When Are Geographic Quantities Semantically Compatible with Each Other?

Towards a Constructive Theory of Amounts, Magnitudes, and Proportions

Simon Scheider[1]
Department of Human Geography and Spatial Planning, Utrecht University, The Netherlands

Shuai Wang
Department of Human Geography and Spatial Planning, Utrecht University, The Netherlands

Eric Top
Department of Human Geography and Spatial Planning, Utrecht University, The Netherlands

Abstract

Geographic quantities are more than just numbers. A particular semantics is required to interpret them, which is best captured by constructive types. From a practical viewpoint, numerical operations are only applicable to certain types of geographic quantities, while for others, they become meaningless, even if they are numerically well-defined. For example, GIS experts know that it is meaningless to multiply some quantities – measured, for instance, in people or square kilometers – with one another. In contrast, areas can meaningfully be multiplied by population densities to obtain the number of people living in that area. We currently lack a semantic theory that allows us to determine whether quantities represented in geodata models are compatible with one another for deriving new quantities by common GIS operations. In this paper, we distinguish three distinct types of quantities that are often collapsed into numbers: amounts, magnitudes, and proportions. For each type, we introduce a constructive first-order logic theory. We then define quantifiable experiments represented by different data sources drawn from a geodata repository, and apply the theory to determine the compatibility of those sources for deriving new quantities.

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1 Introduction

One of the central methodological gaps in geographic modeling and cartographic representation is the absence of explicit *validity criteria* for models of geographic quantities. Data-driven numerical practices alone do not provide explicit justifications for why models of those practices should be considered valid [38]. While GIS users know from training

[1] Corresponding author.

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and experience that numerical operations on geographic data are only valid under certain conditions, the rules governing the selection, combination, and transformation of data are rarely made explicit. Consider, for example, the assessment of solar potential and current renewable energy use across Amsterdam's neighborhoods. To arrive at valid conclusions, it is necessary to account for the fact that mapped quantities differ in semantic type, spatial unit, and level of measurement, and are distributed across multiple geodata sources: neighborhood area, roof area suitable for solar panels, the number of existing rooftop panels, and household energy consumption.[2]

Closing this methodological gap has become increasingly important because GIS lacks general validity criteria [40] as well as widely applicable standards for interdisciplinary use [24]. The problem is further amplified by the growing reliance on data-driven GeoAI [21], where explicit information practices are increasingly replaced by self-supervised machine learning models, including geo-foundational models [22]. These models are expected to support a wide range of downstream purposes, yet it remains unclear *why* the practices they model support valid conclusions [40, 38].

Addressing this issue requires a theory of the semantics of quantities [40, 38]. Such a theory must answer practical questions such as (in our example): How can heterogeneous information sources be combined to model solar potential and its exploitation? Does it make sense to sum, subtract, multiply, or divide areal measurements, energy capacities, and panel numbers? If so, what kinds of quantities and spatial units result? Whether a workflow yields valid conclusions depends on whether its numerical inferences are valid, which in turn depends on the semantic compatibility of the quantities involved. Without such a theory, we risk producing quantifications that may be automated but are not justifiable. At a minimum, it must distinguish between valid and invalid numerical inferences, and between compatible and incompatible data sources in the context of a question [40].

Existing theories remain insufficient because they address only parts of this problem. We know, for example, that extensive quantities may be summed and represented in graduated symbol maps, whereas intensive quantities should not [37]. Yet current theory still amounts largely to an extended list of measurement levels [6]. Measurement theory [32] has a long tradition and has strongly influenced GIScience [6], while foundational ontologies of quantities have been proposed more recently [12, 36]. In parallel, fuzzy set theory laid the foundations for Qualitative Reasoning (QR), which seeks to automate qualitative reasoning about intervals on linear scales [50]. However, these approaches are neither sufficiently specific to geographic information nor do they provide criteria for assessing the compatibility of geographic quantities. In particular, they typically require committing to formalizations of quantities on the real number line [50]. Yet we know that extensive geographic quantities represent different mathematical structures, including experimental controls over mathematical lattices, cf. [48, 49]. Inspired by [43], we recently proposed a corresponding experimental model of geographic information [41], which also accounts for model validity [40]. However, we are still lacking criteria for assessing the semantic compatibility of quantities, leaving a central prerequisite for model validity unresolved.

To address this gap, this paper investigates the following research question: *Under what conditions can geographic quantities represented in geodata sources be combined to derive new valid quantities?* We answer this question by proposing a *constructive theory* of geographic quantities that specifies their *typology* (types of geographic quantities and their experimental structures), *formalization* (reconstructions constraining permissible operations and valid outcomes), and *compatibility* (criteria for determining valid combinations of geodata

[2] Such data is e.g. available at `https://maps.amsterdam.nl/`

sources). While our theory targets geographic quantity construction based on an experimental interpretation of geographic information concepts, it may have broader applicability beyond geography and GIS. To demonstrate its use for geographic quantitative reasoning, we revisit the Amsterdam example to determine the solar energy potential of roofs in the city center.

2 Preliminaries

This paper extends prior work on extensive quantities [48] and experimental measurement [41]. Because terms like "mass" [35], "quantity" [13], and "measure" [43, 7] are used ambiguously in geography and GIScience, we clarify our terminology first. Our approach diverges from conventions in physics and standard representational measurement theory [8, 32], which treat quantities as mappings from the world to measurement scales. Instead, it aligns with pluralist accounts of mereological quantities[3] [1, 12] and models them based on a constructivist epistemology[4].

2.1 Quantities

We consider a *quantity* to be an entity that can be used within a *controlled experiment* and that can be *quantified*.[5] For example, in a census (see examples in Table 1), we may control spatial regions and register their inhabitants. As this example shows, quantities need not be numbers or lie on a linear scale. The region of Amsterdam, its area, its population, and its population density are all different kinds of quantities.

A *quantity domain* is the set of all possible quantities of the same type that can be used in an experiment. Quantity domains are constructed bottom-up from measurement practices [18]. For example, *linear measurement scales*, such as the length (m) or area (m^2) scale, are special cases of quantity domains. We call their quantities *magnitudes*, and distinguish them from domains constructed by ratios of magnitudes (*proportions*, such as densities). Both magnitudes and proportions are *linear quantities* because they can be linearly ordered.

Quantities whose domains form a non-linearly ordered part–whole structure that is closed under sums, subtractions, intersections, and unions are called *amounts* (domain symbol A). Examples include amounts of material, mass, water, soil, energy, people, space, and time. For example, we can sum amounts of people to obtain larger ones, or intersect them to determine which individuals they have in common. Similarly, in a GIS, amounts of space (called regions) can be summed or intersected to generate new amounts of space, e.g., using spatial overlay [7]. In other words, amounts are *homeomerous* concepts [13]: parts, sums, and intersections of amounts are themselves amounts. This fundamentally distinguishes them from *objects*, such as buildings or social groups: if we divide a building in half, we do not obtain another building, and likewise, dividing a social group does not yield another group. In both cases, however, we do obtain the underlying amount of material or people

[3] The idea that multiple types of parthood – and hence multiple types of quantity – may coincide. For example, a forest can be conceptualized as an object as well as a quantity of trees, which can be measured by area, number of trees, and biomass, depending on which aspect of "forest" is relevant.

[4] More precisely, we use prior work on protophysics [18] and constructive logic [29, 31, 28] developed in the Erlangen school of methodical constructivism.

[5] This reflects ordinary linguistic usage of quantities (count and mass nouns) more closely [46] and, more importantly, makes explicit the mereological and constructive origins of quantities in geographic information [48]. In doing so, we draw on constructivist proposals by Janich [18], Gruninger's foundational ontology of units and measures [12], and key insights from Simons [42].

Table 1 Comparing amount, magnitude, and proportion with examples.[6]

	Amount (A)	**Linear quantity**	
		Magnitude (M)	**Proportion (P)**
construction	a mereological quantity whose domain forms a boolean lattice.	a linearly ordered quantity constructed from some amount.	a linearly ordered quantity constructed from pairs of magnitudes.
operations	can be summed, subtracted, intersected, and partially ordered by parts.	can be linearly ordered, summed, and subtracted, but does not form products or ratios.	can be linearly ordered, summed, subtracted, and form products and ratios.
properties	boolean lattice	linear (total) order	
formerly [48]	amount	archimedean magnitude	proportional magnitude
example	the region or population of a neighborhood in Amsterdam.	the size of the region or the number of people of a neighborhood in Amsterdam.	population density: the proportion of population and region size (area) of a neighborhood in Amsterdam.

that constitute them. Formally (see Appendix) [48], amount domains are partially ordered by *parthood* $\subseteq$, satisfying reflexivity, antisymmetry, and transitivity. They support *sums* $(+)$ and *intersections* $(*)$, forming a complete *Boolean lattice* [1]. *Overlapping* amounts $O(a_1, a_2)$ have non-empty intersections (see Appendix). Two distinguished elements are important: the empty amount 0^A ("nothing"), which is neutral for sums and absorbing for intersections, and the total amount 1^A ("everything"), which is neutral for intersections and absorbing for sums. Complements $(-a)$ are what is left when removing a from everything. *Subtraction* $(\backslash)$ is defined via complement and intersection. Importantly, amount lattices are non-linear (i.e., non-overlapping amounts exist). The resulting mereology is extensional and satisfies the supplementation principle [5]. More details can be found in the Appendix.

Magnitudes (domain symbol M), in contrast, allow us to measure the sizes of amounts and to compare them even when the corresponding amounts cannot be directly compared [48]. For this reason, magnitudes behave fundamentally differently from amounts. Because sums of amounts may overlap, their magnitudes do not necessarily sum in the same way as the amounts themselves. For example, summing a neighborhood region in Amsterdam with the larger region of Amsterdam that contains it yields the region of Amsterdam itself, but summing their magnitudes (areas) produces a magnitude larger than the area of Amsterdam. Furthermore, magnitudes do not form products (intersections) or ratios that yield new magnitudes. While intersecting regions is a basic technique of GIS, multiplying their sizes to generate new sizes is nonsense [42]. However, multiplying an area magnitude with a *proportion* (domain P) such as population density can be very useful.

Table 1 compares the terms *amount*, *magnitude*, and *proportion* as used in this paper, highlighting differences in their construction, admissible operations, and formal properties. The underlying idea originates in [48], though we simplified the terminology[7]. Magnitudes must also be distinguished from their *quantifications*. Quantification requires *measurement units*, which are specific kinds of *linear reference systems* [7]. For example, the gram specifies

[6] The table was adapted from [48] with the permission of the authors.
[7] We use magnitude instead of "Archimedean magnitude" and proportion instead of "Proportional magnitude".

a linear reference system for mass magnitudes [8]. It can be used to assign a number to a magnitude relative to the chosen unit. Thus *lb* and *g* correspond to different units that refer to the same magnitude domain (mass) [6]. Magnitudes are abstractions that are independent of any scale, representing a quantity under any chosen unit.

2.2 Experimental measurement of geographic quantities

Understanding spatial information requires more than examining a quantity of interest; it also requires analyzing its experimental structure, in particular what is *controlled*, what is *fixed* (held constant), and what is *measured* [43]. As suggested in [41], we thereby control, fix, or measure situations in an underlying experiment [26]. For example, interpreting data from a *tide gauge*, which *measures water level heights* in a time series, requires more than knowledge of the vertical reference system [7]. One also needs to know that the measurements were taken at a *fixed location* (the gauge's position) and that each height measurement was *controlled by a specific time instant* [7]. This idea was originally introduced by Sinton as a model for different types of thematic cartography [43] and has since been adopted in the context of spatial questions [27, 39]. [41] introduced an *experimental model* that formalizes Sinton's idea from an explicitly experimental perspective. The aim is not only to account for geodata models [39], but more fundamentally, to make explicit the experimental structure underlying any form of geographic information, regardless of how it is represented. In this approach, $exp : X, Y \rightarrow Z$ denotes an experimental function that *fixes* situation variables ranging over X, *controls* situation variables over Y, and *measures* situation variables over Z. Applying this function to examples yields: $ekb \in EKB(f : X, c : Y \rightarrow m : Z)$ a *particular experimental knowledge base* with the same structure, resulting from a finite set of experimental performances – that is, from applying exp to some controls in domain Y and recording the resulting outcomes in Z. A geo-dataset can represent such a knowledge base, which makes its inherent experimental structure explicit [41]. Core concepts of spatial information [24], for example, can be interpreted as types of experiments [41]. This includes not only *spatial fields* [23] as functions of spatial location (experiments controlled by vectors in a spatial reference frame) [10], a view that underlies many geodata models [4], but also *coverages* (experiments measuring the spatial area covered by some controlled amount, e.g., of landcover, or of animals) and *lattices* (measuring amounts controlled by spatially tessellated regions, e.g., of administrative territories) (cf. [39, 41]). Previous attempts include applying this idea to geographic quantities (including amounts) in order to define the notion of (spatial) *extensivity* [48], which is fundamental for understanding geographic quantities and their cartographic representation [37]. It can be defined by the additivity of experimental sums in controls and measures [48]. However, while [41] laid the conceptual foundations for the present paper, it remains to construct magnitudes, their quantifications and proportions based on amounts in the context of geographic information. Such constructions are the basis for assessing their *compatibility when deriving new quantities* in a type-safe manner.

3 A theory of quantifiable magnitudes and proportions

Magnitudes are not amounts; rather, they are used to *record and compare the size* of amounts while abstracting from their identity – for example, when quantifying a determinate amount of water that has evaporated from a container. Comparisons between magnitudes are formalized using *proportions*. In standard measurement theory [32], however, magnitudes and proportions are typically collapsed into a single rational number scale, and amounts disappear [45]. By separating amounts, magnitudes, and proportions, we distinguish levels that must be kept apart for valid geographic measurement: amounts can be aggregated and intersected, magnitudes linearly ordered, and proportions used to scale magnitudes. And unlike classical dimensional analysis [2] or modern quantity calculus [11, 9], which presuppose quantities as elements of a given multi-dimensional structure or of some vector space [51, 34], our theory derives the algebra of linear quantities constructively from amount domains via inductive closure under ratios and products. Using Lorenzen's abstraction principle [31, 29, 28], new domains are generated as equivalence classes of equivalence relations specified on previously constructed domains. Each level of abstraction is specified by constructive norms and definitions in domain-restricted first-order logic. Lorenzen's quantifiers are abstractions over domain-indexed conjunctions and disjunctions: $\bigwedge_{x\in D}\varphi(x)$ and $\bigvee_{x\in D}\varphi(x)$ denote universal and existential quantification over elements x of domain D, while $\wedge$, $\vee$, and $\rightarrow$ denote conjunction, disjunction, and implication. The theory is provably consistent and theorems were proven semi-automatically[8].

3.1 Constructing magnitude scales from amounts

Our reconstruction begins with the establishment of size equivalences over different domains of amounts [19], and then establishes the domains of magnitudes based on such equivalences.

Amount domains

Following methodical constructivist proposals [31, 18], amounts can be reconstructed either *bottom-up*, starting from the smallest atoms and using *set-abstraction* [31], or *top-down*, i.e., by *experimentally controlling* the stuff from which amounts are formed [18].

In top-down construction, amounts are identified by *homogenization norms* and *equivalences* (cf. the research program of *proto-physics* [18]), where different norms result in different kinds of amounts. For *material* amounts (substances), techniques such as sieving, sedimentation, stirring, or measurement are used to separate, mix, or standardize portions. For amounts of *mass*, equivalence among material amounts is established via standard weights and balances, thereby defining mass via observable pull-equivalence using, e.g., rope balances [19]. For *energy* – whether thermal, electrical, or kinetic – amounts are homogenized by bringing container objects into equilibrium by controlling energy transformation (work, heat, motion). For *money or economic value*, amounts are socially constructed through actions, namely (market) exchange trade [31]: goods or services are treated as equivalent if they can be exchanged at agreed ratios, thereby constructing amounts of monetary value. Because these kinds of quantities are constructed from *homogenization norms* and *equivalence relations* rather than from smallest elements, they are considered *atomless.* Any given amount can, practically, be divided further without necessarily reaching the smallest possible unit.

For *bottom-up* constructions of amounts, atoms are explicitly given, because amounts are reconstructed as sets of particular kinds of elements. Each element corresponds, by construction, to a singleton set, forming some smallest part in the partonomy of sets reconstructed from those elements by means of *set abstraction* [30, 31]. In this way, we can reconstruct *spatial regions* as (regular) sets of spatial vectors within a spatial reference frame [20]. We construct *time intervals* as (regular) sets of temporal references (intervals) based on events or processes (e.g., flood intervals) [17]. Finally, we define *amounts of objects* as sets either based on their properties or their relations. For example, amounts of elderly people are defined by age (those whose age is above a threshold), and amounts of citizens are defined by role. Still, the atomic amount of persons always consists of precisely one person.

[8] A Mace4/Prover9 (`https://www.cs.unm.edu/~mccune/prover9/`) script which is called (`quantity_amount_magnitude.txt`) is available under `https://zenodo.org/records/20753038`.

Size equivalences and amount multiples

From a constructivist perspective, sizes are abstractions of *size equivalence relations* in terms of equivalence classes [31, 18]. In top-down constructions of material amounts, such equivalences are established experimentally: *mass equivalence* via rope balances [19], *energy equivalence* via energy transformations, or *value equivalence* via market exchange. As for *bottom-up constructions*, equivalences depend on the kind of set construction involved. For finite sets – such as *amounts of objects* in population counts – size is determined by *cardinality*. However, this strategy fails for (potentially) infinite sets, such as amounts of space or time. In these cases, sizes are determined by comparison with *standard linear, areal, or volumetric forms*, depending on dimensionality. For example, reconstructing the size of a two-dimensional region requires standard geometric forms that *partition* that region. In GIS, e.g., measuring area can be done using Gauss' *shoelace formula*, which presupposes that a region can be decomposed into *triangles* or *trapezoids* [3]. Alternatively, regions may be approximated using grid cells of fixed size. Similarly, *durations* are obtained by partitioning time intervals of equal duration [17]. Essentially, all these methods rely on the ability to determine equivalent sizes, governed by the following norm:

▶ **Constructive Norm 1** (size equivalence). $\bigwedge_{\in A} a, b, c.$

$a \cong_{AE} a$ *(reflexivity)*

$a \cong_{AE} b \to b \cong_{AE} a$ *(symmetry)*

$a \cong_{AE} b \wedge b \cong_{AE} c \to a \cong_{AE} c$ *(transitivity)*

where AE may denote *mass*, *energy*, *value*, *cardinality*, *spatial size* (length, area, volume), or *duration* equivalence. Integers enable the definition of *iterative sums* of amounts. Together with size equivalence, this allows the construction of *multiples*. A multiple [25] is an iterative sum of equally sized[9], mutually non-overlapping amounts:

▶ **Definition 1.** $\bigwedge_{\in A} a, a_1, \dots, a_n.$

$Multiple^n_{\cong_{AE}}(\{a_1, \dots, a_n\}, a) \leftrightarrow$

$(\bigwedge_{\in \{a_1,\dots,a_n\}} x, y.\ x \cong_{AE} y \wedge \neg O(x, y)) \wedge a = ((a_1 + a_2) + \dots) + a_n$

An amount a is an *n-multiple* if it is the sum of a set of n equal-sized, non-overlapping amounts, where n is the cardinality of the set. For example, a *5-multiple* of the mass of a standard weight is obtained by producing five distinct weights that are mass-equivalent and summing up their amounts. Here, *multiplication* corresponds precisely to the craft of multiplying standard amounts, illustrating that measurement capacities depend on material craftsmanship [18]. In an idealized form, multiples provide the basis for constructing both magnitudes and proportions.

Constructing magnitude domains

Constructing *magnitude domains* requires us to abstract over *size-equivalent amounts*. In this way, the lattice structure of amounts is preserved in part but collapsed into a linear domain corresponding to a *scale*. Consider the *mass magnitude domain*, which is based on an equivalence relation $\cong_{mass}$ between amounts of material determined by a balance [19]. The magnitude domain M_{mass} can then be reconstructed as the set of size-equivalence classes over the amount lattice; each class corresponds to a magnitude in the image of a measurement function μ:

[9] Multiples need to be equally sized to avoid circular reasoning in the construction [20]. Reconstructing quantities from unequally sized amounts by numeric comparison would already presume their quantifiability.

▶ **Definition 2** (magnitude measurement of amounts).
$\mu : A \to M_{AE}$ *with* $\bigwedge_{\in A} x, y.\ x \cong_{AE} y \leftrightarrow \mu(x) = \mu(y)$

Further magnitude properties can be *induced* from amount relations by suitable norms:

▶ **Definition 3** (ordering of magnitudes).
$\bigwedge_{\in A} x, y.\ \mu(x) \leq \mu(y) \leftrightarrow \bigvee_{\in A} z, z'.\ x \subseteq z \wedge z \cong_{AE} z' \wedge z' \subseteq y$

Here, magnitude order is defined by combining amount parthood and equivalence, which requires complete amount lattices (see appendix A). As a constructive norm for magnitude domains, $\cong_{AE}$ must *span* the amount domain so that all amounts become comparable by magnitude. This yields a *total* (linear) ordering:

▶ **Constructive Norm 2** (totality of ordering of magnitudes).
$\bigwedge_{\in A} x, y.\ \mu(x) \leq \mu(y) \vee \mu(y) \leq \mu(x)$

The magnitude domain M_{AE} is equipped with sum and difference operations $+, -$ and neutral elements whose meaning derives from – and preserves – the structure of the amount lattice (a *homeomorphism*):

▶ **Constructive Norm 3** (homeomorphism of magnitudes).
$\bigwedge_{\in A} x, y, z.$
$\mu(0^A) < \mu(1^A)$ *(magnitude bounds)*
$\mu(-x) = (\mu(1^A) + -\mu(x))$ *(magnitude inverse)*
$\mu(x \setminus y) = \mu(x) + -\mu(x * y)$ *(magnitude minus)*
$\mu(x) = \mu(x \setminus y) + \mu(x * y)$ *(magnitude minus neutral)*
$\mu(x + y) = \mu(x) + \mu(y \setminus x)$ *(magnitude addition)*

In mathematical measure theory, such a μ is called a *measure function* [14]. Since we require amount lattices to be non-degenerate (see appendix), it follows that the induced magnitude domain is *non-degenerate*, too:

▶ **Theorem 1** (non-degenerate). $\bigvee_{\in A} x, y.\ (-O(x, y) \wedge \mu(y) = \mu(x))$

That is, there exist at least two non-overlapping amounts of identical magnitude. Furthermore, magnitudes inherit bounds[10], complements, inverses, and an additive identity from their respective amount domains:

▶ **Theorem 2** (Magnitude bounds). $\bigwedge_{\in A} x.\ \mu(0^A) \leq \mu(x) \wedge \mu(x) \leq \mu(1^A)$

▶ **Theorem 3** (Magnitude complement). $\bigwedge_{\in A} x.\ \mu(x) + \mu(-x) = \mu(1^A)$

▶ **Theorem 4** (Additive inverse). $\bigwedge_{\in A} x.\ \mu(x) + -\mu(x) = \mu(0^A)$

▶ **Theorem 5** (Additive identity). $\bigwedge_{\in A} x.\ \mu(x) + \mu(0^A) = \mu(x)$

In this respect, magnitude domains resemble linear vector spaces, though equipped with only a *restricted* form of addition and subtraction:

▶ **Theorem 6** (Restricted addition). $\bigwedge_{\in A} x, y.\ \neg O(x, y) \to \mu(x) + \mu(y) = \mu(x + y)$

▶ **Theorem 7** (Restricted subtraction). $\bigwedge_{\in A} x, y, y'.\ x \subseteq y \wedge y \cong_{AE} y' \to \mu(y') + -\mu(x) = \mu(y \setminus x)$

[10] Note that these bounds are not absolute because they depend on the construction of the underlying amount domain. As long as larger amount domains can be constructed, there are no absolute magnitude limits. Nevertheless, every particular amount domain is bounded by construction and therefore induces bounded magnitudes.

Consequently, no magnitude exceeds $\mu(1^A)$ or falls below $\mu(0^A)$; these bounds are inherited from the amount domain. For example, no mass magnitude can be smaller than the zero amount (0^A). To *quantify magnitudes* on ratio scales and construct *proportions*, however, we need to express magnitudes as numerical multiples of others. This involves *idealizing* [29, 31] multiples using numbers.

3.2 Quantifying magnitudes

In our framework, numbers are not quantities themselves. Rational numbers can be reconstructed inductively from integers[11] in a way that is independent of amounts (cf. [30]). In the following, we will assume fractions $\frac{n'}{n''}$ to represent rational numbers ($\mathbb{Q}$) constructed from integers $n', n'' \in \mathbb{Z}$ in the usual way. Magnitudes are quantified by multiplying and dividing their underlying amounts by *base units*. Integer multiplication is defined as n-fold magnitude addition:[12]

▶ **Definition 4** (integer multiplication of magnitudes). $\bigwedge_{\in M_{AE}} m.\ \bigwedge_{\in I} n$
$m \times n = (((\mu(0^A) + m)_1 + m)_2 \cdots + m)_n$

Thus m is added n times. This extends restricted magnitude addition by allowing for the unrestricted generation of multiples (cf. Sec 3.1). It follows that $m \times 0 = \mu(0^A)$ and $\mu(0^A) \times n = \mu(0^A)$. For positive m and n, integer multiplication is injective:

▶ **Theorem 8** (injectivity of integer multiplication). $\bigwedge_{\in M_{AE}} m, m', m''.\ \bigwedge_{\in I} n, n'$
$(0 < n, n' \wedge \mu(0^A) < m, m', m'') \rightarrow$
$(m' \times n = m \wedge m'' \times n = m \rightarrow m' = m'') \wedge (m' \times n = m \wedge m' \times n' = m \rightarrow n = n')$

Because of injectivity, integer division can be defined as the inverse of multiplication:

▶ **Definition 5** (integer division of magnitudes). $\bigwedge_{\in M_{AE}} m.\ \bigwedge_{\in I} n.$
$(0 < n \wedge \mu(0^A) < m) \rightarrow m \times n^{-1} = \iota_{\in M_{AE}} m'.\ m' \times n = m$

(Hereafter, this *positivity condition* is left implicit for all m occurring in formulas with magnitude quantification.)

Rational multiplication combines division and multiplication:

▶ **Definition 6** (rational multiplication of magnitudes). $\bigwedge_{\in M_{AE}} m.\ \bigwedge_{\in I} n_1, n_2$
$m \times \frac{n_1}{n_2} = (m \times n_2^{-1}) \times n_1$

For example, $1\,kg \times \frac{3}{4}$ is practically obtained by dividing an underlying amount of magnitude 1 kg into four equivalent parts and summing up three equivalent amounts. We require all positive magnitudes to be *quantifiable*: there must be a base magnitude from which these magnitudes can be regenerated by multiplication with a unique rational number.

▶ **Constructive Norm 4** (quantifiability of magnitudes). $\bigvee_{\in M_{AE}} m_{base}.$
$\bigwedge_{\in M_{AE}} m_1, m_2.\ \bigvee_{\in I} n_1, n_2, n_3, n_4.$
$m_{base} \times \frac{n_1}{n_2} = m_1 \wedge m_{base} \times \frac{n_3}{n_4} = m_2 \wedge (\bigwedge_{\in I} n_1, n_2, n_3, n_4. m_{base} \times \frac{n_1}{n_2} = m_{base} \times \frac{n_3}{n_4} \rightarrow \frac{n_1}{n_2} = \frac{n_3}{n_4})$

[11] And in a similar way, irrationals, differentials, and infinitesimals can be reconstructed from rational numbers, cf. Lorenzen [30].
[12] To avoid confusion, we use "×" instead of "*" to distinguish multiplication from intersection.

This guarantees (a) the existence of a common base m_{base} for all magnitudes of a domain, (b) there is a unique rational quantification of all magnitudes relative to that base, and (c) the transformability of any two magnitudes via the ratio of their quantifications. The *Archimedean property* for ratio scales [32] follows: any non-zero magnitude can be obtained from any other by multiplication with a suitable rational number. This is not an independent axiom but a consequence of quantifiability and the existence of magnitudes greater than $\mu(0^A)$. To quantify a magnitude is to assign the unique rational number required to obtain it from a chosen base:

▶ **Definition 7** (quantification using base units). $\bigwedge_{\in M_{AE}} m, m_{base}.$
$qu_{m_{base}}(m) = \iota_{\in R}\frac{n_1}{n_2}.\ m_{base} \times \frac{n_1}{n_2} = m$

Hence, every magnitude can be decomposed into a multiple of some base unit:

▶ **Theorem 9** (magnitude unit decomposition). $\bigwedge_{\in M_{AE}} m, m_{base}.$
$m = m_{base} \times qu_{m_{base}}(m)$

What we call base units are the same thing as *scale units* (e.g., 1 *kg*, 1 *meter*). Multiple bases may exist, yielding different but semantically equivalent *measurement scales* (*kg*, *lb*, etc.). In bottom-up constructions of finite sets, a natural base is the magnitude of singleton sets. For example, population numbers use the magnitude of one person as base. All bases, like all quantifiable magnitudes, must exceed $\mu(0^A)$.

3.3 Magnitude ratios and products as linear quantities

Pairs of magnitudes under division ($m_1 : m_2$) are *ratios* and pairs under multiplication ($m_1 \times m_2$) are *products*. Note that products and ratios *are not magnitudes*, just like population density is not the same kind of quantity as a population number. We generalize magnitudes with their iterated ratios and products under the notion of *linear quantities*[13], forming a domain Q, which is inductively generated from magnitude domains by repeated formation of ratios and products:

▶ **Constructive Norm 5** (ratios, products and linear quantities). $\bigwedge_{\in M_{AE1}} m_1.\ \bigwedge_{\in M_{AE2}} m_2.$
$m_1, m_2 \in Q \wedge \bigwedge_{\in Q} q_1, q_2.\ (q_1 \times q_2) \in Q \wedge (q_1 : q_2) \in Q$

Q must furthermore be constructed in a way such that multiplication is symmetric and associative, admits neutral elements ($m : m$), and allows the inversion of ratios:

▶ **Constructive Norm 6** (multiplication of linear quantities). $\bigwedge_{\in Q} m_1, m_2, m_3.$
$m_1 \times m_2 = m_2 \times m_1$ *(symmetry)*
$(m_1 \times m_2) \times m_3 = m_1 \times (m_2 \times m_3)$ *(associativity)*
$m_1 \times (m_2 : m_2) = m_1$ *(neutral element)*
$m_1 \times (m_2 : m_3) = (m_1 \times m_2) : m_3$ *(ratio associativity)*
$m_1 : (m_2 : m_3) = m_1 \times (m_3 : m_2)$ *(ratio inversion)*

These *rewriting rules* make $(Q, \times)$ a commutative Abelian group. Standard simplifications of ratios follow from this:

▶ **Theorem 10** (simplifying ratios). $\bigwedge_{\in M_{AE1}} m_1.\ \bigwedge_{\in M_{AE2}} m_2.\ \bigwedge_{\in M_{AE3}} m_3.$
$(m_1 : m_2) : m_3 = m_1 : (m_2 \times m_3)$
$(m_1 \times m_2) : m_1 = m_2$

[13] Ratios and products are called linear quantities because they can be totally ordered inductively, based on the ordering of their constituting magnitudes.

Thus linear quantities can be algebraically reduced whenever their constituent magnitudes *cancel out*. Unlike rational numbers, however, products and ratios of magnitudes need not themselves yield magnitudes; only cancellation reduces them back to a particular magnitude.

Quantification of linear quantities and unit decomposition

Ratios from different magnitude domains (e.g., persons per area) are *heterolithic*; those from a single domain are called *monolithic*. How can we quantity such ratios? In general, rational multiplication of linear quantities is done componentwise:

▶ **Constructive Norm 7** (multiplication of ratios and products). $\bigwedge_{\in Q} m_1.\ \bigwedge_{Q} m_2.\ \bigwedge_{\in I} n_1, n_2.$
$(m_1 : m_2) \times \frac{n_1}{n_2} = ((m_1 \times \frac{n_1}{n_2}) : m_2) = (m_1 : (m_2 \times \frac{n_2}{n_1}))$
$(m_1 \times m_2) \times \frac{n_1}{n_2} = (m_1 \times \frac{n_1}{n_2}) \times m_2$

Combining unit decomposition with ratio and product multiplication then yields:

▶ **Theorem 11** (Linear quantity unit decomposition). $\bigwedge_{\in M_{AE1}} m_1.\ \bigwedge_{\in M_{AE2}} m_2. \bigwedge_{\in M_{AE3}} m_3.$
$m_1 \times (m_2 : m_3) = (m^1_{base} \times (m^2_{base} : m^3_{base})) \times (qu_{m^1_{base}}(m_1) \times \frac{qu_{m^2_{base}}(m_2)}{qu_{m^3_{base}}(m_3)})$

Multiplication thus separates into: (i) base-unit algebra (where cancellation of base magnitudes may occur), and (ii) ordinary rational arithmetic on their quantifications. Cancellation rules yield special cases, such as monolithic multiplication:

▶ **Theorem 12** (monolithic multiplication). $\bigwedge_{\in M_{AE1}} m_1.\ \bigwedge_{\in M_{AE2}} m_2, m_3.$
$m_1 \times (m_2 : m_3) = m^1_{base} \times (qu_{m^1_{base}}(m_1) \times \frac{qu_{m^2_{base}}(m_2)}{qu_{m^2_{base}}(m_3)})$

Heterolithic multiplication transforms magnitudes across domains whenever units cancel; otherwise compound units remain:

▶ **Theorem 13** (heterolithic multiplication). $\bigwedge_{\in M_{AE1}} m_1. \bigwedge_{\in M_{AE2}} m_2, m_3.$
$m_2 \times (m_1 : m_3) = m^1_{base} \times (qu_{m^2_{base}}(m_2) \times \frac{qu_{m^1_{base}}(m_1)}{qu_{m^2_{base}}(m_3)})$

For example, $100km^2 \times (60\text{person} : 1km^2) = 1km^2 \times (1\text{person} : 1km^2) \times 100 \times \frac{60}{1} =$ 6000 person. We can use heterolithic ratios to quantify important types of proportions, e.g., *density*, in terms of fractions (60 people per square km), and to *transform magnitudes from one domain to another* (here: from areal sizes to number of people).

Equivalence and invariance of ratios

Ratios are equivalent if they only differ by a rational factor in numerator and denominator:

▶ **Definition 8** (equivalence of ratios). $\bigwedge_{\in M_{AE1}} m_1, m_3.\ \bigwedge_{\in M_{AE2}} m_2, m_4.$
$(m_1 : m_2) \cong_{ratio} (m_3 : m_4) \leftrightarrow$
$\bigvee_{\in I} n_1, n_2.\ (m_1 \times \frac{n_1}{n_2}) : (m_2 \times \frac{n_1}{n_2}) = (m_3 : m_4)$

It follows that multiplication with ratios is invariant under equivalent ratios and under changes of base units:

▶ **Theorem 14** (invariance of ratio multiplication). $\bigwedge_{\in M_{AE}} m, m_1, m_2, m'_1, m'_2.$
$(m_1 : m_2) \cong_{ratio} (m'_1 : m'_2) \to m \times (m_1 : m_2) = m \times (m'_1 : m'_2)$

Proof. All equivalent ratios differ only by a rational factor $\frac{n_1}{n_2}$ in the numerator and the denominator. This means that their quantifications also differ by a constant rational factor, and so this factor cancels out by the rules of rational number fractions. Suppose we use two different constructive base units for quantifying these two ratios. Note that by constr. norm 4, one needs to be expressible as a rational multiple of the other. So the ratios differ only by a constant rational factor in the numerator and denominator, too, and the same argument above applies. ◀

Hence magnitude-ratio multiplication depends only on the proportion represented, not on particular ratio pairs or chosen units.

3.4 Proportions

Theorem 14 allows us to define a domain of *proportions* as equivalence classes of magnitude ratios:

▶ **Definition 9** (proportions of magnitude ratios). $\mu : M_{A'E} \times M_{A''E} \to (M_{A'E} : M_{A''E})$
$\bigwedge_{\in M_{A'E}} x, v. \bigwedge_{\in M_{A''E}} y, w. \ (x : y) \cong_{ratio} (v : w) \leftrightarrow \mu(x : y) = \mu(v : w)$

Proportion domains $M_{A'E} : M_{A''E}$ thus represent ratios independently of any chosen base or equivalent scale. Multiplication of magnitudes with proportions follows the same rules as for ratios, and domains of magnitude products with proportions $M_{A'E} \times (M_{A''E} : M_{A'''E})$ obey the same combination and cancellation principles. This construction parallels the *Eudoxean theory of ratios* [15, book V].

3.5 Types of quantity domains

Following our constructions introduced above, monolithic proportions arise within a single amount domain, e.g., population percentages or shares of energy. Heterolithic proportions span distinct domains, e.g., population density (persons per area), and require separate treatment for the numerator and denominator when computing with them. Based on this,

Table 2 Monolithic and heterolithic proportions. The sign " is the ditto mark. It denotes that the previous type (in the numerator) is repeated in the denominator.

Proportion variant	Type	Num-amount	Denom-amount	Example
Monolithic	Continuous share	Atomless	"	Renewable energy fraction
	Discrete share	Objects	"	Fraction of elderly persons
	Space proportion	Space	"	Fraction of urban land
	Time proportion	Time	"	Commuting time fraction
Heterolithic	Density	Any	Space	Population density
	Speed	Space	Time	Travel speed
	Power	Energy	Time	Energy per hour
	PPer object	Any	Objects	CO_2 per person
	PPer unit	Any	Atomless	Price per kg

we introduce the following terminology (cf. Table 2): *Monolithic proportions* of atomless amounts are called *continuous shares*; object counts yield *discrete shares*. Fractions of space or time yield *space* or *time proportions*. *Heterolithic proportions* generalize familiar spatio-temporal measures: *density*, *flow*, *speed*, *tempo*, and per-object or per-unit rates.

Table 3 Grammar for a terminology of quantity domains.

Types of quantity domains	Abbreviations
object ::= person / panel / neighborhood / ...	*region ::= A(2d)*
atomless ::= material / mass / energy / ...	*interval ::= A(time)*
space ::= 1d / 2d / 3d	*number ::= MA(object)*
atomic ::= space / time / object / ...	*volume ::= MA(3d)*
amount ::= A(atomic / atomless)	*area ::= M(region)*
magnitude ::= M(amount)	*length ::= MA(1d)*
proportion ::= P(magnitude : magnitude)	*duration ::= M(interval)*
cshare ::= P(MA(atomless) : MA(atomless))	*density ::= MA(atomic / atomless) : MA(space)*
dshare ::= P(MA(atomic) : MA(atomic))	*speed ::= length : duration*
	power ::= MA(energy) : duration
	avg ::= MA(atomic / atomless) : number

Table 3 summarizes the types of quantity domains we use. These distinctions capture practically relevant consequences: atomic/object-based amounts have minimal countable units restricting fractions, while atomless amounts do not. Explicit incorporation of units enables us to distinguish *unit conversions* (km → miles) from *quantity transformations* (density → population), supporting transparent, type-safe spatio-temporal modeling.

4 Inference and compatibility of quantifiable experiments

So far, we have treated quantities as independently measurable. In practice, every measurement depends on an experimental setup relating amount domains. By re-embedding quantities into a prior experimental framework [41] (see Sec. 2.2), we can define both the procedures used to generate measurements and how new experiments can be inferred from existing ones. *Quantifiable experiments* are those that fix, control, or measure *amounts*. These amounts can be measured as magnitudes, which can in turn be quantified using rational numbers. The *semantics of GIS operations* for quantities can be understood in terms of *experimental inferences* [41]. Furthermore, whether a set of experimental knowledge bases is *compatible* with respect to a goal experiment reduces to the question whether there is an inference that produces the goal from the sources.

4.1 Experimental combinations

▶ **Definition 10** (experimental combination). $\bigwedge_{\in EKB(f:x,c:Y \to m:M_{A'E})} ekb_1.$
$\bigwedge_{\in EKB(f:x,c:Y \to m:M_{A''})} ekb_2. \bigwedge_{\in EKB(f:x,c:Y \to m:M_{A'''})} ekb_3.$
$ekb_1 \times (ekb_2 : ekb_3) \in EKB(f : x, c : Y \to m : M_{A'E} \times (M_{A''E} : M_{A'''E}))\wedge$
$\bigwedge_{\in Y} y. \bigwedge_{\in M_{A'}} m'. \bigwedge_{\in M_{A''}} m''. \bigwedge_{\in M_{A'''}} m'''.$
$[x, y, (m' \times (m'' : m'''))] \in ekb_1 \times (ekb_2 : ekb_3)) \leftrightarrow ([x, y, m'] \in ekb_1 \wedge [x, y, m''] \in ekb_2) \wedge$
$[x, y, m'''] \in ekb_3)$

This definition captures how new experimental recordings can be inferred by combining existing experimental knowledge bases, matching controls, and combining measurements via magnitude multiplication. For example, consider multiplying a database of region sizes: $rs \in EKB(f : time, c : region \to m : area)$ with a database of regional mass densities $mp \in EKB(f : time, c : region \to m : P(MA(mass) : area))$ at a fixed time t. The latter could represent vegetation mass density in kg/km^2 for each region of a spatial lattice. If the controlled regions in both databases share a common subset, $rs \times mp$ yields a non-empty database of proportional mass magnitudes for this subset: $rs \times mp \in \mathit{EKB}(f : \mathit{time}, c : \mathit{region} \rightarrow m : \mathit{MA}(\mathit{mass}))$. If additionally the measurement units of area are compatible, they cancel out, allowing the mass to be quantified as a rational number: $(0.5km^2 \times (1000kg : 1km^2) = (1kg \times 1000) \times ((1km^2 \times \frac{1}{2}) : 1km^2) = 1kg \times (1000 \times \frac{1}{2}) = 500kg)$. This outcome can be inferred purely from the types of domains and the units of quantification.

Experimental combinations correspond to a generalized form of *local map algebra* [47], allowing computations with different magnitude and proportional units, as well as various experimental control domains beyond raster cells.

4.2 Experimental transformations

In case either the control regions or the measurement units are *incompatible*, one first would need to apply some *transformation* to the experiments in order to render them compatible with each other. The simplest transformation is a *unit conversion.* Since every base magnitude m''_{base} of some domain must be a rational multiple of every other base unit m'_{base} of the same domain, we can always convert units into each other: $m'_{base} \times qu_{m'_{base}}(m''_{base}) = m''_{base}$. The rational number $qu_{m'_{base}}(m''_{base})$ can then be used to transform different quantifications of the same magnitude into each other. For example, suppose the area is given in m^2 instead of km^2, then $qu_{1m}(1km) = \frac{1000}{1}$. More complex transformations may occur in a GIS [44, 7].

5 Case study: estimating the solar potential on roofs in Amsterdam

To evaluate and test the proposed constructive formalism about geographic quantities, we apply it to a case study.[14] Suppose your task as a GIS analyst is to estimate the solar energy potential and its current exploitation on roofs in the city center of Amsterdam, using currently available geodata sources. We have four questions *for each neighborhood* in the city center of Amsterdam, to be answered for the year 2022 (a common fix):

(Q1) *What is the potential amount of solar panels that could be installed on roofs?* $(\mathit{EKB}(f : \mathit{2022}, c : A(\mathit{space}) \rightarrow m : \mathit{number}))$

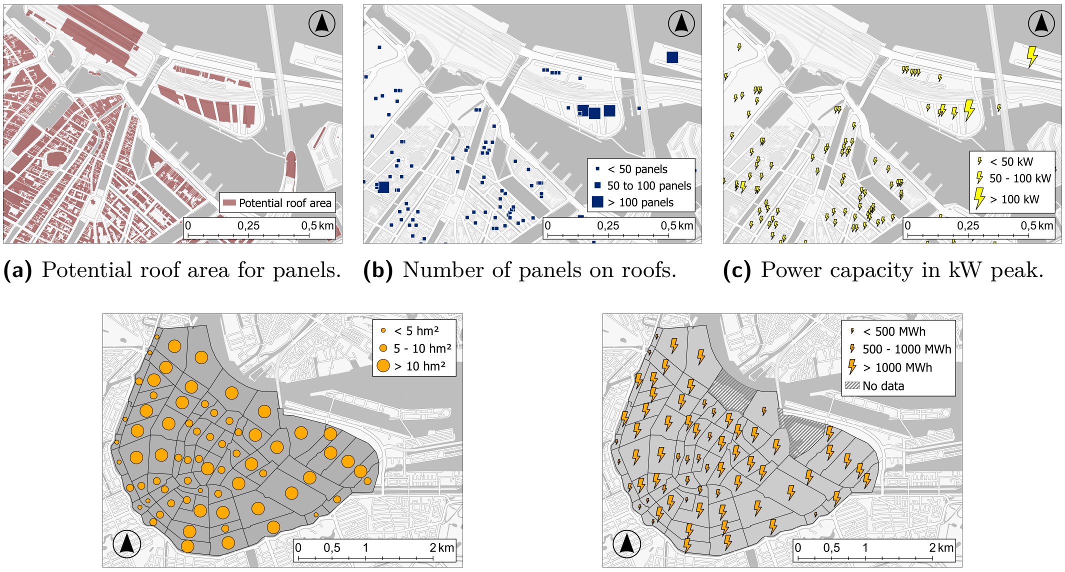
(Q2) *Which proportion of the potential of solar panels is currently installed?* $(\mathit{EKB}(f : \mathit{2022}, c : A(\mathit{space}) \rightarrow m : P(\mathit{number} : \mathit{number})))$

(Q3) *What is the current amount of solar energy produced?* $(\mathit{EKB}(f : \mathit{2022}, c : A(\mathit{space}) \rightarrow m : M(\mathit{energy})))$

(Q4) *What proportion of the currently used electricity is produced by solar panels?* $(\mathit{EKB}(f : \mathit{2022}, c : A(\mathit{space}) \rightarrow m : P(M(\mathit{energy}) : M(\mathit{energy}))))$.

Q1 and Q3 ask for magnitudes: the first counts an object amount (number of panels), the second an energy magnitude (solar energy production per year). Q2 and Q4 ask for proportions: the former is a discrete share (installed panels relative to potential panels), the latter a continuous energy share (solar production relative to total electricity consumption). We aim to *infer* which combinations of available sources can answer these questions.

[14] The ArcGIS workspace, a proof script, and related files are available on Zenodo with DOI: `10.5281/zenodo.20753038`.



(a) Potential roof area for panels. **(b)** Number of panels on roofs. **(c)** Power capacity in kW peak.

(d) Area of neighborhoods in Amsterdam. **(e)** Electric energy consumption per year.

Figure 1 Various thematic source maps of geographic quantities in Amsterdam, the Netherlands, 2022. Data sources: Amsterdam Data Lab and CBS.

5.1 Geodata sources and their interpretation as quantifiable experiments

To answer these questions, we will reuse data sources from two geodata repositories (illustrated in Fig. 1): the Centraal Bureau voor de Statistiek (CBS)[15] and the data portal of the Municipality of Amsterdam.[16]

(a) *Potential roof area for solar panels*[17]. Assesses the potential roof area suitable for placing solar panels from a spatial and cultural-historical perspective, taking into account the value of building monuments (Fig. 1a). Data format is vector polygons with their size in km^2. This gives an EKB that controls for amounts of space, fixes the year 2022, and measures their area: $\textit{roof-ekb}^{area}_{A(space)} \in EKB(c{:}A(space),\ f{:}interval \rightarrow m{:}area)$.

(b) *Solar panels*[18]. This dataset represents the spatial distribution of existing solar panels on rooftops in Amsterdam, including their number and power capacities (Fig. 1b and Fig. 1c). The data are vector points with two attributes: panel count and power capacity in kilowatt peak (kW_{peak}).[19]. The EKBs have controls for amounts of panels and measure the spatial area, the number of panels, and power capacity:

$$ekb^{A(space)}_{A(panel)} \in EKB(c{:}A(panel),\ f{:}interval \rightarrow m{:}A(space))$$

$$ekb^{number}_{A(panel)} \in EKB(c{:}A(panel),\ f{:}interval \rightarrow m{:}number)$$

$$ekb^{power}_{A(panel)} \in EKB(c{:}A(panel),\ f{:}interval \rightarrow m{:}power)$$

[15] The neighborhood statistics ("Kerncijfers wijken en buurten 2022"): `https://dataportal.cbs.nl/detail/CBS/85318NED`. Date accessed: 11th Feb 2026.
[16] `https://maps.amsterdam.nl`
[17] `https://maps.amsterdam.nl/zonnedaken_erfgoed/`. Date accessed: 11th Feb 2026.
[18] `https://maps.amsterdam.nl/zonnepanelen/`. Date accessed: 11th Feb 2026.
[19] Kilowatt peak denotes the power output of solar panels under standardized ideal insolation conditions.

From an experimental perspective, the first constitutes an example of a *coverage*, in which space is used as a measure rather than as a control.

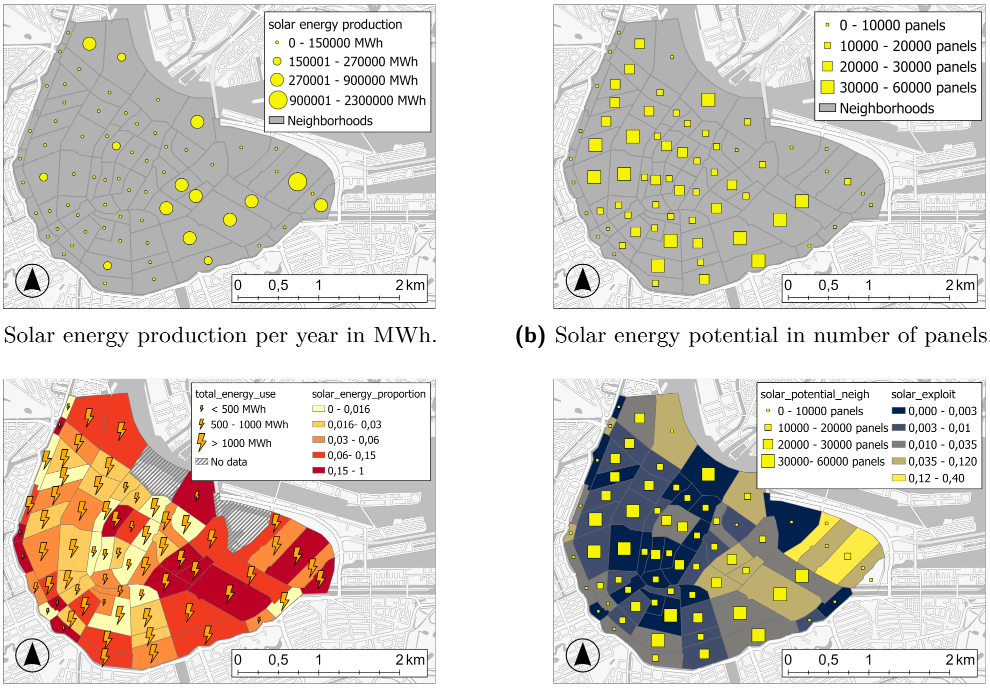
(c) *Neighborhoods*[20]. Represents the regions of administrative units on the "Buurt" level with quantities that are statistical aggregations over these regions (Fig. 1d, Fig. 1e), including their areas in hm^2 and the electric energy consumption of households in MWh.

$$neigh\text{-}ekb^{M(energy)}_{A(space)} \in EKB(c{:}A(space),\ f{:}interval \rightarrow m{:}M(energy))$$

$$neigh\text{-}ekb^{area}_{A(space)} \in EKB(c{:}A(space),\ f{:}interval \rightarrow m{:}area)$$

From an experimental perspective, these are examples of a spatial lattice, where we control tessellated amounts of space via administrative objects and measure various magnitude sums over space.

These sources differ not only in the types of quantities but also in their spatial support units, which serve as experimental controls. We shall therefore reason about the compatibility of both the quantities and their units, as well as their experimental structures (see Fig. 3).



(a) Solar energy production per year in MWh.

(b) Solar energy potential in number of panels.

(c) Solar energy proportion (fraction of energy use).

(d) Solar panel exploitation (fraction of potential).

Figure 2 Thematic maps of derived quantities assessing the solar energy production, the solar potential, and its exploitation on roofs in Amsterdam neighborhoods.

5.2 Compatibility and inference of quantity types for example maps

We showed how quantities can be transformed and inferred to answer the four questions. A key challenge is that the source and target quantities use different controls: potential roof areas and installed rooftop panels do not share controls with the neighborhood-based target

[20] `https://www.cbs.nl/nl-nl/visualisaties/cijfers-op-de-kaart`. Date accessed: 11th Feb 2026.

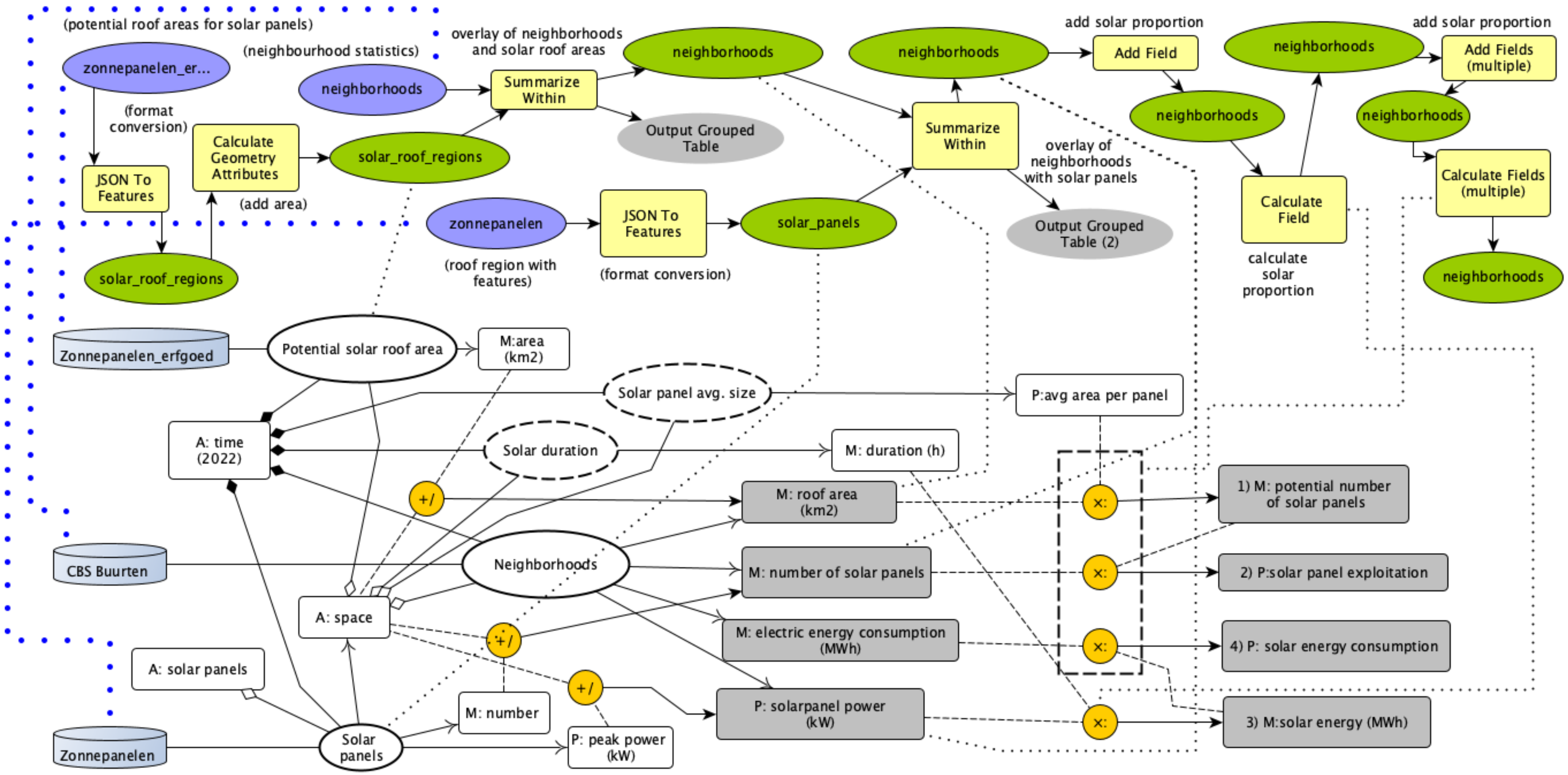


Figure 3 Experimental inference of geographic quantities for the solar energy scenario and its correspondence with a GIS workflow. The upper part is the workflow of data processing in ArcGIS. The lower part is the modelling of quantities. Blue dotted lines are about the resource-data correspondence. The black dotted lines between the two parts are about the variable and operations correspondence. For the upper part, the ellipses nodes denote data and the boxes denote data processing (data format conversion, summarization, calculation, etc). Grey ellipses are for outputs. For the lower part, thick ellipses denote EKBs. Dashed ellipses denote (implicit) attributes. Arrows with black diamonds indicate fixes, those with white diamonds represent controls, and curly arrows correspond to measures. White boxes are primary source quantities, and grey boxes denote derived quantities. The abbreviations $A:$, $M:$, $P:$ stand for *amount of*, *magnitude* and *proportion*. Orange circles denote experimental transformations (+/) (using sums and differences of either amounts or magnitudes) and combinations (×:) (using magnitude products and ratios).

experiments. As a result, straightforward experimental combinations or concatenations of these EKBs yield empty results because there are no common controls. Yet, we can use *experimental aggregations* that are based on *amount/magnitude sums and differences* (the orange circles with +/ in Fig. 3) to transform both the solar panels and the potential solar roof area experiments into corresponding neighborhood experiments. Such transformations are implemented as spatial overlay operations in a GIS, which can be used to cut and merge spatial regions into larger ones to fit into "cookie cutter" regions, and to simultaneously sum or subtract magnitude values of their measures. An example would be a *vector overlay* with a *sum function*, such as ArcGIS's *Summarize Within* tool. In the example above, we used this tool to summarize both the potential roof areas and the solar panel point data into neighborhood polygons, yielding three new neighborhood attributes representing: magnitudes of roof area per neighborhood ($roof\text{-}neigh\text{-}ekb^{area}_{A(space)}$), aggregated from $roof\text{-}ekb^{area}_{A(space)}$; number of solar panels per neighborhood ($panel\text{-}neigh\text{-}ekb^{number}_{A(space)}$), aggregated from $ekb^{number}_{A(panel)}$; solar power per neighborhood ($panel\text{-}neigh\text{-}ekb^{power}_{A(space)}$), aggregated from $ekb^{power}_{A(panel)}$. Conceptually, these aggregation steps replace object-amount based controls (e.g., panels or roofs) by spatial controls (neighborhood regions) and infer magnitude sums.

Recall that our EKBs are conceptual models for interpreting geodata, not data models (in Fig. 3, we distinguish between EKBs and actual data sources), since not all aspects needed for interpretation are explicitly represented in the data. Power is defined as a proportion of magnitude of energy per duration. To convert power back into solar energy, we need to have information about the duration of insolation within the respective control region for the fixed year, which can be obtained from weather data. We assume 4 hours of sunshine per day, i.e. $365 \times 4 = 1460$ sun hours per year. Thus, an entire *separate experiment* (*solar duration*, see Fig. 3) hides behind these simple numbers, which we represent by: $solar\text{-}neigh\text{-}ekb^{duration}_{A(space)}$. Similarly, to estimate a potential number of solar panels, we need to combine the aggregated roof area with a panel size, which could be obtained from another data source. Yet, this case study adopts a rough estimate of $2m^2$ per panel. We represent this implicit knowledge by: $panel\text{-}neigh\text{-}ekb^{P(area:number)}_{A(space)}$. The remaining steps to derive the target maps are based on *experimental combinations* of magnitudes and proportion measures using the following magnitude multiplications (the orange circles with $\times$: in Fig. 3), including quantifications and necessary unit conversions:

1. potential number of solar panels on roofs per neighborhood:
   $roof\text{-}neigh\text{-}ekb^{area}_{A(space)} \;:\; panel\text{-}neigh\text{-}ekb^{P(area:number)}_{A(space)} \;=\; roof\text{-}panel\text{-}neigh\text{-}ekb^{number}_{A(space)}$,
   since
   $area : (area : number) = area \times (number : area) = number \times (area : area) = number$
   The roof areas are measured in km^2, while panel sizes are given in m^2. Since $1\,km^2 = 1{,}000{,}000, m^2$, the units are first converted to the same scale and then cancel out, yielding the number of panels.
2. solar exploitation (discrete share of number of panels) per neighborhood:
   $panel\text{-}neigh\text{-}ekb^{number}_{A(space)} : roof\text{-}panel\text{-}neigh\text{-}ekb^{number}_{A(space)} = panel\text{-}neigh\text{-}ekb^{P(number:number)}_{A(space)}$
3. solar panel energy production per neighborhood:
   $panel\text{-}neigh\text{-}ekb^{power}_{A(space)} \times solar\text{-}neigh\text{-}ekb^{duration}_{A(space)} = solar\text{-}neigh\text{-}ekb^{M(energy)}_{A(space)}$, since
   $power \times duration = (M(energy) : duration) \times duration = M(energy) \times (duration : duration) = M(energy)$. The resulting energy is obtained in kWh and subsequently converted to MWh by dividing by 1000.
4. solar energy proportion per neighborhood:
   $solar\text{-}neigh\text{-}ekb^{M(energy)}_{A(space)} : neigh\text{-}ekb^{M(energy)}_{A(space)} = solar\text{-}neigh\text{-}ekb^{P(M(energy):M(energy))}_{A(space)}$

They provide valid answers to our four questions under the practical restrictions of using approximate panel sizes and solar hours.

Our theory also allows for identifying incompatible combinations. For example, multiplying solar power by the number of solar panels or by the roof area instead of solar duration does not yield any meaningful magnitude or proportion. Formally, $power \times area = (M(energy) : duration) \times area = M(energy) \times (area : duration)$ produces a fictive compound unit (e.g., $MWh \times km^2/h$) that not only lacks experimental interpretation, but also cannot be reduced to any magnitude or proportion of interest for answering any of our 4 questions.

6 Conclusion

This paper proposes a constructive semantic theory for determining when geographic quantities can be validly combined. Note that the purpose of our theory does not lie in enabling new kinds of geo-computations. Rather it makes an implicit numeric practice explicit and thereby delivers, for the first time, validity criteria for forming geographic quantities. In contrast to standard measurement theory, we distinguish amounts, magnitudes, and proportions, provide a constructive formalization of these domains, and introduce a notion of compatibility that accounts for the operations available on each domain and the principled derivation of linear quantities from amount domains. The relevance of our theory thereby goes beyond geography, since it enables explicit compatibility constraints for quantities and programmatic validation of numeric workflows. Yet, embedding these quantity types into our experimental framework offers a conceptual lens on GIS operations and their capacity to derive new quantities. The Amsterdam solar case study demonstrated how the theory supports type-safe transformations, unit conversions, and proportional reasoning, thereby establishing a formal basis for assessing quantitative compatibility and the validity of GIS workflows. In increasingly automated AI-driven contexts, our theory might form the backbone of a *neuro-symbolic* AI [16, 33] for valid computing with geographic quantities. Future work should strengthen empirical evaluation, for example by applying the theory to GeoAI or GeoQA benchmark corpora, further investigate its logical properties, extend it to more complex experimental transformations (including concatenations, spatial interpolations, and conversions) in the context of GIS workflow semantics [44], and translate it into metadata standards and algorithms for automated assessment of geodata compatibility in indirect question answering [39].

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A Detailed preliminaries for amounts

Amount domains constitute non-linearly ordered *additive part–whole structures* closed under sums, subtractions, intersections and unions [48]:

Norms for portioning amounts. We assume a partially ordered algebra, where the symbol $\subseteq$ ranges over amounts of a particular amount domain A (such as sand, gravel, or clay): For all $a, b, c \in A$, we have $a \subseteq a$ (Amount order reflexivity), $(a \subseteq b) \leftrightarrow (b \supseteq a)$ (Amount order gte/lte), $(a \subseteq b \wedge a \supseteq b) \rightarrow a = b$ (Amount order antisymmetry), and $(a \subseteq b \wedge b \subseteq c) \rightarrow a \subseteq c$ (Amount order transitivity). Based on these norms, we define strict order: $(a \subset b) \leftrightarrow (a \subseteq b \wedge \neg(b \subseteq a))$. Strictly ordered amounts are not identical – one is a proper part of the other.

Summing, subtracting, and intersecting amounts. We define "+" for adding two amounts to form a bigger one and "*" for giving only the overlapping (common) part of two amounts (i.e., intersection/product). We have the constructive norm: $\bigwedge_{\in A} a, b, c.\ a+b = b+a$ (Amount sum symmetry), $a*b = b*a$ (Amount prod symmetry), $(a+b)+c = a+(b+c)$ (Amount sum associativity), $(a*b)*c = a*(b*c)$ (Amount prod associativity), $a*(b+c) = (a*b)+(a*c)$, (Amount prod distributivity), $a+(b*c) = (a+b)*(a+c)$ (Amount sum distributivity). The order of summing or intersecting amounts is irrelevant, leading to symmetry and associativity norms as well as distributivity norms. Amounts can also be *subtracted* from each other (symbol $\backslash$), and thereby have certain well defined neutral elements. As introduced in Section 2, the empty amount 0^A ("nothing") is neutral for sums and absorbing for intersections, and the total amount 1^A ("everything") is neutral for intersections and absorbing for sums. If one removes an amount equal to that of itself, nothing remains (i.e., the inverse amount shows $a \backslash a = 0^A$.). Subtracting/adding this empty amount to another does not change it: $b \backslash 0^A = b$ and $a + 0^A = a$. Furthermore, 1^A ($1^A \in A$) is the neutral element for amount products, and it corresponds to "everything" ($1^A \in A$). For amounts $a, b \in A$, removing everything from something leaves nothing ($b \backslash 1^A = 0^A$), and adding it results in everything ($a + 1^A = 1^A$). The product of nothing with anything is nothing ($a * 0^A = 0^A$), while the product of everything with something does not make a difference to something ($a * 1^A = a$). Based on this, we can define overlaps of amounts in terms of non-empty intersections: $\bigwedge_{\in A} a, b, c.O(a, b) \leftrightarrow (0^A \subset a * b))$.

Amount lattices. Amounts also obey clear rules about how parts combine under sums and products, making them, in mathematical terms, *Boolean lattices*.[21] This means, that for $a, b, c \in A$, the existence of joins and meet always exists: $a \subseteq a + b$, and $a * b \subseteq a$, and we

[21] https://www.umsl.edu/~siegelj/SetTheoryandTopology/The_algebra_of_sets.html

always have $(a \subseteq c \wedge b \subseteq c) \rightarrow a + b \subseteq c$ and $(c \subseteq a \wedge c \subseteq b) \rightarrow c \subseteq a * b$. We can always form the *complement* (symbol $-$) of a given amount, which is everything that is not part of that amount. Thus, we have $a + (-a) = 1^A$ and $a * (-a) = 0^A$. The complement of a complement is the amount itself: $-(-a) = a$ (Amount involution). Finally, we have two laws of De Morgan: $-(a + b) = -a * -b$ and $-(a * b) = -a + -b$. It follows by construction that the empty amount needs to be part of every other amount, which means that there are no "negative" amounts: $(0^A \subseteq a)$ for every $a \in A$. This was used to show that magnitudes are also non-negative. We have subtraction defined as $a \backslash b := a * (-(b * a))$ for all $a, b \in A$. An important norm that prevents degenerate amount lattices is that their parts are always *non-linearly ordered*. This means there are always some parts that are not part of each other. Thus, it has the non-totality axiom: $\bigvee_{a,b \in A} \neg(a \subseteq b)$.[22] More details can be found in [48].

[22] In consequence, amount lattices consist of at least two different amounts on the same hierarchical level.