|$.

By the induction hypothesis, G contains some paths

$$P_1 = (b^*, \dots, \tilde{r}) \quad \text{and} \quad P_2 = (\tilde{r}, \dots, b),$$



■ **Figure 4** An rb -path when $N(i, j)$ is monochromatic and $N(i + 10, j)$ is bichromatic.

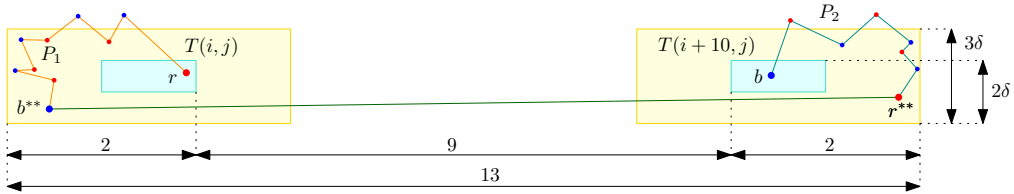
resp., of length at most $(3 + \varepsilon)|b^*\tilde{r}|$ and $(3 + \varepsilon)|\tilde{r}b|$. Then, the concatenation of $rb^* \oplus P_1 \oplus P_2$ is an rb -path in G and its length can be bounded above as follows

$$\begin{aligned}
 |rb^*| + |P_1| + |P_2| &\leq \sqrt{11^2 + (2\delta)^2} + (3 + \varepsilon)|b^*\tilde{r}| + (3 + \varepsilon)|\tilde{r}b| \\
 &\leq 11 + 2\delta^2 + 3(3 + \varepsilon)\text{diam}(T(i, j)) + 2(3 + \varepsilon)\text{diam}(T(i, j)) \\
 &= 11 + 2\varepsilon/3 + 5(3 + \varepsilon)(1 + \varepsilon/6) \\
 &= 11 + 2\varepsilon/3 + 5(3 + 3\varepsilon/2 + \varepsilon^2/6).
 \end{aligned}$$

Recall that $0 < \varepsilon < 1$, hence $\varepsilon^2 < \varepsilon$. It follows that

$$\begin{aligned}
 |rb^*| + |P_1| + |P_2| &\leq 11 + 2\varepsilon/3 + 5(3 + 5\varepsilon/3) \\
 &= 26 + 9\varepsilon \\
 &< 9(3 + \varepsilon) \\
 &\leq (3 + \varepsilon)|rb|.
 \end{aligned}$$

So, the concatenation of $rb^* \oplus P_1 \oplus P_2$ is an rb -path in G of length at most $(3 + \varepsilon)|rb|$.



■ **Figure 5** An rb -path when both $N(i, j)$ and $N(i + 10, j)$ are bichromatic.

- **Case 3:** Both $N(i, j)$ and $N(i + 10, j)$ are bichromatic; see Figure 5. Similarly to Case 2, we have $|rb^{**}| \leq \text{diam } N(i, j) < |rb|$ and $|r^{**}b| \leq \text{diam } N(i + 10, j) < |rb|$. By the induction hypothesis, G contains some paths

$$P_1 = (r, \dots, b^{**}) \quad \text{and} \quad P_2 = (r^{**}, \dots, b),$$

resp., of length at most $(3 + \varepsilon)|rb^{**}|$ and $(3 + \varepsilon)|r^{**}b|$. Then, the concatenation $P = P_1 \oplus b^{**}r^{**} \oplus P_2$ is an rb -path of G , and observe that by a similar process as in Case 2, its length be bounded above as follows

$$\begin{aligned}
|P| &= |P_1| + |b^{**}r^{**}| + |P_2| \\
&\leq (3 + \varepsilon)|rb^{**}| + \text{diam}(N(i, j) \cup N(i + 10, j)) + (3 + \varepsilon)|r^{**}b| \\
&\leq 2(3 + \varepsilon) \text{diam}(T(i, j)) + \sqrt{13^2 + (3\delta)^2} + 2(3 + \varepsilon) \text{diam}(T(i + 10, j)) \\
&\leq 4(3 + \varepsilon)(1 + \varepsilon/6) + 13 + 3\varepsilon/2 \\
&\leq 4(3 + 5\varepsilon/3) + 13 + 3\varepsilon \\
&= 25 + 49\varepsilon/6 \\
&< 9(3 + \varepsilon) \\
&\leq (3 + \varepsilon)|rb|.
\end{aligned}$$

So, the concatenation $P_1 \oplus b^{**}r^{**} \oplus P_2$ is an rb -path in G of length at most $(3 + \varepsilon)|rb|$. Overall, the bipartite graph G is a $(3 + \varepsilon)$ -spanner of $K(R, B)$. ◀

2.4 Sparsity Analysis

Fix an angle $\theta \in \Phi$. We distinguish between two types of edges in G . An edge is of *Type 1* if it was added when the construction processed a pair of tiles $T_{\theta, \lambda}(i, j), T_{\theta, \lambda}(i + 10, j)$ where one of the tiles had monochromatic neighborhood; otherwise it is of *Type 2*. In particular, edges of Type 1 correspond to Cases 1-2 of the construction, and Type 2 to Case 3.

Overview. We count edges of each type separately in Lemmas 6 and 8 below. Edges of Type 1 are the main concern, since our spanner may contain many edges of Type 1 for each pair $T_{\theta, \lambda}(i, j), T_{\theta, \lambda}(i + 10, j)$. However, even if the neighborhood of one of the tiles is monochromatic, their union $N_{\theta, \lambda}(i, j) \cup N_{\theta, \lambda}(i + 10, j)$ contains both colors. We show (Lemma 5) that if we go up 26 scales higher, then every point in $T_{\theta, \lambda}(i, j) \cup T_{\theta, \lambda}(i + 10, j)$ is contained in a bichromatic neighborhood: It follows that each point $s \in S$ is incident to an edge of Type 1 in $O(1)$ scales; hence there are $O(n)$ edges of Type 1 (for each $\theta \in \Phi$).

For every pair $T_{\theta, \lambda}(i, j), T_{\theta, \lambda}(i + 10, j)$ with bichromatic neighborhoods, we add only two edges of Type 2, so it is enough to analyze the number of such pairs. We rely on the classic result that a quadtree-based WSPD has $m = O(n)$ well-separated pairs [20, Theorem 4.1.4]. The proof of this result hinges on properties of quadtrees and compressed quadtrees: The compressed quadtree for n points is known to have $O(n)$ nodes, but chains of single-child nodes in the uncompressed quadtree must be handled carefully. We adapt this approach to our pairs of tiles (Lemma 7).

It is easy to arrange the tiles $T_{\theta, \lambda}$, $\lambda \in \Lambda$, into groups in which the scales differ by integer factors – so we can obtain a laminar system, which supports a quadtree. However, for tiles in a standard quadtree, the translated tiles $T'_{\theta, \lambda}$ would not be laminar. For this reason, we use a variation of quadtrees: nonatrees [10], in which each tile is subdivided into nine congruent subtiles. Using nonatrees, the tiles in $T_{\theta, \lambda}$ and $T'_{\theta, \lambda}$, over all $\lambda \in \Lambda$, can be grouped into laminar systems.

Edges of Type 1. We introduce a charging scheme: We charge each edge st of Type 1 to one of its vertices as follows. Assume st was added to G when processing the pair tiles $T_{\theta, \lambda}(i, j)$ and $T_{\theta, \lambda}(i + 10, j)$. If $N_{\theta, \lambda}(i, j)$ is monochromatic, we charge st to s , otherwise (when $N_{\theta, \lambda}(i + 10, j)$ is monochromatic), we charge st to t .

► **Lemma 5.** *Let $s \in S$. Assume that s lies in tiles $T_{\theta,\lambda}(i, j)$ and $T_{\theta,\lambda'}(i', j')$ with monochromatic neighborhoods, where $\lambda = \gamma^p, \lambda' = \gamma^{p'}$ with $p < p'$, and our construction added an edge st of Type 1 when processing the pair of tiles $T_{\theta,\lambda}(i, j)$ and $T_{\theta,\lambda}(i + 10, j)$. Then we have $p' < p + 26$.*

Proof. Assume w.l.o.g. that $\theta = 0$ and s is red (i.e., $s \in R$). Since we added an edge st for some $t \in N_{\theta,\lambda}(i + 10, j)$, then t is blue. Note that

$$\begin{aligned} |x(s) - x(t)| &\leq |(i - 1)\gamma^p - (i + 12)\gamma^p| = 13\gamma^p < 3^{26/11} \gamma^p = \gamma^{p+26}, \\ |y(s) - y(t)| &\leq \delta\gamma^p. \end{aligned}$$

Recall that $s \in T_{\theta,\lambda'}(i', j')$. For any $q \in \mathbb{R}^2$ such that $|x(s) - x(q)| \leq \gamma^{p'}$ and $|y(s) - y(q)| \leq \delta \cdot \gamma^{p'}$, we have $q \in N_{\theta,\lambda'}(i', j')$. Hence $p' \geq p + 26$ implies $t \in N_{\theta,\lambda'}(i', j')$, and so the neighborhood $N_{\theta,\lambda'}(i', j')$ is bichromatic, and st cannot be charged to s . We conclude that $p' < p + 26$, as claimed. ◀

► **Lemma 6.** *For every bichromatic set $S = R \cup B$ of n points in the plane, and for every $\theta \in \Phi$, the graph G has $O(n)$ edges of Type 1.*

Proof. Each vertex $s \in S$ receives charges from at most 26 scales by Lemma 5. Consequently, there are $O(|S|) = O(n)$ edges of Type 1. ◀

Edges of Type 2. For the purpose of analyzing the edges of Type 2, we arrange the nonempty tiles $T_{\theta,\lambda}(i, j)$ into eleven nonatrees as follows. For $\sigma = 0, 1, \dots, 10$, let

$$\Lambda_\sigma = \{\gamma^\sigma \cdot 3^p : p \in \mathbb{Z}\} = \{3^{\sigma/11} \cdot 3^p : p \in \mathbb{Z}\}.$$

Then the set of tiles $\{T_{\theta,\lambda}(i, j) : \lambda \in \Lambda_\sigma\}$ is a laminar set system (i.e., any two tiles are either interior-disjoint or one contains the other). Importantly, the translated tiles $\{T'_{\theta,\lambda}(i, j) : \lambda \in \Lambda_\sigma\}$ also form a laminar system [10] (this is why we use nonatrees, instead of quadrees). Let $\mathcal{Q}_\sigma$ denote the set of tiles $T_{\theta,\lambda}(i, j)$ such that $\lambda \in \Lambda_\sigma$ and $T_{\theta,\lambda}(i, j) \cap S \neq \emptyset$ (the case of the tiles $T'_{\theta,\lambda}(i, j)$ is analogous). Containment among the tiles in $\mathcal{Q}_\sigma$ defines a hierarchy (i.e., poset or tree); and we refer to $\mathcal{Q}_\sigma$ as a *nonatree*. We can now define a parent-child relationship between the tiles in $\mathcal{T}_\sigma$: For $T_{\theta,\lambda}(i, j), T_{\theta,\lambda'}(i', j') \in \mathcal{T}_\sigma$, we say that $T_{\theta,\lambda}(i, j)$ is a *child* of $T_{\theta,\lambda'}(i', j')$ if $T_{\theta,\lambda}(i, j) \subset T_{\theta,\lambda'}(i', j')$ and $\lambda' = 3\lambda$. It is clear from the definition that every tile has at most nine children. Let $\mathcal{R}_\sigma$ be the set of tiles $T \in \mathcal{Q}_\sigma$ that have two or more children. Containment among the tiles in $\mathcal{R}_\sigma$ gives a partition tree on S for every $\sigma \in \{0, \dots, 10\}$, consequently

$$|\mathcal{R}_\sigma| \leq n. \tag{4}$$

Note that every tile $T \in \mathcal{Q}_\sigma$ is contained in some tile $\hat{T} \in \mathcal{R}_\sigma$; possibly $\hat{T} = T$. For every $T \in \mathcal{Q}_\sigma$, let $\bar{p}(T)$ be the smallest tile in $\mathcal{R}_\sigma$ that contains T (i.e., $\bar{p}(T)$ is the closest ancestor of T in $\mathcal{R}_\sigma$).

We can now introduce a charging scheme: Consider the edges r^*b^* and $r^{**}b^{**}$ of Type 2 that were added to G when we processed the pair of tiles $\{T_1, T_2\} = \{T_{\theta,\lambda}(i, j), T_{\theta,\lambda}(i + 10, j)\}$ with bichromatic neighborhoods. Let λ_1 and λ_2 be the scales of $\bar{p}(T_1)$ and $\bar{p}(T_2)$, respectively. If $\lambda_1 \leq \lambda_2$, then we *charge* both edges to $\bar{p}(T_1)$; otherwise (if $\lambda_1 > \lambda_2$), we charge them to $\bar{p}(T_2)$.

► **Lemma 7.** *In the charging scheme above, every tile $T \in \bigcup_{\sigma=0}^{10} \mathcal{R}_\sigma$ receives $O(1)$ charges.*

Let $\mathcal{P}_1$ be the collection of pairs of tiles (T_1, T_2) such that two bichromatic edges corresponding to (T_1, T_2) are charged to T and $\bar{p}(T_1) = T$. By assumption, we have $|\mathcal{P}_1| \geq \lceil 149/2 \rceil = 75$. In each pair $(T_1, T_2) \in \mathcal{P}_1$, the tile T_1 is either *left* (i.e., $T_1 = T(i, j)$ and $T_2 = T(i + 10, j)$) or *right* (i.e., $T_1 = T(i + 10, j)$ and $T_2 = T(i, j)$). We may assume w.l.o.g. that in at least half of these pairs, T_1 is left. Let $\mathcal{P}_2 \subseteq \mathcal{P}_1$ be the set of these pairs; note that $|\mathcal{P}_2| \geq \lceil |\mathcal{P}_1|/2 \rceil = 38$.

Note for all pairs $(T_1, T_2) \in \mathcal{P}_3$, the tiles T_1 are at different scales (since they are in the same single-child chain). Let (T_1, T_2) and (T'_1, T'_2) be the pairs in $\mathcal{P}_3$ at the lowest and the second lowest scales, respectively. Since $|\mathcal{P}_3| \geq 5$ and $\text{width}(T) = 1$, then $\text{width}(T_1) \leq 3^{-5}$ and $\text{width}(T'_1) \leq 3^{-4}$. Since $T_1 \subseteq T'_1$, then $\text{dist}(T'_1, T_2) \leq \text{dist}(T_1, T_2) = 9 \text{width}(T_1) \leq 3 \text{width}(T'_1)$. It follows that

and, in particular, the tiles T_2 and T'_2 are disjoint. Furthermore, $T_1 \subset T'_1 \subseteq T_{\text{child}}$ implies

As $\text{width}(T_{\text{child}}) = \frac{1}{3}$, both T_2 and T'_2 are contained in some tile $T^* \in \mathcal{Q}_\sigma$ at the same scale as T_{child} . But $\text{chain}(T_1)$ is a single-child chain of tiles, so $T_2, T'_2 \not\subseteq T_{\text{child}}$, and $T^* \neq T_{\text{child}}$ (although, T^* could be a sibling of T_{child}). Note that the least common ancestor $\text{lca}(T_2, T'_2)$ of T_2 and T'_2 in the nonatree $\mathcal{Q}_\sigma$ satisfies $\text{lca}(T_2, T'_2) \subseteq T^*$. In particular, $\text{lca}(T_2, T'_2)$ is at a scale below T ; and $\text{lca}(T_2, T'_2)$ has at least two children, which implies $\text{lca}(T_2, T'_2) \in \mathcal{R}_\sigma$.

Consequently, both $\bar{p}(T_2)$ and $\bar{p}(T'_2)$ are at a scale below $T = \bar{p}(T_1) = \bar{p}(T'_1)$; and so our charging scheme could not have charged (T_1, T_2) and (T'_1, T'_2) to T : A contradiction. We conclude that at most 148 edges were charged to the tile T . ◀

► **Lemma 8.** *For every bichromatic set $S = R \cup B$ of n points in the plane, and for every $\theta \in \Phi$, the graph G has $O(n)$ edges of Type 2.*

Proof. This readily follows from the combination of Equation (4) and Lemma 7. ◀

2.5 Running Time Analysis

For every set S of n bichromatic points in the plane and every $\varepsilon > 0$, our spanner construction (Section 2.2) can be implemented in $O(\sqrt{1/\varepsilon} \cdot n \log n)$ time. The construction is done independently for each $\theta \in \Phi$, where $|\Phi| = O(\sqrt{1/\varepsilon})$, so it is enough to show that the construction takes $O(n \log n)$ time for each $\theta \in \Phi$. Assume w.l.o.g. that $\theta = 0$. Instead of infinite tilings, we can use nonatrees (the generalization of quadrees, where each cell is split into 9 congruent cells) after an affine transformation $f(x, y) = (x, y/\delta)$, which maps an $1 \times \delta$ rectangle into a unit square. Let $S_0 := f(S)$. Let B_0 be an axis-aligned bounding box of S_0 where $\text{width}(B_0) = 2 \text{height}(B_0)$, and scale B_0 centrally up by factors of $3^{\sigma/11}$ for $\sigma = 0, \dots, 10$. Each scaled copy of B_0 can be expressed as an intersection of two axis-aligned squares $B_0 = B \cap B'$, where $\text{width}(B) = \text{width}(B') = \text{width}(B_0)$; note that $B' = B + (0, \frac{1}{2} \text{height}(B))$.

Starting from each of these 22 squares (B and B' with $\sigma = 0, \dots, 10$), as root cells, we can build compressed nonatrees $\mathcal{R}_\sigma$ and $\mathcal{R}'_\sigma$ for S_0 in $O(n \log n)$ time; analogously to the construction of a quadrees [10, 9, 20]. The compressed nonatree for n points has $O(n)$ cells, but the (uncompressed) nonatree can have arbitrarily many cells (as it may have descending chains of arbitrarily length). By Lemmas 5 and 7, our construction only uses (uncompressed) nonatree nodes that have an ascending chain of length at most 5 to a nodes of the compressed nonatree $\mathcal{R}_\sigma$: We augment $\mathcal{R}_\sigma$ with up to 5 single-child descendants per node to obtain a tree $\mathcal{R}_\sigma^*$ with $O(n)$ nodes. In a bottom-up traversal of $\mathcal{R}_\sigma^*$, we can find the leftmost and rightmost red (blue) points in each cell in $O(n)$ time. For each cell $T(i, j)$, we can find their eight adjacent cells in $N(i, j)$ in $O(\log n)$ time, and classify $N(i, j)$ as either monochromatic or bichromatic. In another bottom-up traversal, we find all pairs of tiles $T(i, j)$ and $T(i + 10, j)$ and find the edges of G , according to Cases 1–3 in Section 2.2, in $O(n)$ time.

2.6 Generalization to Higher Dimensions

It is straightforward to generalize our construction to Euclidean d -space: In $\mathbb{R}^2$, we used basic tiles of size $1 \times \sqrt{\varepsilon/3}$, and rotated the tilings with a set $\Phi \subset [0, \pi]$ of $\Theta(\sqrt{1/\varepsilon})$ angles at distance $\arctan(\sqrt{\varepsilon/3}/22) = O(\sqrt{\varepsilon})$ apart. In d -space, for $d \geq 3$, we use basic tiles of size $1 \times \sqrt{\varepsilon/(d+1)} \times \dots \times \sqrt{\varepsilon/(d+1)}$, and we use a set Φ of rotations constructed as follows: We choose a set $U_d \subset \mathbb{S}^{d-1}$ of unit vectors such that the $O(\sqrt{\varepsilon})$ -neighborhoods of the vectors in $\mathbb{S}^{d-1}$ (where each neighborhood is a spherical cap) cover U_d . A standard packing argument shows that $O(1/\varepsilon^{(d-1)/2})$ directions suffice. Then for each $\vec{u} \in U_d$, we rotate the basic tiling such that the positive x_1 -axis is parallel to $\vec{u}$. With this setup, Lemma 3, generalizes. The construction, as well as the stretch and sparsity analyses readily extend to $\mathbb{R}^d$. The number of edges is $O(|U_d| \cdot n) = O(n/\varepsilon^{(d-1)/2})$.

3 Bichromatic Spanners on the Line

In this section, we study spanners for n bichromatic points on the real line $\mathbb{R}$. The lower bound construction in [15] shows, that for every $\varepsilon > 0$, there is a bichromatic point set such that a $(3 - \varepsilon)$ -spanner must contain all edges of the complete bipartite graph $K(A, B)$; this construction consists of two monochromatic clusters of diameter $\frac{\varepsilon}{3}$ at unit distance apart, and can be realized already in 1-dimension. We show that there is 3-spanner with at most $2n - 3$ edges (Proposition 9; furthermore, the minimum spanning tree of $K(R, B)$ (denoted $\text{minBST}(S)$) is a 7-spanner, and stretch bound 7 is the best possible (Proposition 10).

We start by introducing some notation. Let $S = R \cup B$ be a set of n bichromatic points on a line. Assume that $S = \{s_1, \dots, s_n\}$, where the points are labeled in linear order along the line. Partition S into maximal subsets of consecutive points of the same color $S = \bigcup_{i=1}^k S_i$; we call these subsets *blocks*. Clearly, the blocks span disjoint intervals along the line, which are also ordered linearly. For any nonempty subset $S' \subseteq S$, denote by $\ell(S')$ and $r(S')$ the leftmost and rightmost point in S' , respectively.

► **Proposition 9.** *For a set $S = B \cup R$ of n bichromatic points on a line, there is a 3-spanner with at most $2n - 3$ edges that can be constructed in $O(n \log n)$ time.*

Proof. We construct a subgraph H of $K(A, B)$ as follows. Consider the partition of S into blocks, $S = \bigcup_{i=1}^k S_i$. For every $i = 1, \dots, k$, and for every point $s \in S_i$, add edges $sr(S_{i-1})$ and $s\ell(S_{i+1})$ to H (if these edges exist). In particular, we add only one edge for every $s \in S_1 \cup S_k$, and two edges for all other vertices. Note also that the $k - 1$ edges of the form $r(S_i)\ell(S_{i+1})$ are counted twice. Since $|S_1| \geq 1$, $|S_k| \geq 1$, and $k \geq 2$, then the total number of edges in H is $2n - (|S_1| + |S_k|) - (k - 1) \leq 2n - 2 - 1 = 2n - 3$.

Next we show that H is a 3-spanner. Let $a, b \in S$ be two points of opposite colors, and assume w.l.o.g. that $a \in S_i$, $b \in S_j$ for some $1 \leq i < j \leq k$. Then H contains the path

$$P = (a, \ell(S_{i+1}), r(S_i), r(S_{i+1}), \dots, r(S_{j-1}), b).$$

The first three edges of the path P intersect in the interval, $r(S_i), \ell(S_{i+1})$. All subsequent edges are pairwise interior disjoint (from each other and from the first three edges). This implies that every point on the line is contained in at most 3 edges of P , which immediately implies that $|P| \leq 3|ab|$, as required. ◀

Bichromatic Minimum Spanning Tree. Bandyapadhyay et al. [7] showed that if n bichromatic points are already sorted along the line, then $\text{minBST}(S)$ can be computed in $O(n)$ time. (Biniaz et al. [11] showed how to compute $\text{minBST}(S)$ in $O(n \log n)$ in the plane.)

We start with a characterization of $\text{minBST}(S)$, based on the observations in [7]. Consider a block S_i , $1 \leq i \leq k$, defined above. For every point $s \in S_i$, the closest point of the opposite color is either $r(S_{i-1})$ or $\ell(S_{i+1})$. We define a partition $S_i = S_i^- \cup S_i^+$, where S_i^- (resp., S_i^+) is the set of points in S_i for which the closest point of the opposite color is in S_{i-1} (resp., S_{i+1}). Note that $S_1^- = \emptyset$ and $S_k^+ = \emptyset$, but S_i^- or S_i^+ may be empty for any $1 < i < k$.

Recall, that if $G = (V, E, w)$ is an edge-weighted connected graph with positive edge weights, then for every nontrivial cut $V = V_1 \cup V_2$, the MST of G contains a minimum-weight edge between V_1 and V_2 . We apply this observation for nontrivial cuts $S = \{s_1, \dots, s_j\} \cup \{s_{j+1}, \dots, s_n\}$ for $i = 1, \dots, n - 1$ in the complete bipartite graph $K(R, B)$. Specifically, if s_j and s_{j+1} are in different blocks, then the shortest bichromatic edge across the cut is $s_j s_{j+1}$; in this case $s_j = r(S_i)$ and $\ell(S_{i+1})$ for some $1 \leq i \leq k$. Now assume that both s_j and s_{j+1} are in the same block S_i . If $s_j, s_{j+1} \in S_i^-$ (resp., $s_j, s_{j+1} \in S_i^+$), then $r(S_{i-1})s_j$

(resp., $\ell(S_{j+1})s_{i+1}$) is the shortest bichromatic edge across the cut. Finally, if $s_i \in S_j^-$ and $s_{i+1} \in S_j^+$, then the shortest bichromatic edge across the cut is the shorter of two candidates: $r(S_{i-1})s_j$ or $\ell(S_{j+1})s_{i+1}$. Note that we considered $n - 1$ nontrivial cuts, and obtained $n - 1$ distinct edges: Let H be the graph formed by these edges. Since H has $n - 1$ edges and it is a subgraph of $\text{minBST}(S)$, then $H = \text{minBST}(S)$.

We give a tight bound on the stretch of the bichromatic MST on a line.

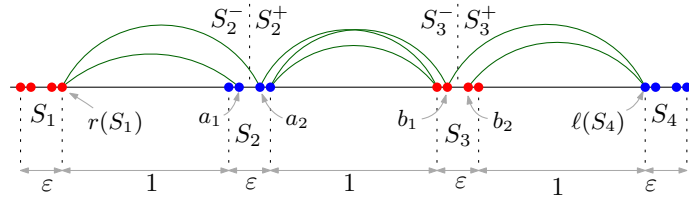
► **Proposition 10.** *For a set $S = B \cup R$ of n bichromatic points on a line, $\text{minBST}(S)$ is a 7-spanner; and for every $\varepsilon > 0$, there is a bichromatic point set $S \subset \mathbb{R}$ such that the stretch of $\text{minBST}(S)$ is at least $7 - \varepsilon$.*

Proof. It is enough to show that for any pair of points a, b in consecutive blocks, $\text{minBST}(S)$ contains a path of length at most $7|ab|$. Indeed, if $a \in S_i$ and $b \in S_j$, where $i < j$, then we can first consider the path $P = (a, \ell(S_{i+1}), \dots, \ell(S_{j-1}), b)$ in $K(R, B)$. Note that $|P| = |ab|$, and every edge of P is between points in consecutive blocks. By replacing every edge $e \subseteq P$ by a path in minBST of length at most $7|e|$, we obtain an ab -path in minBST of length at most $\sum_{e \subseteq P} 7|e| = 7|P| = 7|ab|$.

We may assume w.l.o.g. that $a \in S_i$ and $b \in S_{i+1}$. We claim that $\text{minBST}(S)$ contains a path P_1 from a to $\ell(S_{i+1})$ of length $|P_1| \leq 3|ab|$; and symmetrically a path P_2 from $r(S_i)$ to b of length $|P_2| \leq 3|ab|$. Then the concatenation $P_1 \oplus \ell(S_{i+1})r(S_i) \oplus P_2$ is an ab -path in $\text{minBST}(S)$, and its length is at most $|P_1| + |\ell(S_{i+1})r(S_i)| + |P_2| \leq (3 + 1 + 3)|ab| = 7|ab|$, as required. It remains to prove the claim. We distinguish between two cases:

- **Case 1:** $a \in S_i^+$ or $a = r(S_i^-)$ and $r(S_i^-)\ell(S_{i+1}) \in \text{minBST}(S)$. In this case, P_1 is a single edge path, and $|P_1| = |a\ell(S_{i+1})| \leq |ab|$.
- **Case 2:** $a \in S_i^-$ and if $a = r(S_i^-)$, then $r(S_i^-)\ell(S_{i+1}) \notin \text{minBST}(S)$. The latter condition implies that $r(S_{i-1})\ell(S_i^+) \in \text{minBST}(S)$ because $r(S_{i-1})\ell(S_i^+)$ is shorter than $r(S_i^-)\ell(S_{i+1})$. In particular, we have $|r(S_{i-1})\ell(S_i^+)| \leq |r(S_i^-)\ell(S_{i+1})| \leq |ab|$. In this case, let $P_1 = (a, r(S_{i-1}), \ell(S_i^+), r(S_{i+1}))$. Now each of the three edges of P_1 has length at most $|ab|$, and so $|P_1| \leq 3|ab|$, as required.

This confirms the claim in both cases; and completes the proof of the upper bound.



■ **Figure 7** Lower bound construction.

For a lower bound consider a bichromatic point set with 4 blocks, $S = \bigcup_{i=1}^4 S_i$, each of 4 points, such that each block spans an interval of length $\varepsilon > 0$ for a sufficiently small $\varepsilon > 0$, and the distance between any two consecutive blocks is 1; see Figure 7. Let a_1, a_2 be the two interior points in S_2 such that $a_1 \in S_2^-$, $a_2 \in S_2^+$, and $|r(S_1)a_2| < |a_1\ell(S_3)|$. Symmetrically, let b_1, b_2 be the two interior points in S_3 such that $b_1 \in S_3^-$, $b_2 \in S_3^+$, and $|b_1\ell(S_4)| < |r(S_2)b_2|$. Then a_1 has degree 1 in $\text{minBST}(S)$, and its only neighbor is $r(S_1)$. Similarly, the only neighbor of b_2 is $\ell(S_4)$. The set $S_2^+ \cup S_3^-$ induces a bistar centered at $r(S_2)$ and $\ell(S_3)$: A bistar on 4 vertices forms a spanning path on $S_2^+ \cup S_3^-$ between a_2 and b_1 . Overall, the a_1b_2 -path in $\text{minBST}(S)$ is

$$P = (a_1, r(S_1), a_2, \ell(S_3), r(S_2), b_1, \ell(S_4), b_2).$$

We have $|a_1b_2| \leq 1 + 2\varepsilon$, but every edge of P has length at least 1. Consequently the stretch of $\text{minBST}(S)$ is at least $|P|/|a_1b_2| \geq 7/(1 + 2\varepsilon)$, which tends to 7 as $\varepsilon \rightarrow 0$. ◀

4 Conclusions

We conclude with a few open problems. What is the tight dependence on $\varepsilon > 0$ for the size of a $(3 + \varepsilon)$ -spanner for n bichromatic points in the plane? We have proved an upper bounds of $O(\sqrt{1/\varepsilon} \cdot n)$ in the plane, and $2n - 3$ (i.e., no dependence on ε) on the line.

The *lightness* of a spanner H for an edge-weighted graph G is the ratio $w(H)/w(\text{MST}(G))$ of the weight of the spanner to the weight of a minimum spanning tree of G . On bichromatic points on the line, we have shown (Proposition 10) that there is a 7-spanner of lightness 1. It remains an open to obtain upper bounds for stretch $3 \leq t < 7$ on the line, and for any $t > 3$ in the plane.

Our results give existential bounds on the size of a bichromatic spanners. The corresponding *optimization problems* remain open: For a given bichromatic point set $S = R \cup B$ and stretch $t \geq 1$, find a t -spanner of $K(R, S)$ of minimum size or minimum weight (note that for a given instance, stretch $1 \leq t < 3$ can be feasible). We conjecture that these problems are NP-hard already in Euclidean plane. Finding good approximations is left for future research. For monochromatic spanners in the plane, these optimization problems are known to be NP-hard, and bicriteria approximation algorithms were obtained only recently [24]. However, for bichromatic points, the optimization problem is already open for n red-blue points on the line with stretch $1 < t < 7$.

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