The Second Chvátal Closure Can Yield Better Railway Timetables*

Christian Liebchen and Elmar Swarat

Technische Universität Berlin, Institute of Mathematics,
Straße des 17. Juni 136, D-10623 Berlin, Germany
{liebchen,swarat}@math.tu-berlin.de

Abstract. We investigate the polyhedral structure of the Periodic Event Scheduling Problem (PESP), which is commonly used in periodic railway timetable optimization. This is the first investigation of Chvátal closures and of the Chvátal rank of PESP instances.
In most detail, we first provide a PESP instance on only two events, whose Chvátal rank is *very* large. Second, we identify an instance for which we prove that it is feasible over the first Chvátal closure, and also feasible for another prominent class of known valid inequalities, which we reveal to live in much larger Chvátal closures. In contrast, this instance turns out to be infeasible already over the second Chvátal closure. We obtain the latter result by introducing new valid inequalities for the PESP, the multi-circuit cuts.
In the past, for other classes of valid inequalities for the PESP, it had been observed that these do not have any effect in practical computations. In contrast, the new multi-circuit cuts that we are introducing here indeed show some effect in the computations that we perform on several real-world instances – a positive effect, in most of the cases.

1 Introduction

It has been only recently that combinatorial optimization entered the practice of service design in public transport. The 2005 timetable of Berlin Underground is the first optimized timetable that was put into service [9]. It had been computed with integer programming techniques, namely profiting from several different classes of valid inequalities. Today, also the Dutch railways are operating a timetable that was designed with the help of techniques from combinatorial optimization and constraint programming [7]. Both projects build upon the Periodic Event Scheduling Problem (PESP).

The PESP, in its pure formulation of a feasibility problem, had been introduced by Serafini and Ukovich [18] and it generalizes the vertex coloring problem. In particular, for the two most natural optimization problems that are investigated on top of the PESP, MAXSNP-hardness has been established [8, 9]. In practice, this results in the following typical behavior of MIP solvers on medium to large sized instances. Known valid inequalities are able to close 60–90% of the initial gap between the integer optimum value and the optimum value of the LP relaxation. Still, solving this tightened IP risks to take several hours, if it is solvable at all.

* Supported by the DFG Research Center MATHEON in Berlin.

ATMOS 2008
8th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems
http://drops.dagstuhl.de/opus/volltexte/2008/1580

There are of course much larger transportation networks in practice, which are beyond the computational limits of the methods that were used so far. As a consequence, at present there are several other research groups trying to tackle the periodic railway timetabling problem, and they are sharing the PESP as their model of choice [2, 17, 19]. For instance, Villumsen put the polyhedral approach that was suggested by Lindner [14] into practical computations for the commuter train network of Copenhagen. Unfortunately, he had to make the observation that

> "the chain cuts [14] have no effect on the solution" [19].

This is one motivation for us to have a closer look at the polyhedral structure of the feasible region of PESP instances. We do so by following the methodology that has been suggested recently by Fischetti and Lodi [6] for optimizing over the first Chvátal closure. Notice that one of the first instances to which they applied their method was the "hard MIPLIB instance `timtab1`", which is in fact a PESP model [10].

As a motivation, we first generalize an infeasible PESP instance – which is due to Lindner [14] – to a family of instances that are defined on wheel graphs. In Section 6 we will prove that these instances are feasible over the first Chvátal closure. Still worse, even the change-cycle inequalities that have been introduced by Nachtigall [15], of which in Section 4 we prove that, in general, they lie in much larger Chvátal closures, are not suited to certify infeasibility. Nevertheless, the techniques of Fischetti and Lodi suggested that these particular instances might be infeasible already over the *second* Chvátal closure. Indeed, by exploiting problem-specific insight, in the second Chvátal closure we identify general new valid inequalities for the PESP (Section 5) by which we prove that these particular instances are infeasible. We call these new inequalities the *multi-circuit cuts*.

In Section 7 we add multi-circuit cuts to the IP formulations of several timetabling instances that we took from practice. Although we have to admit that the results are not fully striking, on many instances we observe a perceptible speed-up in the solution time. In turn, on more complex instances, for which up to now no optimal solution has been found, our new cuts from the second Chvátal closure might indeed yield better railway timetables.

2 An IP for PESP

Initially, the Periodic Event Scheduling Problem (PESP, [18]) has been stated as a pure feasibility problem. We are given a directed graph $D = (V, A)$, which may feature (anti-) parallel arcs. For each arc a, there are defined some lower bound ℓ_a and some upper bound u_a. The PESP then asks whether for the given fixed period time T, the instance admits a *(periodically) feasible node potential* $\pi \in [0, T)^V$, i.e.,

$$(\pi_j - \pi_i - \ell_a) \bmod T \leq u_a - \ell_a, \ \forall a = (i, j) \in A. \tag{1}$$

In a railway timetabling context, the value T is the period time of the railway system, e.g., 60 minutes. A node i represents an arrival or departure of some specific directed line in the network, and we must assign a time value π_i to this event. For instance, in the current timetable, the direct ICE trains from Berlin to Karlsruhe leave Berlin main station 33 minutes past the hour. Finally, in the constraint parameters ℓ and u one may encode lower and upper bounds on time durations to ensure safety requirements, transfer quality requirements, as well as many other features [11].

In a mixed-integer linear programming formulation, the modulo-operator in (1) is resolved by introducing integer variables p_a for the arcs, which we denote *periodical offsets*. Furthermore, we penalize any slack on the lower bounds ℓ_a in a linear objective function,

$$\begin{array}{lll} \min & \sum_{a=(i,j)\in A} w_a(\pi_j - \pi_i + Tp_a) & \\ \text{s.t.} & \pi_j - \pi_i + Tp_a \geq \ell_a, & \forall a = (i,j) \in A \\ & \pi_j - \pi_i + Tp_a \leq u_a, & \forall a = (i,j) \in A \\ & \pi_i \in [0, T), & \forall i \in V \\ & p_a \in \mathbb{Z}, & \forall a \in A. \end{array} \tag{2}$$

Other formulations for this problem had been stated in terms of so-called tension variables $y_a = \pi_j - \pi_i$, or even *periodic tension variables* $x_a = \pi_j - \pi_i + Tp_a$, see e.g. [4, 11]. Observe that we always have $\ell_a \leq x_a$. In particular, the resulting MIPs, in which we can make the node potential variables π redundant, already perform considerably better [13]. Yet, their performance can even be enhanced—and it has to!—by adding valid inequalities. In this spirit, in the remainder of the paper we illustrate the limits of known valid inequalities, and introduce new classes of valid inequalities, which let us go beyond.

In Section 4, when we provide a relatively large lower bound on the Chvátal rank of PESP polyhedra, we will also find it most convenient to make use of the periodic tension variables x_a. Throughout the other parts of this article, however, we stay with (2). This is because we consider this formulation being more accessible, in particular for the newcomer, and it is a straightforward computation to adapt the classes of valid inequalities that we identify there to other equivalent mixed-integer programming formulations of the PESP.

The following lemma reveals that we are in fact dealing with pure integer programs.

Lemma 1 ([16]). *If ℓ, u, and T are integers, then in* (2) *w.l.o.g. we may replace $\pi_i \in [0, T)$ with $\pi_i \in \{0, \ldots, T-1\}$.*

Proof. Consider an optimum solution (π^*, p^*) of (2). Now, fix the vector p^*. The resulting problem is a linear optimization problem with twice the node-arc incidence matrix of the constraint graph D as constraint matrix, which is thus totally unimodular. Since the right-hand side is integer, the LP has some integer optimum solution π°, and (π°, p^*) is feasible for (2) and not worse than the optimum solution (π^*, p^*). □

Note that the periodical offset variables p_a are either binary, or may in addition take the value two, provided that $u_a > \left\lceil \frac{\ell_a + \varepsilon}{T} \right\rceil T$. Nevertheless, w.l.o.g. we forget about any explicit bound on any of the variables in (2), and just keep their integrality requirements.

3 Chvátal Closures

Let M be an $m \times n$ matrix and consider the general rational polyhedron

$$P = \{x \mid Mx \leq b\}.$$

The *(first) Chvátal closure* P' of P is characterized by

$$P' = \{x \mid \lambda^\mathsf{T} Mx \leq \lfloor \lambda^\mathsf{T} b \rfloor, \text{ for all } \lambda \geq \mathbf{0} \text{ with } \lambda^\mathsf{T} M \text{ integer}\}.$$

Also, set $P^{(0)} := P$ and recursively define $P^{(i+1)} = (P^{(i)})'$. In integer programming, we are interested in the *integer hull* P_I of P,

$$P_I := \text{conv}(\{x \in \mathbb{Z}^n \mid Mx \leq b\}).$$

The following is a key theorem in integer programming.

Theorem 1 ([3]). *For each rational polytope P there exists some integer t such that $P^{(t)} = P_I$.*

Note that in the sequel, we will switch back to $n = |V|$ and $m = |A|$, of course.

Now, denote by B the node-arc incidence matrix of a PESP constraint graph D. Then, consider the matrix

$$M := \begin{bmatrix} -B^\mathsf{T} & -T \cdot I_m \\ B^\mathsf{T} & T \cdot I_m \end{bmatrix}, \tag{3}$$

where I_m refers to the m-dimensional unit matrix. Together with the right-hand side vector

$$b := \begin{bmatrix} -\ell \\ u \end{bmatrix}, \tag{4}$$

the convex hull of the feasible solutions of (2) is nothing but P_I.

Also for the PESP, several specific studies of its polyhedral structure have been conducted [14–16]. In the sequel, we summarize some of their results and relate them to the general concept of Chvátal closures. To this end, define an *oriented circuit* $C = C^+ \dot\cup C^-$ as a subset of the arcs of D such that reorienting the elements of C^- would result in a directed circuit. The arcs in C^+ are called the *forward arcs*, and the arcs in C^- are the *backward arcs*. In particular, we distinguish the two oriented circuits that map onto the same circuit in the underlying undirected graph.

The following valid inequalities for PESP have been identified by Odijk [16].

Theorem 2 ([16]). *Let D be the constraint graph of a PESP instance and consider some oriented circuit C in D. Then the* cycle inequality

$$\sum_{a \in C^+} p_a - \sum_{a \in C^-} p_a \leq \left\lfloor \sum_{a \in C^+} \frac{u_a}{T} - \sum_{a \in C^-} \frac{\ell_a}{T} \right\rfloor \tag{5}$$

is valid for (2). *More precisely, the cycle inequalities show up as early as in the first Chvátal closure $P^{(1)}$ of the LP-relaxation P of a PESP-polytope P_I.*

Proof. We combine these inequalities from the ones in (2). To this end, for each forward arc in C, multiply the less-than inequality of its upper bound u_a with $\frac{1}{T}$. Similarly, for each backward arc in C, multiply the greater-than inequality of its lower bound ℓ_a with $-\frac{1}{T}$, which translates into a positive coefficient in the vector λ. It is a simple observation that the node variables π all cancel out in a telescope sum. Finally, we round down the right-hand side and obtain (5). □

Both the potential strength of the cycle inequalities and the key role of the periodical offset variables p are reflected by the following theorem.

Theorem 3 ([16]). *An instance of PESP is feasible if and only if there exists an* integer *vector* p *such that* p *satisfies* all *the cycle inequalities.*

This is why we are seeking stronger valid inequalities in terms of the periodical offset variables p. In the next theorem we show that doing so we need to investigate the second Chvátal closure. This will be the main topic from Section 5 on. There, we start by highlighting that there exist some oriented circuits C in which the upper bound in (5) can even be decreased, still being valid for P_I, of course. In fact, Lindner [14] proved that the coefficients of *any* valid inequality for the PESP that only features periodical offset variables p, have to constitute a circulation in the constraint graph. Let us already mention that in Section 4 we provide an explicit proof that the Chvátal rank of a PESP instance may be at least $\frac{T}{2}$.

Denote by Q the polyhedron that is defined by taking all the inequalities from $P^{(1)}$ that do not feature any of the node variables π. Observe that formally the support of these inequalities may differ from circuits, as they are required in (5).

Theorem 4. *The cycle inequalities* (5) *constitute the complete description of* Q*, i.e.,*

$$Q = \{p \mid p \text{ satisfies all cycle inequalities } (5)\}.$$

Proof (idea). Basically, the proof makes use of the decomposition of an integer circulation into oriented circuits. However, due to space limitations we have to omit further details here. □

Notice that we are aware of instances on which Q does *not* equal the projection of $P^{(1)}$ onto the periodical offset variables p. In particular, there the p-part of some reversed-arc cut, which is defined in the next section, is necessary to certify the emptiness of $P^{(1)}$, while $Q \neq \emptyset$.

4 A Lower Bound on the Chvátal Rank of PESP

In this section we present the change-cycle inequalities, which were introduced by Nachtigall [15]. We provide a PESP-instance on two vertices, on which the change-cycle inequalities appear first in the $\frac{T}{2}$-th Chvátal closure, where T denotes the period time. To the best of our knowledge, this is the strongest explicit lower bound on the Chvátal rank of PESP. Unfortunately, due to space limitations we have to omit details of the proof here.

Before formulating the change-cycle inequalities, we introduce a few notation. Let C be some oriented circuit in the constraint graph of a PESP-instance. We sum the periodic tension values of the forward arcs in x^+ and the periodic tension values of the backward arcs in x^-, i.e.,

$$x^+ := \sum_{a \in C^+} x_a \qquad \text{and} \qquad x^- := \sum_{a \in C^-} x_a.$$

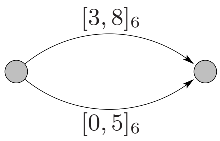

Fig. 1. A feasible PESP instance on the 2-circuit C_2 with $T = 6$

Analogously, we define

$$\ell^+ := \sum_{a \in C^+} \ell_a, \quad \text{and} \quad \ell^- := \sum_{a \in C^-} \ell_a.$$

Last, we define the slope μ and the ordinate intercept ν of the line that induces the change-cycle inequality as

$$\mu := 1 - \frac{T}{\ell^- - \ell^+ + T\tilde{z}} \qquad \text{and} \qquad \nu := (1-\mu)\ell^+ - T(\tilde{z}-1), \tag{6}$$

where $\tilde{z} := \left\lceil \frac{1}{T}(\ell^+ - \ell^-) \right\rceil$.

Theorem 5 ([15]). *The following* change-cycle inequalities

$$x^- \geq \mu x^+ + \nu \tag{7}$$

are valid for feasible instances of (2)*.*

Notice that a similar inequality, which involves the upper bounds u_a of the arcs, is valid, too. Moreover, it had been observed in [12, Fig. 5.1] that change-cycle inequalities (7) are in a sense complementary to cycle inequalities (5).

In the remainder of this section we provide a two vertices instance of PESP, of which we prove that its Chvátal rank is $\frac{T}{2}$. In particular, the change-cycle inequality (7) of this instance does only appear in the $\frac{T}{2}$-th Chvátal closure. To this end, let T be a fixed period time and consider the following PESP-instance on two vertices: Let a_1 and a_2 be two parallel arcs, where $\ell_{a_1} = \frac{T}{2}$, $u_{a_1} = \left(\frac{3}{2}T\right) - 1$, $\ell_{a_2} = 0$, and $u_{a_2} = T - 1$. See Figure 1 for the example that corresponds to the period time $T = 6$.

In particular, in terms of periodic tension variables x_a we are dealing with the following polytope

$$P = \{(x_{a_1}, x_{a_2}, z)^\mathsf{T} \mid \frac{T}{2} \leq x_{a_1} \leq \left(\frac{3}{2}T\right) - 1,\ 0 \leq x_{a_2} \leq T - 1,\ x_{a_1} - x_{a_2} = Tz\}, \tag{8}$$

where the variable z is in fact a shorthand for $p_{a_1} - p_{a_2}$. Observe that P_I corresponds to the convex hull of this PESP instance's solutions.

Proposition 1. *Consider the point* $Q_i = (\frac{T}{2} + i \cdot \frac{1}{2},\ i \cdot \frac{1}{2},\ \frac{1}{2})$*. Then* $Q_i \in P^{(i)} \setminus P^{(i-1)}$*, for all* $i \in \{1, \ldots, \frac{T}{2}\}$*. Moreover, for* $i < \frac{T}{2}$ *the points* Q_i *violate the change-cycle inequality* (7)*. In particular, the change-cycle inequality* (7) *cannot be generated prior to the* $\frac{T}{2}$*-th Chvátal closure.*

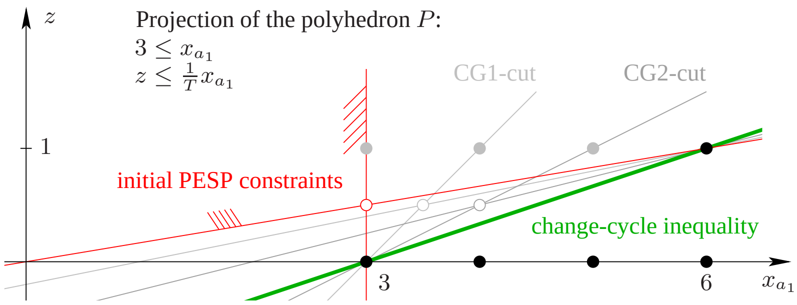

Fig. 2. A visualization of a change-cycle inequality for PESP, and its relation to Chvátal closures, here $T = 6$

Proof (sketch). In this context, the situation can be inspected best by exploiting the redundancy of the equation $x_{a_1} - x_{a_2} = Tz$ to only consider the projection into the $x_{a_1}z$-plane. In this space, the relevant inequalities of P are the initial inequality $x_{a_1} \geq \frac{T}{2}$ as well as $z \leq \frac{1}{T}x_{a_1}$, which is obtained by plugging $0 \leq x_{a_2}$ into $x_{a_1} - x_{a_2} = Tz$. Observe that the point $(x_{a_1}, z)^\intercal = (\frac{T}{2}, 0)^\intercal$ makes the former inequality tight, while $(x_{a_1}, z)^\intercal = (T, 1)^\intercal$ makes the latter inequality tight. In Figure 2, the corresponding half-spaces are drawn in red, while our ultimate goal, the change-cycle inequality (7), is drawn in green.

Then, here we can only summarize that by going from one Chvátal closure $P^{(i-1)}$ to the subsequent one $P^{(i)}$, both these inequalities are "rotated" around the points $(\frac{T}{2}, 0)^\intercal$ and $(T, 1)^\intercal$, respectively, such that the point Q_i become tight. □

Corollary 1. *The Chvátal rank of PESP is at least* $\frac{T}{2}$.

5 New Valid Inequalities for the PESP

The next section will reveal the need for new valid inequalities for the PESP: There, we present an instance for which all cycle inequalities (5) *and* change-cycle inequalities (7) are valid, although the instance is infeasible. Also, in practical computations adding these two types of valid inequalities we typically close no more than 60-90% of the initial gap between the IP optimum and its LP relaxation, and the resulting refined IPs still risk to be hard to solve. This is why here, we identify two new types of valid inequalities for the PESP polyhedron.

The first one is defined exclusively on the periodical offset variables p. By Theorem 4 we know that these cannot stem from the first Chvátal closure of the feasible region P of the LP relaxation of (2). In more detail, we specify situations in which we may decrease the right-hand side of the cycle inequalities (5). And with these new inequalities, we can easily prove the infeasibility of the instance that we discuss in depth in the next Section 6. In Section 7, we complement this analysis with promising empirical computations.

The second type of valid inequalities lives in the first Chvátal closure, and hence may now contain both types of variables, π and p. Unfortunately, due to space limitations we cannot illustrate in-depth their respective contribution here.

5.1 Multi-circuit Cuts

We start by presenting new PESP cuts from the second Chvátal closure $P^{(2)}$ of P.

Theorem 6. *Let $C_0, \ldots, C_k$ be oriented circuits with incidence vectors γ_i. Let $\lambda_i \in (0,1)$ such that $\gamma_0 = \lambda_1\gamma_1 + \cdots + \lambda_k\gamma_k$. Finally, let β_i be the right-hand sides in the cycle inequalities* (5) *of $C_1, \ldots, C_k$. Then*

$$\gamma_0^T p \leq \lfloor \lambda_1\beta_1 + ... + \lambda_k\beta_k \rfloor \tag{9}$$

is a valid inequality for $P^{(2)}$.

The proof follows immediately from Theorem 2 together with the definition of the second Chvátal closure. For some oriented circuits we may not be lucky at all, and (9) is the same as (5). However, for other cycles, the right-hand side in (9) may be much smaller than the one in (5), see Remark 1 on Page 14 for one such example. Since these cuts are obtained by representing an oriented circuit as the fractional sum of multiple other circuits, we refer to (9) as *multi-circuit cuts*.

Despite the fact that these inequalities are somehow straightforward, they are indeed useful. We will illustrate this in a detailed example in the next section, where in particular we find that

$$P^{(1)} \neq \emptyset \qquad \text{but} \qquad P^{(2)} = \emptyset.$$

5.2 Reversed-Arc Cuts

Here, we introduce one further new class of valid inequalities for the PESP, which stems from the first Chvátal closure. These inequalities were inspired by the results that we obtained by applying the methods of Fischetti and Lodi [6].

Theorem 7. *Let C be an oriented circuit, and take some backward arc $a_0 = (i,j) \in C^-$. The following inequality is valid for $P^{(1)}$*

$$\begin{aligned}\pi_j - \pi_i + (T-1)p_{a_0} + \sum_{a \in C^+} p_a - \sum_{a \in C^- \setminus a_0} p_a \\ \leq \left\lfloor \frac{1}{T}\left((T-1)u_{a_0} + \sum_{a \in C^+} u_a - \sum_{a \in C^- \setminus a_0} \ell_a\right)\right\rfloor.\end{aligned} \tag{10}$$

Proof. We provide the vector λ that combines (10) for some circuit C out of the initial matrix M. To this end, for $k \in \{0, \ldots, m\}$ consider the arc $a_k = (v,w) \in C$. Then, the rows k and $m+k$ of the matrix M correspond to the following two PESP inequalities

$$\begin{aligned}-\pi_w + \pi_v - Tp_{a_k} &\leq -\ell_{a_k}, \\ \pi_w - \pi_v + Tp_{a_k} &\leq u_{a_k}.\end{aligned}$$

Finally, choosing the components of the coefficient vector λ as

$$\lambda_k = \begin{cases} \frac{T-1}{T}, & k = m + c, \text{ where } a_c = a_0, \\ \frac{1}{T}, & k = c, \text{ where } a_c \in C^- \setminus \{a_0\}, \\ \frac{1}{T}, & k = m + c, \text{ where } a_c \in C^+, \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

yields (10). □

In fact, these inequalities emerge from cycle inequalities by reversing one of their backward arcs. Hence, we refer to (10) as *reversed-arc cuts*. Observe that in some special cases, these inequalities can coincide with what Lindner [14] called *chain cutting planes*. However, for the latter Villumsen [19] had to observe in practical computations that these have "no effect" on the solution of his PESP instances. In addition to Theorem 3, this is another motivation for us to focus in our exposition on the multi-circuit cuts.

6 PESP Instances on Wheel Graphs

We introduce a family of infeasible PESP instances, for which the first Chvátal closure is still nonempty. Since the pioneering work of Edmonds [5], we are not aware of too many explicit such results. Here, even adding the change-cycle inequalities (7) does not change this status. Only adding two appropriate multi-circuit cuts (9) provides a certificate for the infeasibility of these instances. Let us annotate that these instances were inspired by an infeasible PESP instance which was studied by Lindner [14] and whose constraint graph is the wheel graph W_4 on four vertices.

We consider one fixed period time $T \geq 6$ for any of the instances that are defined below. Let $n \geq 4$ be some even number and consider the wheel graph W_n, see Figure 3 for an example with $n = 6$. We set the feasible intervals of the spoke arcs to $[0, 1]_T$, while we require $[1, T - 1]_T$ for the remaining outer arcs.

We start investigating this class of instances by first giving a simple proof for the infeasibility of these instances. Hereafter, we establish that $P^{(1)} \neq \emptyset$, but $P^{(2)} = \emptyset$.

Lemma 2. *Let $T \geq 2$ and $n \geq 4$ be an even number. The PESP instance that is defined on the wheel graph W_n with feasible intervals $[0, 1]_T$ on the spokes and $[1, T - 1]_T$ on the arcs of the outer circuit is infeasible.*

Proof. We may assume w.l.o.g. that $\pi_h = 0$, where h is the hub vertex in W_n. The constraints on the spokes restrict the π values of the other vertices to $\{0, 1\}$. The constraints on the remaining arcs require these two values to be used alternatingly around the outer circuit of W_n. Since we chose n to be even, the outer circuit has an odd number of vertices. But this is not compatible with the π values of all the vertices on the outer circuit taking the values zero and one alternatingly. □

The next lemma slightly simplifies the argumentation in the proof of the main theorem of this section, namely that $P^{(1)}$ is not empty.

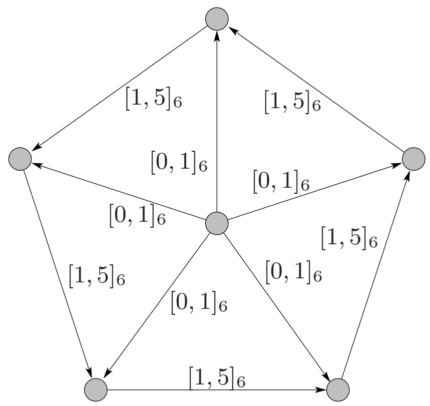

Fig. 3. An infeasible PESP instance on the wheel graph W_6 with $T = 6$

Lemma 3. *Consider some coefficient vector* $\lambda \geq 0$. *Let* λ_a *and* $\lambda_{a^{-1}}$ *correspond to two components whose PESP inequalities refer to the very same arc* a *and define* $c := \min\{\lambda_a, \lambda_{a^{-1}}\}$. *Derive* λ' *from* λ *by subtracting* c *from the components of both,* a *and* a^{-1}. *Now, if* ${\lambda'}^\mathsf{T} M \leq \left\lfloor {\lambda'}^\mathsf{T} b \right\rfloor$ *then* $\lambda^\mathsf{T} M \leq \lfloor \lambda^\mathsf{T} b \rfloor$.

Proof. First, observe that $(\lambda - \lambda')^\mathsf{T} M = 0$. Second, $(\lambda - \lambda')^\mathsf{T} b = c \cdot (-\ell_a + u_a) \geq 0$. Thus, rounding down cannot provide any negative value. Finally, because of $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor$ we may add $(\lambda - \lambda')$ to λ' while keeping any valid inequality valid. $\square$

As a consequence, for investigating $P^{(1)}$ we may assume w.l.o.g. that in any (relevant) valid inequality for $P^{(1)}$ none of the arcs shows up with both its inequalities for its respective lower and upper bounds.

Theorem 8. $P^{(1)} \neq \emptyset$. *In particular, all the cycle inequalities* (5) *and reversed-arc cuts* (10) *are valid for the same particular vector, in the case of* $T \geq 6$.

Proof. Before starting, in the vector p we distinguish the components that correspond to the $n-1$ spoke arcs from the components that correspond to the $n-1$ arcs of the outer circuit, $p^\mathsf{T} = (p_s^\mathsf{T}, p_c^\mathsf{T})$. Moreover, with $\mathbf{1}$ we denote the all-one vector of appropriate dimension. Our goal is to establish that

$$y_1 := (\pi^\mathsf{T}, p_s^\mathsf{T}, p_c^\mathsf{T}) = (\mathbf{0}^\mathsf{T}, \frac{1}{2T} \cdot \mathbf{1}^\mathsf{T}, \frac{1}{2} \cdot \mathbf{1}^\mathsf{T}) \in P^{(1)}. \tag{11}$$

To this end, let $\lambda^\mathsf{T} M x \leq \lfloor \lambda^\mathsf{T} b \rfloor$ be an arbitrary valid inequality of $P^{(1)}$, where M and b are as defined in (3) and (4), respectively. We have to check y_1 against this general inequality.

For ease of notation we rewrite the coefficient vector λ as $\lambda^\mathsf{T} = (\lambda_1^\mathsf{T}, \lambda_2^\mathsf{T}, \lambda_3^\mathsf{T}, \lambda_4^\mathsf{T})$, where λ_1 and λ_3 refer to the rows that correspond to the spokes, while λ_2 and λ_4 refer to the rows that correspond to the outer circuit of the wheel graph W_n. Moreover, λ_3 and λ_4 refer to the initial PESP-inequalities that define the upper bounds u_a, but λ_1 and

λ_2 refer to the initial PESP-inequalities that define the lower bounds ℓ_a, after having multiplied these with minus one.

Using these definitions, we find that

$$\begin{aligned}\lambda^\mathsf{T} M y_1 &= (\lambda_1^\mathsf{T}, \lambda_2^\mathsf{T}, \lambda_3^\mathsf{T}, \lambda_4^\mathsf{T}) \cdot (-\frac{1}{2} \cdot \mathbf{1}^\mathsf{T}, -\frac{T}{2} \cdot \mathbf{1}^\mathsf{T}, \frac{1}{2} \cdot \mathbf{1}^\mathsf{T}, \frac{T}{2} \cdot \mathbf{1}^\mathsf{T})^\mathsf{T} \\ &= -\frac{1}{2}||\lambda_1||_1 - \frac{T}{2}||\lambda_2||_1 + \frac{1}{2}||\lambda_3||_1 + \frac{T}{2}||\lambda_4||_1\end{aligned}$$

and

$$\begin{aligned}\left\lfloor \lambda^\mathsf{T} b \right\rfloor &= \left\lfloor (\lambda_1^\mathsf{T}, \lambda_2^\mathsf{T}, \lambda_3^\mathsf{T}, \lambda_4^\mathsf{T}) \cdot (\mathbf{0}^\mathsf{T}, -1 \cdot \mathbf{1}^\mathsf{T}, 1 \cdot \mathbf{1}^\mathsf{T}, (T-1) \cdot \mathbf{1}^\mathsf{T})^\mathsf{T} \right\rfloor \\ &= \lfloor -||\lambda_2||_1 + ||\lambda_3||_1 + (T-1)||\lambda_4||_1 \rfloor .\end{aligned}$$

In particular, for the point y_1 the initial inequality $\lambda^\mathsf{T} M y_1 \leq \left\lfloor \lambda^\mathsf{T} b \right\rfloor$ is equivalent to

$$\begin{aligned}-||\lambda_2||_1 + ||\lambda_3||_1 + (T-1)||\lambda_4||_1 - \lfloor -||\lambda_2||_1 + ||\lambda_3||_1 + (T-1)||\lambda_4||_1 \rfloor \quad (12) \\ \leq \frac{1}{2}||\lambda_1||_1 + \left(\frac{T}{2} - 1\right) ||\lambda_2||_1 + \frac{1}{2}||\lambda_3||_1 + \left(\frac{T}{2} - 1\right) ||\lambda_4||_1 \quad (13) \\ = \frac{1}{2}(||\lambda_1||_1 + ||\lambda_3||_1) + \left(\frac{T}{2} - 1\right) (||\lambda_2||_1 + ||\lambda_4||_1). \quad (14)\end{aligned}$$

In order to prove that (12-13) is valid, observe first that the left-hand side (12) has values in the interval $[0, 1)$. So, we first identify some coefficient vectors λ for which (14) is at least one. Hereafter, we investigate the remaining vectors λ.

From Lemma 3, $\lambda \geq \mathbf{0}$, $\lambda^\mathsf{T} M$ being integer, and the coefficients of the periodical offsets p having value $|T|$, we conclude that for each component i of λ we have $\lambda_i = \frac{k}{T}$, with $k = 0, 1, 2, \ldots$.

Case "$||\lambda_2||_1 + ||\lambda_4||_1 \geq \frac{3}{T}$". We find immediately that (14) is at least as large as $\frac{3}{2} - \frac{3}{T}$. Now, recall that we chose the period time $T \geq 6$, and in particular (14) is at least one, establishing the theorem in this case.

Case "$||\lambda_2||_1 + ||\lambda_4||_1 = \frac{1}{T}$". In other words, the Chvátal-Gomory coefficient vector λ does only involve exactly one inequality of one arc $a = (i, j)$ of the outer circuit of W_n. In this case we are not aiming at showing that (12-13) was indeed valid. Rather, we enumerate all the eight relevant valid inequalities of $P^{(1)}$ that involve the arc a as the only arc of the outer circuit.

For that the requirement of $\lambda^\mathsf{T} M$ being integer is fulfilled, in particular for the node variables π, some of the initial PESP constraints in which π_i or π_j appear must have non-zero components in the coefficient vector λ. Because of $||\lambda_2||_1 + ||\lambda_4||_1 = \frac{1}{T}$, these must correspond to the spokes (h, i) and (h, j), where h denotes the hub of the wheel graph W_n, see Figure 4 for an illustration.

Depending on whether we use the lower bound or the upper bound inequalities of the spokes, w.l.o.g. the CG-multipliers are either $\frac{1}{T}$ or $\frac{T-1}{T}$.

First, if we choose twice the $\frac{1}{T}$, we end with the two standard cycle inequalities (5) for this triangle,

$$0 \leq p_a - p_{(h,j)} + p_{(h,i)} \leq 1. \quad (15)$$

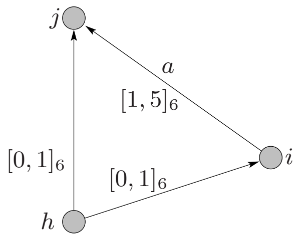

Fig. 4. A triangle in W_n with $T = 6$

For the values $p_s \equiv \frac{1}{2T}$ and $p_c \equiv \frac{1}{2}$ that we chose in our particular vector y_1, these inequalities are of course valid, because $0 \leq \frac{1}{2} \leq 1$.

Second, if for the spokes we chose once the value $\frac{1}{T}$ and once the value $\frac{T-1}{T}$, we obtain the following four reversed-arc cuts,

$$1 \leq \pi_j - \pi_h + p_a + (T-1)p_{(h,j)} + p_{(h,i)} \leq 1 \tag{16}$$
$$0 \leq \pi_i - \pi_h - p_a + p_{(h,j)} + (T-1)p_{(h,i)} \leq 0, \tag{17}$$

which are valid for our choice of y_1, too.

Last, taking $\frac{T-1}{T}$ as the coefficient for both spokes yields

$$0 \leq \pi_j - \pi_i + p_a + (T-1)p_{(h,j)} - (T-1)p_{(h,i)} \leq 1. \tag{18}$$

Also these two inequalities are valid for the vector y_1 as defined, $0 \leq \frac{1}{2} \leq 1$.

To summarize, in the case of $||\lambda_2||_1 + ||\lambda_4||_1 = \frac{1}{T}$ we considered all the eight relevant valid inequalities of $P^{(1)}$ and verified that the vector $(\pi^\intercal, p_s^\intercal, p_c^\intercal) = (\mathbf{0}^\intercal, \frac{1}{2T} \cdot \mathbf{1}^\intercal, \frac{1}{2} \cdot \mathbf{1}^\intercal)$ is valid for any of them.

Case “$||\lambda_2||_1 + ||\lambda_4||_1 = \frac{2}{T}$”. We distinguish between several subcases. First, we may have two non-incident arcs a_1 and a_2 of the outer circuit being involved in the cut that is defined by the coefficient vector λ. But then we are done, because we are in fact twice in the case of $||\lambda_2||_1 + ||\lambda_4||_1 = \frac{1}{T}$.

Second, we may have just one arc of the outer circuit being involved. The two cycle inequalities (5) that emerge from multiplying all its three initial constraints with $\frac{2}{T}$ are in fact nothing but just scaled versions of (15). Hence, here we need to consider valid inequalities in which some of the initial constraints are multiplied with $\frac{2}{T}$, while others are multiplied with $\frac{T-2}{T}$. The counterparts of (16) and (17) read

$$1 \leq \pi_j - \pi_h + 2p_a + (T-2)p_{(h,j)} + 2p_{(h,i)} \leq 2$$
$$-1 \leq \pi_i - \pi_h - 2p_a + 2p_{(h,j)} + (T-2)p_{(h,i)} \leq 0.$$

For the particular point y_1 these terms evaluate to $\frac{3}{2}$ and $-\frac{1}{2}$, respectively, and all the four inequalities are thus feasible. The same holds for the counterpart of (18), where y_1 yields one, which is feasible in

$$0 \leq \pi_j - \pi_i + 2p_a + (T-2)p_{(h,j)} - (T-2)p_{(h,i)} \leq 2.$$

Last, what we still have to investigate is the case in that two consecutive arcs a_1 and a_2 of the outer circuit are activated by the coefficient vector λ. Due to their orientation in W_n, in the valid inequality that is induced by λ, both arcs contribute either with their PESP inequalities that define their lower bounds, or both contribute with their PESP inequalities that define their upper bounds. In particular, the π variable of their common vertex has coefficient zero in the cut.

Hence, we are in a situation that is quite similar to the one that we already discussed in the case of $||\lambda_2||_1 + ||\lambda_4||_1 = \frac{1}{T}$. The only difference is that for the outer arcs we are now summing *twice* their lower or upper bounds in the inequalities. We summarize the relevant computations by providing the eight resulting valid inequalities – using the same notation as in the previous case – in which the reader will have no difficulty to verify that y_1 is indeed feasible,

$$\begin{aligned}
1 \leq \quad & p_{a_1} + p_{a_2} - p_{(h,j)} + p_{(h,i)} && \leq 1, \\
1 \leq \quad & \pi_j - \pi_h + p_{a_1} + p_{a_2} + (T-1)p_{(h,j)} + p_{(h,i)} && \leq 2, \\
-1 \leq \quad & \pi_i - \pi_h - p_{a_1} - p_{a_2} + p_{(h,j)} + (T-1)p_{(h,i)} && \leq 0, \text{ and} \\
0 \leq \quad & \pi_j - \pi_i + p_{a_1} + p_{a_2} + (T-1)p_{(h,j)} - (T-1)p_{(h,i)} && \leq 2.
\end{aligned}$$

This concludes the last case for the coefficient vector λ and thus establishes (11). □

Proposition 2. *The change-cycle inequalities* (7) *are valid for the infeasible PESP instance that we consider on the wheel graphs W_n.*

Proof (sketch). We must omit the full proof due to space limitations. Nevertheless, let us compute the relevant quantities of the particular fractional solution

$$y_1 = (\pi^\mathsf{T}, p_s^\mathsf{T}, p_c^\mathsf{T}) = (\mathbf{0}^\mathsf{T}, \frac{1}{2T} \cdot \mathbf{1}^\mathsf{T}, \frac{1}{2} \cdot \mathbf{1}^\mathsf{T}) :$$

For a spoke arc a, here, the periodic tension variable is $x_a = \frac{1}{2}$, and for any other arc a, its periodic tension variable is $x_a = \frac{T}{2}$. In the most interesting case, namely the case of a triangle, cf. Figure 4 for an illustration in the case of $T = 6$, the integer variable z of this triangle evaluates to $\frac{1}{2}$. And with these values, the reader might not have any difficulties to compute the slope $\mu = -\frac{1}{T-1}$ and ordinate intersect $\nu = \frac{T}{T-1}$, and thus verify that the corresponding change-cycle inequality (7) is tight. For longer circuits, there is even some positive slack. □

Theorem 9. $P^{(2)} = \emptyset$. *In particular, two multi-circuit cuts* (9) *certify the emptiness of $P^{(2)}$.*

Proof. We apply Theorem 6 to the outer circuit C of the wheel graph W_n. We combine it linearly by summing over all the $|C|$ oriented 4-circuits that contain two consecutive edges of C.

Let C_i be one of these 4-circuits. Consider the cycle inequalities (5) of C_i and of its opposite counterpart C_i^{-1},

$$p_1 + p_2 + p_3 - p_4 \leq \left\lfloor \frac{1}{T}(1 + (T-1) + (T-1) - 0) \right\rfloor = \left\lfloor \tfrac{2T-1}{T} \right\rfloor = 1, \quad (19)$$

$$-p_1 - p_2 - p_3 + p_4 \leq \left\lfloor \frac{1}{T}(0 - 1 - 1 + 1) \right\rfloor = \left\lfloor \tfrac{-1}{T} \right\rfloor = -1, \quad (20)$$

where p_1 and p_4 are the periodical offset variables that we introduced for the two spokes of C_i. In other words, $p_1 + p_2 + p_3 - p_4 = 1$.

For that the oriented circuits C_i linearly combine C, we have to multiply each of them with $\frac{1}{2}$. Recall that we selected n to be even, thus $|C| = n - 1$ being odd. Doing so for their initial orientation, using (19) we find that

$$\sum_{a \in C} p_a \leq \left\lfloor |C| \cdot \frac{1}{2} \cdot 1 \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor \overset{n \text{ odd}}{=} \frac{n}{2} - 1, \tag{21}$$

because the periodical offset variables p of all the spokes cancel out. Similarly, summing (20) for all their opposite counterparts C_i^{-1} yields

$$\sum_{a \in C} -p_a \leq \left\lfloor |C| \cdot \frac{1}{2} \cdot (-1) \right\rfloor = \left\lfloor \frac{-n+1}{2} \right\rfloor \overset{n \text{ odd}}{=} -\frac{n}{2}. \tag{22}$$

Finally, multiplying (22) with minus one and comparing it to (21) yields $\frac{n}{2} \leq \frac{n}{2} - 1$ and thus reveals that indeed $P^{(2)} = \emptyset$. □

Remark 1. It is highly interesting to compare the resulting pair of inequalities (9) to their initial counterparts (5) in $P^{(1)}$:

$$\begin{array}{lll} P^{(1)}: & \lceil (n-1)\frac{1}{T} \rceil \leq \sum_{a \in C} p_a \leq \lfloor (n-1)\frac{T-1}{T} \rfloor & \text{vs.} \\ P^{(2)}: & \frac{n}{2} \leq \sum_{a \in C} p_a \leq \frac{n}{2} - 1. & \end{array}$$

Hence, in a sense on the wheel graph instances the multi-circuit cuts propagate to $P^{(2)}$ the rounding benefit that particular cycle inequalities achieved already in $P^{(1)}$. □

This is our main motivation for the separation heuristic that we apply in the next section.

7 Computational Results

For the PESP, we investigate the change in the solution behavior of CPLEX 11, when adding multi-circuit cuts (9) to its IP models. To this end, we need to separate these cuts. In Remark (1) we observed that if we combine valid inequalities (5) of the first Chvátal closure in which the rounding was strong, i.e., $b - \lfloor b \rfloor \approx 1 - \varepsilon$, then, in the second Chvátal closure we can achieve much stronger multi-circuit cuts (9) than their corresponding cycle inequalities (5) in the first Chvátal closure.

In most detail, we generate multi-circuit cuts (9) in the following way.

1. Build an initial IP model of an optimization instance of PESP.
   Actually, instead of immediately using (2) we are using a purely tension-based formulation here, because in [13] it was reported that these performed best.
2. Generate valid inequalities for this IP.
   These are cycle inequalities (5) and change-cycle inequalities (7). For the separation heuristic we made the same experience as Nachtigall, namely that considering the fundamental circuits subject to a minimum spanning tree with the periodic tension values of the current LP relaxation as weights, empirically is the most efficient deterministic solution heuristic. Denote the resulting LP by LP_1.

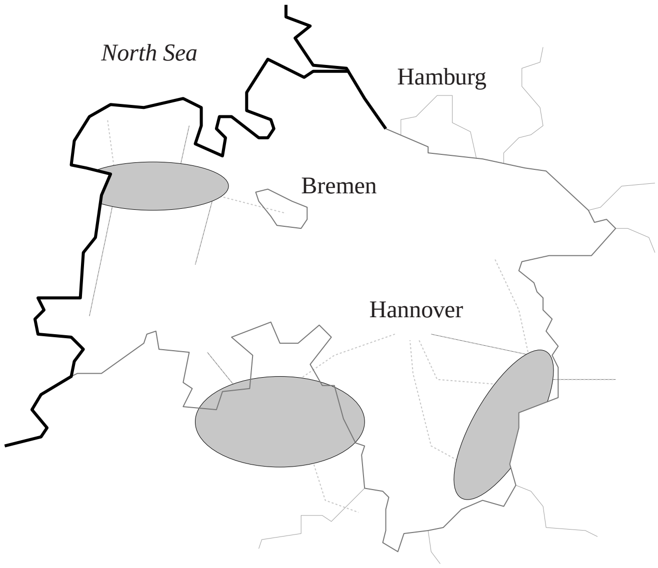

Fig. 5. The subregions of Lower Saxony and Westfalia (north-western part of Germany) of which we distill our three test instances

3. Store "strong" cycle inequalities in a pool $\mathcal{P}$.
   While computing LP_1, we record for each cycle inequality (5) that we generate its rounding benefit $\beta := b - \lfloor b \rfloor$, no matter whether it is added to LP_1 or not. If β is larger than some threshold value – we used $\beta \geq 0.7$ – then this cycle inequality is added to a pool $\mathcal{P}$ of "strong" cycle inequalities.
4. Add multi-circuit cuts (9) to LP_1.
   After Steps 2 and 3 have been accomplished, denote by x^* the optimum fractional solution of the final LP relaxation LP_1. To cut this point x^* off with some multi-circuit cut (9), we formulate the Chvátal-Gomory IP, that Fischetti and Lodi proposed in [6], for the cycle inequalities (5) in $\mathcal{P}$. Since the cycle inequalities already live in the first Chvátal closure, this way we are exploring parts of the second Chvátal closure. We iterate this CG-procedure until for some subsequent linear program LP_2 (LP_1 plus some multi-circuit cuts) its optimal solution can no more be separated by this procedure, or a time limit applies.
5. Solve the IP.
   In LP_2, switch on the integrality requirements on the periodical offset variables p and let CPLEX 11 solve this (mixed) integer linear program.

Data. We investigate the performance of the multi-circuit cuts (9) on several real-world data sets. Unfortunately, there is still not available any public library of real-world periodic railway timetabling instances. Hence, we need to resort to instances that have been available at our institute, e.g., some that had already been used in [8, 10]. In particular, all are subnetworks of the German passenger railway network.

More precisely, we consider three regions within Lower Saxony and Westfalia: Harz (H), Ostfriesland (O), and Ostwestfalen-Lippe (L), see Figure 5. All these net-

Table 1. Size of our test instances. Here, ν is the cyclomatic number $|A| - |V| + 1$, i.e., the number of integer variables in the tension-based IP models that we apply [13].

Instance name	service lines	$\|V\|$	$\|A\|$	ν	tight arcs	width
Harz 1 (H1)	17	54	309	256	26	10^{120}
Harz 2 (H2)	16	30	308	279	7	10^{149}
Harz 3 (H3)	12	43	226	184	23	10^{93}
Harz 4 (H4)	22	58	432	375	26	10^{183}
Harz 5 (H5)	15	55	332	278	29	10^{153}
Ostfriesland 1 (O1)	10	77	281	205	58	10^{99}
Ostfriesland 2 (O2)	13	107	380	274	86	10^{128}
Ostwestfalen-Lippe 1 (L1)	12	60	295	236	45	10^{108}
Ostwestfalen-Lippe 2 (L2)	12	65	289	225	48	10^{112}
Ostwestfalen-Lippe 3 (L3)	13	66	357	292	49	10^{145}

works are operated at a period time of two hours. Together with the standard time precision that is used by Deutsche Bahn AG, and which is 0.1 minutes, in our models this yields $T = 1200$. It is a general observation that cycle inequalities (5) tend to be stronger if the spans $u_a - \ell_a$ of the PESP constraints are smaller. Obviously, multi-circuit cuts (9) inherit this property. Hence, if these new valid inequalities bear any computational benefit, we hope to reveal it on instances where railway capacity is rather scarce. This is done by modeling the complete passenger traffic in the respective regions (regional and long-distance trains), and by considering single tracks. The sizes of the resulting PESP instances, after eliminating redundancies such as contracting fixed arcs with zero span, are reported in Table 1. There, in the column "tight arcs" we counted the number of arcs a with relatively small span, i.e., $u_a - \ell_a \leq \frac{T}{10}$. In the column "width", we provide a (rough) upper bound on the size of the Branch-and-Bound tree that had already been considered in [13], which is the product of the possible number of values over all the integer variables z.

Results. We summarize our computational results in Table 2. There, we compare three different policies for solving PESP instances. First, take the pure initial model as is, with no problem-specific valid inequalities being added. Its LP relaxation admits a trivial optimal solution: simply take $\pi \equiv 0$ and $p_a := \frac{\ell_a}{T}$. When reporting on values of refined LP relaxations, we scale the values such that this trivial solution has value zero, and the optimum value is 100.[1] Second, we add the problem-specific cycle inequalities (5) plus some change-cycle inequalities (7), as described above. Last, we also add multi-circuit cuts (9).

We start by giving the optimum solutions of the respective (refined) LP relaxations in the columns "LP bound". Next to this, we put the solution time under standard settings of CPLEX 11 on an Intel Core2 with 2.13 GHz and a 2GB RAM running Linux. In the last but third column we report how many multi-circuit cuts (9) could be found by the separation heuristic that we sketched above, and which was based on [6].

[1] In the tension-based IP (see [13]) we add cycle inequalities (5) as bounds on the integer variables, which typically yields values slightly larger than zero, e.g, 5–25%.

Table 2. Computational Results of adding multi-circuit cuts (9) to PESP IP models. A **boldface** entry indicates that the shortest solution time is achieved by adding multi-circuit cuts (9) (LP bounds indexed to "intopt$\widehat{=}100$", time in seconds)

	pure IP model		IP + (5) + (7)		IP + (5) + (7) + (9)		
Instance	LP bound	opt time	LP bound	opt time	# cuts (9)	LP bound	opt time
H1	4.4	325	86.0	75	2	86.0	**42**
H2	35.6	850	83.0	263	15	83.0	349
H3	4.3	64	77.8	13	64	81.0	**12**
H4	40.8	3059	86.8	2255	1	86.8	2727
H5	4.1	2921	56.7	1221	17	58.9	1663
O1	12.3	216	84.8	197	18	85.3	**79**
O2	16.7	338	84.4	365	25	85.0	**187**
L1	27.2	141	89.0	94	25	89.2	**69**
L2	11.2	203	94.7	71	22	94.7	**56**
L3	19.0	2652	90.3	1010	20	90.7	1226

To summarize, in contrast to what Villumsen [19] had to observe for the chain-cutting planes, which were due to Lindner [14], multi-circuit cuts (9) indeed have an effect on the solution behavior of CPLEX 11 on PESP instances. First of all, on each instance, CPLEX is (still, see below) better off when fed with the full machinery of additional valid inequalities, compared to not adding any cuts at all. Unfortunately, there are some instances, on which adding multi-circuit cuts (9) cause longer solution times, compared to the (5)+(7) setting. Nevertheless, in the majority of the cases, multi-circuit cuts (9) yield an improved solution behavior. In several cases, the solution time drops by more than 40%.

Additional Comments. Let us close by commenting on two interesting effects. First, in Table 2, we voluntarily decided to consider the pure LP bounds instead of the dual bound that CPLEX is able to achieve in its root node preprocessing. This is mainly motivated by the fact that the LP bounds are conceptually better accessible, compared to the result of a powerful "black box". Yet, consider the instance O2. For this, Table 2 contains entries of 16.7% and 85.0% for the LP bounds with and without cuts, respectively. But after the root node preprocessing of CPLEX 11, the respective values get together as close as 82.0% and 85.4%. Now, compare these values to the root node preprocessing of CPLEX 8.1, which is the version that had been used in an extensive computational study on other railway timetabling instances [13]: 28.6% and 85.3%. Similar observations can be made for the respective solution times.

This illustrates the improvements that more recent versions of CPLEX are able to achieve in the preprocessing of PESP IP models. Could this be a consequence of the fact that pure PESP IP models have been included in the MIPLIB [1, 10], in combination with new general IP insight, e.g., the one reported in [6]? Here, it might be interesting to recall that Fischetti and Lodi called the PESP IP models in the MIPLIB "very hard"...

Nevertheless, although the preprocessed dual bounds get closer to each other, problem-specific insight, e.g., in form of the new multi-circuit cuts (9) that we just introduced here, may still cut the solution time by roughly one half.

Second, and last but not least, we point out the high sensitivity that the models show with respect to certain specific multi-circuit cuts (9). As an example, on the instance H2 we had to make the following observation. With just inequalities (5) and (7) being added, a solution time of 263s can be observed, cf. Table 2. Then, adding just the first *two* multi-circuit cuts (9) that our separation heuristic found, the solution time is cut by more than 73% to less than 70s. But adding the next two such cuts, we end with a solution time of even 392s. In other words, if we just added the first two cuts, instead of all the 25 that we were able to separate, in Table 2 we could have replaced the value 349s in the H2 row with only **70**s...

On the one hand, this underlines that multi-circuit cuts (9) indeed have some effect. On the other hand, this asks for an understanding on which particular ones of these cuts are the “right” ones.

8 Conclusions

We introduced multi-circuit cuts as new valid inequalities for the Periodic Event Scheduling Problem (PESP). These live in its second Chvátal closure. For a particular family of infeasible PESP instances, we managed to prove that its first Chvátal closure is nonempty. And even adding all change-cycle inequalities, of which we further proved that in general they appear only in much larger closures, does not turn the status to infeasible. Hence, it is a first theoretical merit of the multi-circuit cuts to certify infeasibility of these particular instances. Complementary to this, in our computational study, we observed that multi-circuit cuts are likely to reduce the solution time of CPLEX 11 on PESP IP models.

We admit that up to now, our separation has not really been tuned. More theoretical insight is needed to distinguish between helpful multi-circuit cuts, and unproductive ones. We are very much confident that with such an additional insight, adding just the helpful multi-circuit cuts will *always* improve on the two other settings that we considered in Table 2. In addition, practically efficient separation heuristics for multi-circuit cuts are required, in particular if we want to use these cuts in a branch-and-cut context, too. But also any further new classes of valid inequalities from whichever Chvátal closure will be equally welcome – given that they have some (positive) effect on the solution behavior of CPLEX 11.

To summarize, of course multi-circuit cuts are not the end of the story in the solution of PESP instances. However, we feel that these are one step forward into a promising direction.

References

1. T. Achterberg, T. Koch, and A. Martin. MIPLIB 2003. *Oper. Res. Lett.*, 34(4):361–372, 2006.
2. G. C. Caimi, M. Fuchsberger, M. Laumanns, and K. Schüpbach. Periodic railway timetabling with event flexibility. In C. Liebchen, R. K. Ahuja, and J. A. Mesa, editors, *ATMOS 2007 - 7th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems*, Dagstuhl, Germany, 2007. Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany.

3. V. Chvátal. Edmond's polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:205–337, 1973.
4. W. Dauscha, H. D. Modrow, and A. Neumann. On cyclic sequence types for constructing cyclic schedules. *Zeitschrift fur Operations Research*, 29(1):1–30, 1985.
5. J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research National Bureau of Standards Section B*, 69:125–130, 1965.
6. M. Fischetti and A. Lodi. Optimizing over the first Chvátal closure. *Mathematical Programming*, 110(1):3–20, 2007.
7. L. G. Kroon. Mathematics for railway timetabling. *ERCIM News*, 68:22–23, January 2007.
8. C. Liebchen. A cut-based heuristic to produce almost feasible periodic railway timetables. In S. E. Nikoletseas, editor, *WEA*, volume 3503 of *Lecture Notes in Computer Science*, pages 354–366. Springer, 2005.
9. C. Liebchen. The first optimized railway timetable in practice. *Transportation Science*, 2008. accepted for publication.
10. C. Liebchen and R. H. Möhring. Information on MIPLIB's timetab-instances. Preprint 049/2003, TU Berlin, Mathematical Institute, 2003.
11. C. Liebchen and R. H. Möhring. The modeling power of the periodic event scheduling problem: Railway timetables - and beyond. In F. Geraets, L. G. Kroon, A. Schöbel, D. Wagner, and C. D. Zaroliagis, editors, *ATMOS*, volume 4359 of *Lecture Notes in Computer Science*, pages 3–40. Springer, 2004.
12. C. Liebchen and L. W. Peeters. On cyclic timetabling and cycles in graphs. Technical Report 761-2002, TU Berlin, Mathematical Institute, 2002.
13. C. Liebchen, M. Proksch, and F. H. Wagner. Performance of algorithms for periodic timetable optimization. In M. Hickman, P. Mirchandani, and S. Voß, editors, *Computer-Aided Systems in Public Transport (CASPT 2004)*, volume 600 of *Lecture Notes in Economics and Mathematical Systems*. Springer, 2008.
14. T. Lindner. *Train Schedule Optimization in Public Rail Transport*. Ph.D. thesis, Technische Universität Braunschweig, 2000.
15. K. Nachtigall. *Periodic Network Optimization and Fixed Interval Timetables*. Habilitation thesis, Universität Hildesheim, 1998.
16. M. A. Odijk. A constraint generation algorithm for the construction of periodic railway timetables. *Transportation Research B*, 30(6):455–464, 1996.
17. J. Opitz and K. Nachtigall. A modulo network simplex method for solving periodic timetable, 2007. A german description of their software system TAKT can be found in *Der Eisenbahningenieur*, 58(7/2007):50–55.
18. P. Serafini and W. Ukovich. A mathematical model for periodic scheduling problems. *SIAM Journal on Discrete Mathematics*, 2(4):550–581, 1989.
19. J. C. Villumsen. *Construction of Timetables Based on Periodic Event Scheduling*. Master's thesis, Danish Technical University, Copenhagen, 2006.