A new concept of robustness

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Abstract. In this paper a new concept of robustness is introduced and the corresponding optimization problem is stated. This new concept is applied to transportation network designs in which the set of scenarios arising from the uncertainty of the parameters follows a probability distribution. The p-robustness concept is aimed to problems where the feasibility of the solutions is not affected by the uncertainty of the parameters. In order to compare the solution with those of other concepts of robustness already known, some computational experiments with real data are included.

1 Introduction

Transportation network design is based on the estimation of the future utilization of the system. Furthermore, the characteristics of the network to be designed also depend on the expected number of trips. Thus a railway for high-speed trains will be constructed if the forecasted patronage is high; otherwise, a more conventional railway will be built. Usually, the estimation of the future demand is based on the current mobility patterns for which the new infrastructure does not exist yet. Therefore, data obtained by samples or some analytical models and gathered in the origin-destination matrix are uncertain. This leads to mathematical programs with uncertain coefficients. Traditionally, this kind of models have been addressed by stochastic programming techniques (Rockafellar and Wets [9]). A classical and different approach is that of the sensitivity analysis, where the sensitivity of the solution regarding the nominal value of the parameters is evaluated.

In the past decade Robust Optimization was introduced. Those models for which small changes of the input data lead to small changes of the solution are called robust counterparts. Different models and techniques have been recently introduced (Ben-Tal and Nemirovski [1], [3]); Bertsimas and Sim [5], [6]). Most of these works have focused on the non feasibility of the solutions and assume that all the scenarios have the same probability. However, there are many problems in which uncertainty does not affect the feasibility of the solutions but their value.

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The concept proposed in this paper is aimed to these cases and is insensitive to outlier scenarios.

The paper is structured as follows. In Section 2 we introduce our new robustness concept. Section 3 presents algorithms to find networks satisfying such new conditions. In Section 4 we show the main results obtained after our computational experience. The paper finishes with some conclusions.

2 A new robustness concept

Traffic network design problems, see [4], in which the parameters and/or the topology of the network are to be determined, are examples of network design problems tackled in this work. Classical formulations assume fixed values in the parameters of the model. In this work we allow some of them to be uncertain, for instance the origin-destination matrix. In this work, we consider that the network desing problem can be formulated as

$$\begin{aligned} \text{maximize} \quad & Z = U(N, \theta), \\ \text{subject to: } & N \in \mathcal{N} \end{aligned} \qquad \text{[NDP]}$$

In the rest of the paper we will consider that each feasible network $N \in \mathcal{N}$ has a utility which depends on the random parameter θ, which might be the origin-destination matrix, the budget,... Let $U(N, \theta)$ denote such utility function.

Since function U depends on the random variable θ, we can state that U itself is also a random variable. Therefore we cannot guarantee that a network is better (meaning that it has greater utility) than another. The concept of p-robustness chooses a network which is better than any other feasible network with probability p.

Definition 1. *Let* $p \in [0, 1]$. $N_i \geqq_p N_j$ *if*

$$\mathsf{Pr}\left\{U(N_i, \theta) \geq U(N_j, \theta)\right\} \geq p.$$

Definition 2. $N^* \in \mathcal{N}$ *is* **p-robust with respect to θ iff:**

$$N^* \geqq_p N \qquad \forall\ N \in \mathcal{N} \tag{1}$$

The concept of p-robustness generalizes the classical optimization problems in network design, in which parameters are assumed to be known. In such cases θ only takes value $\hat{\theta}$ with probability 1 and the probability of a network being better than another is zero or one. Therefore a network is $p > 0$ robust if

$$U(N^*, \hat{\theta}) \geq U(N, \hat{\theta}) \qquad \forall\ N \in \mathcal{N}$$

which is equivalent to the concept of global optimum in a network design problem.

Some considerations on this concept of robustness must be underlined:

1. The concept of p-robustness is not affected by outliers in the parameter θ.

2. The definition of p-robustness is not given from a linear programming problem.
3. Classical design criteria are used to define the utility function and, therefore, the concept of being better.

In order to illustrate the concept of p-robustness we will make use of this example. In Figure 1, the problem of locating a highway under three possible scenarios is considered: S_1, S_2 and S_3. There are four possible locations, $N_1, \dots, N_4$. Three of those locations fit one possible scenario and location N_4 try to satisfy several possible scenarios. That one can be considered as a robust solution, a priori. In this problem the unknown parameter θ represents the demand and is considered a random variable which can take values $\Omega = \{S_1, S_2, S_3\}$ with probability $Pr(S_1) = 0.2$, $Pr(S_2) = Pr(S_3) = 0.4$.

Table 1 reflects the values $U(N_i, S_j)$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. Note that $\theta = S_1$ corresponds with an outlier value, that is, a situation in which the transportation demand is unusually high, and makes the utility of some possible locations to be very high as well.

	S1 (0.2)	S2 (0.4)	S3 (0.4)
N1	10	0	0
N2	1	3	0
N3	0.5	0	3
N4	0.7	1.5	1.5

Fig. 1. Example

This problem has the structure of a decision problem, and we refer to all possible values of θ as *scenarios*.

We consider the following decision criteria:

C1: *Maximizing the expectation.* This criterion, which appears in stochastic mathematical programming, is strongly influenced by outliers, since it chooses the network N_1 with the highest mathematical expectation (because of the outlier) despite of the fact that N_1 has utility 0 with probability 0.8.

C2: *Absolute robustness.* A network N_a is said to be absolute robust if it satisfies:

$$\min_{S_j \in \Omega} U(N_a, S_j) = \max_{N_i \in \mathcal{N}} \min_{S_j \in \Omega} U(N_i, S_j).$$

In this criterion one implicitly assume that that all scenarios are equiprobable. In this example we could have divided scenarios S_2 and S_3 into two other scenarios each, having this way four scenarios with probability 0.2. This is a conservative criterion and chooses the only network having a positive utility in any possible scenario: N_4.

C3: *Robust deviation*: A network N_d is said to satisfy the robust deviation criterion if:

$$\max_{S_j \in \Omega} \left[U(N_j^*, S_j) - U(N_d, S_j)\right] = \min_{N_i \in \mathcal{N}} \max_{S_j \in \Omega} \left[U(N_j^*, S_j) - U(N_i, S_j)\right],$$

where N_j^* is the best network for scenario S_j (in Figure 2 the cells of (N_j^*, S_j) are emphasized by grey circles). The i^{th} component of column $C3$ shows the value $\max_{S_j \in \Omega} \left[U(N_i, S_j) - U(N_j^*, S_j)\right]$. One can observe that the minimum component is achieved in $N_d = N_1$. This criterion is affected by outliers, since scenario S_1 is essential in the final decision.

C4: *Bertsimas-Sim robustness*. This criterion calculates the optimum of the problem so that constraints are satisfied with certain probability, having this way the following problem:

$$\begin{aligned} &\max Y \\ &\text{s.t.: } \Pr\{Y \le U(N, \theta)\} \ge \delta \\ &\quad N \in \mathcal{N} \end{aligned} \tag{2}$$

If in this example we consider the value $\delta = 0.5$, column $C4$ shows the maximum value of the utilities guaranteed with a minimum probability of $\delta = 0.5$. This criterion is robust to outliers but is conservative with respect to the value of the mathematical expectation.

C5: *$p-$robustness*. Applying to this example the value $p = 0.5$ one obtains that the network N_2 is p-robust, that is, one has that

$$N_2 \geqq_p N_1,\ N_2 \geqq_p N_3,\ N_2 \geqq_p N_4.$$

Note that this criterion is robust with respect to outliers and it has a mathematical expectation greater than criterion $C4$.

	S1 (0.2)	S2 (0.4)	S3 (0.4)	C1	C2	C3	C4	C5
N1	10	0	0	2	0	3	0	
N2	1	3	0	1.4	0	9	1	X
N3	0.5	0	3	1.3	0	9.5	0.5	
N4	0.7	1.5	1.5	1.3	0.7	9.3	1.5	

Fig. 2. Criteria

2.1 p^*-robustness

The design problems we have proposed in this work may not have p-robust solutions for certain values of p, which naturally lies in the range $(0,1]$. For instance, a 1-robust solution would be that one which is optimal in all possible values of θ which is, in general, not possible. The concept of p^*-robustness is introduced so as to indicate the maximum value of p for which one can find p-robust solutions, denoted from now on by p^*. Note that for $p \in (0, p^*]$ one can always find p-robust solutions.

Definition 3. *Given $p \in [0,1]$ we define*

$$NDP(p) = \{N \ \in \mathcal{N} \,/\, N \ \textit{is a } p-\textit{robust solution to NDP}\}$$

We will denote

$$p^* = \sup\{p \in [0,1] \,/\, NDP(p) \neq \{\emptyset\}\}$$

Now some considerations on this new concept.

1. $NDP(0) = \mathcal{N}$ and therefore the concept of p^*-robustez is well defined.
2. Note that if $0 \leq p < q \leq 1$ then $NDP(q) \subseteq NDP(p)$. This way one has that for all $p \in [0, p^*)$ there exist p-robust solutions to NDP.
3. An interesting decision criterion is to choose as final solution to NDP the network maximizing the mathematical expectation of the utility among the p-robustness networks. Taking the maximum value p^* could make the set $NDP(p^*)$ too small.

3 Solution algorithms

In this section we propose algorithms, both heuristic and exact, which find a solution to our problem, provided such solution exists.

The first (heuristic) algorithm we propose reduces the set of all feasible networks to a set of networks which are not worse than any other network in all possible scenarios to, later on, find the p-robust networks, if any. Find below a pseudocode of such algorithm:

Notice that the algorithm above does not necessarily return p-robust solutions, for two reasons:

1. Not all possible networks have to be generated, only until a stop criterion is satisfied.
2. In step 4, we find p-robust solutions in the set $\widehat{\mathcal{N}}$ which, as we mentioned before, does not have to be the whole set of feasible networks.

An interesting question that should be addressed is which values of p are considered to be good. Notice that having a 0.3-robust solution might not be desirable. Therefore we now provide an algorithm which finds the maximum p for which there are p-robust solutions of complexity $O(n^2)$ on the number of feasible

Table 1. Heuristic algorithm

0. (Initialization) Let $\{\theta_1, \ldots, \theta_m\}$ a ramdom sample of parameter θ.
   Set $\widehat{\mathcal{N}} = \{\emptyset\}$
1. (Generating solutions) Find N' a feasible solution. $\hat{\mathcal{N}} = \hat{\mathcal{N}} \cup \{N'\}$
2. (Update the solution set) For each $\widehat{N} \in \widehat{\mathcal{N}}$ do
   If $N' \geqq_1 \widehat{N}$ and $N' \neq \widehat{N}$ set $\hat{\mathcal{N}} = \hat{\mathcal{N}} - \widehat{N}$
   If $\widehat{N} \geqq_1 N'$ and $N' \neq \widehat{N}$ set $\hat{\mathcal{N}} = \hat{\mathcal{N}} - N'$. Go to 3.
3. (Stop criterion) If a stop criterion is satisfied go to 4, otherwise go to 1.
4. (Find p-robust solutions in $\widehat{\mathcal{N}}$)

Table 2. Exact algorithm

(Input data)
$\widehat{\mathcal{N}} = \{N_1, \ldots, N_q\}, \{\theta_1, \ldots, \theta_m\}$ possible scenarios,
$P \in \mathbf{R}^{n \times n},\ p_{ij} = 1\ \forall\ i, j$
for $k = 1, \ldots, m$ do
  for $i = 1, \ldots, q$ do
    for $j = 1, \ldots, q$ do
      if $U(N_i, \theta_k) < U(N_j, \theta_k)$ then
        $p_{ij} = p_{ij} - Pr(\theta_k)$
      end if
    end do
  end do
end do

networks, assuming the number of possible scenarios fixed. Such algorithm is exact, provided that the set of all feasible networks is known.

Notice that p_{ij} is the probability of network N_i being better than network N_j. Therefore, network N_i is better than all other networks with probability $p_{i\bullet} = \min_j p_{ij}$. As a conclusion, $p^* = \max_i p_{i\bullet}$ gives us the maximum p for which there are p-robust solutions, the set $\{N_i : p_{i\bullet} = p^*\}$ consisting of all p^*-robust networks. This algorithm can be inserted as the step 4 of the previously introduced heuristic algorithm, all networks N_i such that $p_{i\bullet} \geq p$ being p-robust solutions, if any.

4 Computational experiments

In this section we show the computational results obtained from three different situations.

4.1 $p-$robust location of a highway: uncertainty in the origin-destination matrix

In this section we perform tests in the model of [8]. In such model it is assumed that one can travel directly from the origin to the destination at a speed v or, alternatively, using the highway. It is considered that one can access the highway at a speed of v, and once one is travelling on the highway the speed is $w > v$. All users will choose to take the highway if and only if their travelling times are decreased.

This model has been applied to the Spanish region of Castilla La Mancha, which has 918 councils. Since this number is too high, we have considered three situations. Problem 1 consists of all cities with more than 50000 inhabitants, Problem 2 studies all cities with more than 5000 inhabitants and in Problem 3 we only consider cities with more than 1000 inhabitants. In Table 7 it is shown the number of demand pairs analyzed, the percentage of the demand analyzed over all 918 councils and the number of networks considered. In the first step of the heuristic algorithm previously proposed, we generated highways in a uniform way over the feasible space, without applying any intelligent strategy.

Table 3. Problem definitions

Problem	Cities	# pairs $o-d$	% demand
Problem 1	6	15	27.0
Problem 2	67	2211	65.2
Problem 3	290	41905	91.3

Our uncertain parameter is the origin-destination matrix. In this computational experience we estimated such matrix following those procedures:

S1: *Surveys.* The INE (Spanish Statistics Agency) made a poll in 2000 where it was asked in which city citizens lived and to which city they would go to study or work.

S2: *Equiprobability model.* Trips are done from one site to the others with a probability which depends on their size.

S3: *Gravitational model.* The number of travels from one origin to a destination is proportional to the product of their populations and inversely proportional to their distance squared.

S4: *Exponential model I.* This model is similar to the gravitational model but using as deterrence function $\exp(-\beta d)$, β being a parameter which can be estimated from the average distance between cities and d being the distance between cities. In matrix $S4$ we have taken β considering that the average travel distance is around 90 kilometers.

S5: *Exponential model II.* In this case the chosen parameter β makes the average travel distance be around 150 kilometers.

The total demand in all scenarios has been forced to be the same so that matrices from **S2** to **S5** have the same travel pattern. That is, the attracted-generated demand in each city is the same in the four cases considered, being different in their spacial distribution.

We have used the following criteria:

Ci: Best location with respect to the scenario (matrix) **Si**, $i = 1, \ldots, 5$
C6: Best highway for the mathematical expectation.
C7: Regret Optimization.
C8,C9,C10,C11: Robust Optimization of Bertsimas-Sim) $p = 0.8, 0.6, 0.4, 0.2$, respectively.
C12: Minimum deviation.
C13: $0.5-$robustness.

Table 4 shows the highways chosen for Problem 1 for the considered criteria. The values in the cells are the total travelling time in the network. Since the goal is to minimize such total time, we must maximize the utility $U(N_i, S_j)$. Figure 3 shows the location of the highways and their access points.

Table 4. Solution to Problem 1

Ni	**S1**	**S2**	**S3**	**S4**	**S5**	Criterion
N1	12062839702	28883313021	16154267014	24676276092	18692831932	C13
N2	11988012356	29194839746	16069677561	24845510865	18828968001	C1,C3, C10,C11
N3	12298245628	29100143634	16229629691	24672528156	18656209259	C5,C9
N4	12425246717	28211062229	16308102751	24276679346	18807462054	C2,C4,C6,C7,C8, C12

The same criterion is used in tables 5 and 6 and their corresponding figures 4 y 5.

Table 5. Solution to Problema 2

Ni	**S1**	**S2**	**S3**	**S4**	**S5**	Criterion
N1	16053503414	53019291022	10761024144	45302154763	34988292110	C2,C3-C9,C11-C13
N2	16038156696	53074579244	10762657906	45335914217	35001117559	C1,C10

As a conclusion, in Problem 1 it is observed that our robustness concept and other concepts introduced in the literature choose different corridors. In problems 2 and 3, all criteria locate the highway on the same corridor, different criteria having only small differences between them. Our criteria coincides with the maximization of the mathematical expectation.

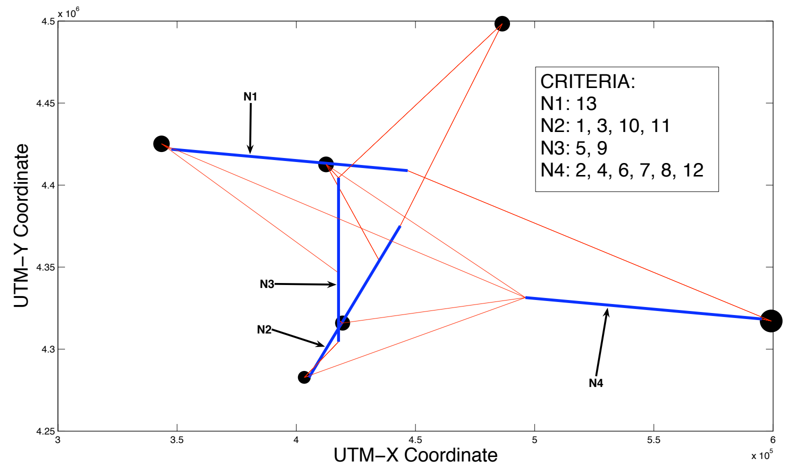

Fig. 3. Problem 1

Table 6. Solution to Problem 3

Ni	**S1**	**S2**	**S3**	**S4**	**S5**	Criterio
N1	18438919997	74925928034	15401578244	64022301823	49479085419	C2,C4,C7,C8
N2	18420801443	74939621002	15394489341	64023930579	49468576625	C3,C5,C6,C9,C11,C12,C13
N3	18399175078	74959084514	15394912582	64032727329	49470962232	C1,C10

4.2 Fitting to a segment

A well studied problem in practice is that of fitting a straight line $y = bx+a$ to a bunch of points. This problem has been modelled as an optimization in which the set of points is known, $\{(x_i, y_i)\}$ desde $i = 1\ldots, m$. The most common criterion is the least square method, which is affected by outliers and has motivated the study of robust estimators of a and b with respect to outliers.

In this section we illustrate the application of our concept of p-robustness to this optimization problem. Note that each solution (straight line) is feasible, so it is not appropriate its use. Nevertheless, our goal is to estimate a straight line which allows us to predict the value of y of a future (unknown) x. In this procedure each observation (x_i, y_i) represents a realization (a city) that can be done in the future. Therefore each point (x_i, y_i) defines a future scenario S_i, Ω being the set of available points. The utility of a straight line $N_j \equiv y = b_j x + a_j$ in scenario $S_i = (x_i, y_i)$ is the negative value of the error:

$$U(N_j, S_i) = -\left|y_i - b_j x_i + a_j\right|. \tag{3}$$

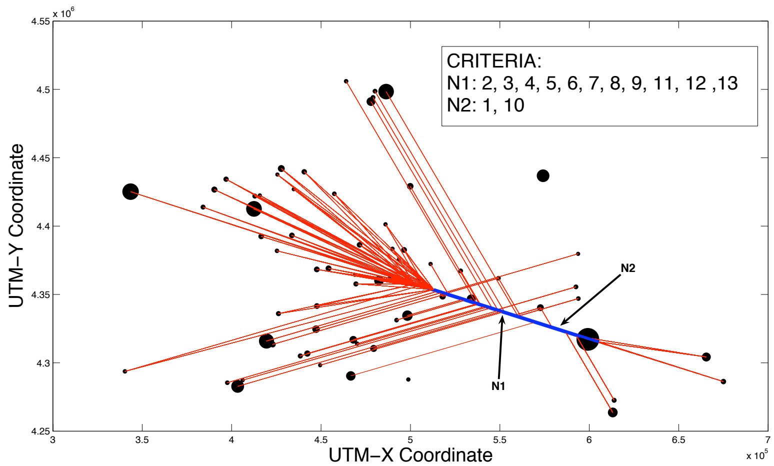

Fig. 4. Problem 2

In this problem we have taken $p = 0.5$, that is, $N_l \geqq_p N_j$ if the absolute error of N_l is lower than that of N_j in at least $\frac{m}{2}$ points (scenarios).

For this situation we have considered that the cities of Problems 2 and 3 define the bunch of points $\{(x_i, y_i)\}$. In such cases we have 67 and 290 cities, respectively. All cities have been considered equally important, that is, they define equiprobable scenarios. In order to maintain the scheme used in previous tests we have represented them according to their size, although it has not been considered in the computational experiments. Figure 6 shows the results of the first least square fit and the 0.5-robustness. That is, in the last set of undominated straight lines, there is no N^* satisfying $N^* \geqq_{0,5} N'$ for every other network N'. This problem has been overcome in two ways:

1. Calculating p^*-robustness in Problem 2 and Problem 3.
2. Calculating 0.5-(nearly)robust solutions. That is, constraints in the definition of 0.5-robustness has been relaxed to: a solution N^* is 0.5-(nearly)robust if $N^* \geqq_{0.5} N'$ for **the highest number** of solutions N'. In Problem 2, a 0.5-(nearly)robust solution has been obtained, which is *better* than 7051 out of 7387 solutions with probability 0.5. For Problem 3 we did find a 0.5-robust solution.

Figure 6 shows the computational results obtained. There are no 0.5-robust solutions for Problem 2. We calculated the p^*-robustness, whose value was $p^* = 0.462$, leading to 11 different p^*-robust solutions. In Problem 3, the value of p^* was 0.5. In this case, the p^*, 0.5 and 0.5-(nearly)robustness coincide. It is worth noting that in general similar solutions were obtained.

Solutions obtained for different methods are similar. In figure 7, the effect of the outliers in the least-squares fit is represented. An outlier has been added and

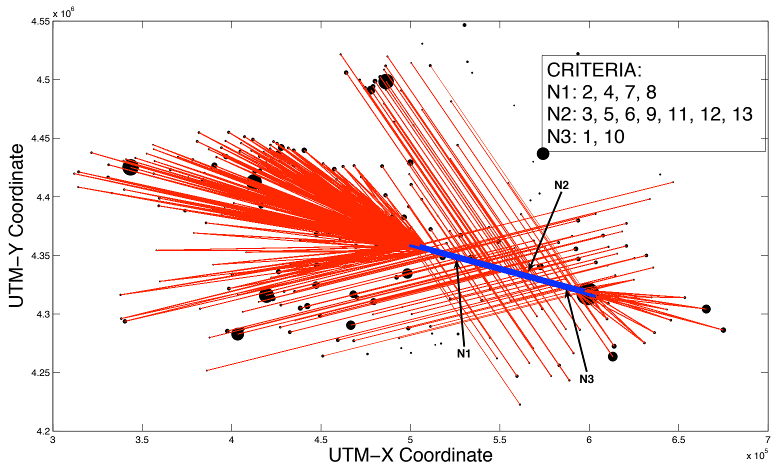

Fig. 5. Problem 3

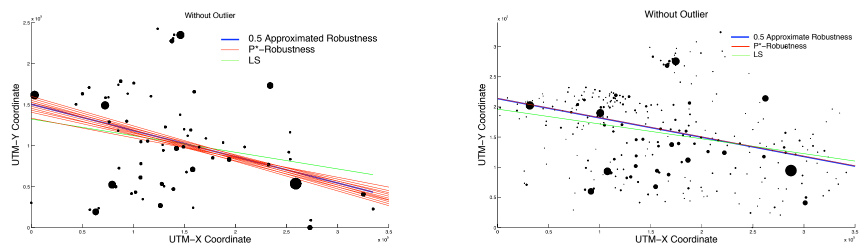

Fig. 6. Solution to the problem of fit to a straight line.

the straight line has been estimated, for five different values of the outlier. All of them have the same component $x = 3.5 * 10^5$, while the y components were $y_1 = 2 * 10^5$, $y_2 = 4 * 10^5$, $y_3 = 6 * 10^5$, $y_4 = 8 * 10^5$, $y_5 = 10 * 10^5$.

The following experiments are meant to investigate the effect of outliers in the p-robust estimation of the regression lines, which is evaluated in Figure 8. For Problem 2 (left hand side graphic), when one adds an outlier the value of p^* changes to $\frac{32}{62}$ and the number of p^*-robust solutions change from 11 to only 2, the two ones closer to the outlier. In Problem 3, the p^*-robust solution is not affected by the outlier. Adding an outlier makes the 0.5-robust solution become (145/291)-robust (145/291 = 0.49), therefore the 0.5-(nearly)robust solution is now a different one.

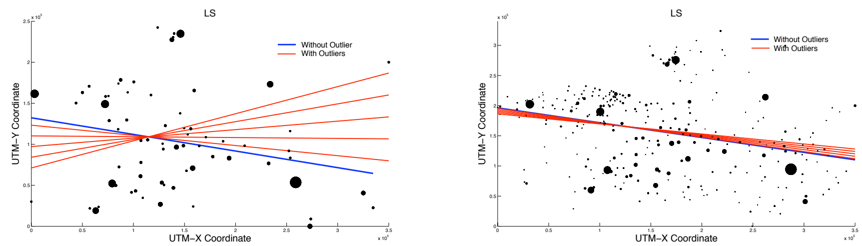

Fig. 7. Effect of the outliers in the solution to the least-square fit problem.

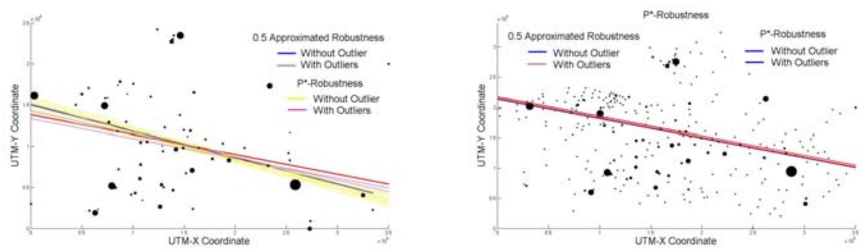

Fig. 8. Effect of the outliers in the solution to the $p-$robustness fit problem.

4.3 Computational considerations

Our goal in this paper was to propose a new robustness concept for a class of network design problems. In further research we will focus on efficient algorithms for its calculation.

In the three different classes of problems presented in sections 4.1 and 4.2, we generated solutions by *sweeping* the feasible set, with the idea of not leaving areas of such feasible set without being explored more than obtaining a good initial solution, because in the definition of p-robustness one has to check all $N \in \mathcal{N}$.

As a note, it is worth underlying that the model developed in Problem 3 in Section 4.1 had a computational time of 5 days. The latter fact made us reduce the number of networks considered for this problem with respect to Section 4.1. This shows the need to develop efficient algorithms, in which we should use selective rules to sweep the feasible set.

A second fact in the complexity of our heuristic algorithm is that the evaluation of the p-robustness requires an effort depending on the number of final solutions considered, which is shown in table 8. One observes that for the regression problem that number is very high, which could cause a computational cost impossible to meet. In this example we only exclude solutions which are dominated in all possible scenarios by any of the previously selected solutions. We will pay special attention to the development of elimination strategies.

Table 7. Number of solutions evaluated in previous sections.

Problem	Model in Section 4.1	Model in Section 4.2
Problem 1	92256	–
Problem 2	93774	10000
Problem 3	95256	10000

Table 8. Number of solutions $\hat{\mathcal{N}}$ kept in the last iteration

Problem	Model in Section 4.1	Model in Section 4.2
Problem 1	43	–
Problem 2	41	7387
Problem 3	47	8707

Conclusions

In this work we have introduced a new robustness concept for network design problems. We show that such new concept gives rise to solutions different from other robustness concepts studied in the literature. We have also proven that, in regression problems, p-robust solutions do not always exist and, therefore, new concepts such as p^*-robustness and p-(nearly)robustness have been introduced.

Algorithms, both heuristic and exact, have been proposed to calculate p-robust and p^*-robust solutions. From our experimental experience we deduce that it is worth investigating new strategies in order to obtain more efficient algorithms.

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References

1. Ben-Tal, A. and Nemirovski, A. Robust solutions of uncertain linear programs. Operations Research Letters 25, 1-13, (1999).

2. Ben-Tal, A., Nomirovski, A.: Robust convex optimization. Mathematics of Operations Research **23** (1998) 769–805
3. Ben-Tal, A. and Nemirovski, A. Robust optimization-methodology and applications. Mathematical Programming, Ser. B 92, 453-480, (2002).
4. Bell, M.G.H., Iidia, Y.: Transportation Network Analysis. Wiley (2002)
5. Bertsimas, D. and Sim M. Robust discrete optimization and networks flows. Mathematical Programming, Ser. B 98, 49-71, (2003).
6. Bertsimas, D. and Sim M. The price of robustness. Operations Research 52, 35-53, (2004).
7. Kouvelis, P., Yu, G.: Robust Discrete Optimization and its Applications. Academic Pubisher (1997)
8. Ohyama, T.: Highway location problem considering demand. In Cruz, M., Lozano, A., Mesa, J., Puerto, J., eds.: tenth International Symposium on Locational Decisions (ISOLDE X). (2005) 125–128
9. Rockafellar, R.T. and Wets, R.J.-B. Scenarios and policy aggregation in optimization under uncertainty. Mathematics of Operations Research 16, 119-147, (1991).