Computable Separation in Topology, from T_0 to T_3

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Abstract. This article continues the study of computable elementary topology started in [7]. We introduce a number of computable versions of the topological T_0 to T_3 separation axioms and solve their logical relation completely. In particular, it turns out that computable T_1 is equivalent to computable T_2. The strongest axiom SCT_3 is used in [2] to construct a computable metric.

1 Preliminaries

We use the representation approach to computable analysis [6] as the basis for our investigation. In particular, we use the terminology and concepts introduced in [7] (which can be considered as a revision and extension of parts from [6]).

Let Σ^* and Σ^ω be the sets of the finite and infinite sequences, respectively, of symbols from a finite alphabet Σ. A function mapping finite or infinite sequences of symbols from Σ is computable, if it can be computed by a Type-2 machine, that is, a Turing machine with finite or infinite input and output tapes. On Σ^* and Σ^ω we use canonical tupling functions $\langle \cdot \rangle$ that are computable and have computable inverses. Computability on finite or infinite sequences of symbols is transferred to other sets by representations, where elements of Σ^* or Σ^ω are used as "concrete names" of abstract objects. For representations $\gamma_i :\subseteq Y_i \to M_i$ we consider the product representation defined by $[\gamma_1, \gamma_2]\langle p, q\rangle := (\gamma_1(p_1), \gamma_2(p_2))$. Let $Y = Y_1 \times \ldots \times Y_n$, $M = M_1 \times \ldots \times M_n$ and $\gamma : Y \to M$, $\gamma(y_1, \ldots, y_n) = \gamma_1(y_1) \times \ldots \times \gamma_n(y_n)$. A partial function $h :\subseteq Y \to Y_0$ realizes the multi-function $f : M \rightrightarrows M_0$ if $\gamma_0 \circ h(y) \in f(x)$ whenever $x = \gamma(y)$ and $f(x)$ exists. This means that $h(y)$ is a name of some $z \in f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. The function f is (γ, γ_0)-computable, if it has a computable realization.

We will consider computable topological spaces as defined in [7]. Various similar definitions have been used, see, for example, [4, 3, 5] and the references in [7]. In particular, the definition in [6] is slightly different. A computable topological space is a 4-tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that (X, τ) is a topological T_0-space, $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base β of τ, $\mathrm{dom}(\nu)$ is recursive and $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in S\}$ for all $u, v \in \mathrm{dom}(\nu)$ for some r.e. set $S \subseteq (\mathrm{dom}(\nu))^3$.

For the points, the open sets and the closed sets we use the representations δ, θ and ψ^- that are defined as follows. For $p \in \Sigma^\omega$ and $x \in X$, $\delta(p) = x$ iff p is

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a list of all $u \in \mathrm{dom}(\nu)$ such that $x \in \nu(u)$, $\theta(p)$ is the union of all $\nu(u)$ where u is listed by p, and $\psi^-(p) := X \setminus \theta(p)$.

2 Axioms of Computable Separation

For a topological space $\mathbf{X} = (X, \tau)$ with set $\mathcal{A}$ of closed sets we consider the following classical separation properties:

Definition 1 (separation axioms).

$$\begin{aligned}
&\mathrm{T}_0 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)((x \in W \wedge y \notin W) \vee (x \notin W \wedge y \in W))),\\
&\mathrm{T}_1 : (\forall x, y \in X,\ x \neq y)(\exists W \in \tau)(x \in W \wedge y \notin W),\\
&\mathrm{T}_2 : (\forall x, y \in X,\ x \neq y)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge y \in V),\\
&\mathrm{T}_3 : (\forall x \in X, \forall A \in \mathcal{A}, x \notin A)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge A \subseteq V),\\
&\mathrm{T}_4 : (\forall A, B \in \mathcal{A}, A \cap B = \emptyset)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge A \subseteq U \wedge B \subseteq V).
\end{aligned}$$

For $i = 0, 1, 2, 3$, we call $\mathbf{X} = (X, \tau)$ a T_i-space iff $\mathrm{T_i}$ is true.

For the four axioms, $\mathrm{T}_2 \Longrightarrow \mathrm{T}_1 \Longrightarrow \mathrm{T}_0$ and $\mathrm{T}_0 + \mathrm{T}_3 \Longrightarrow \mathrm{T}_2$, where all the implications are proper [1]. T_2-spaces are called *Hausdorff spaces* and T_3-spaces are called *regular*. (Many authors, for example [1], call a space T_3-space or regular iff $\mathrm{T}_1 + \mathrm{T}_3$.) We mention that (X, τ) is a T_1-space, iff all sets $\{x\}$ $(x \in X)$ are closed [1]. For computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$, which are countably based T_0-spaces (also called *second countable*), $T_3 \Longrightarrow T_2$.

We introduce computable versions $\mathrm{CT_i}$ of the conditions $\mathrm{T_i}$ by requiring that the existing open neighborhoods can be computed. For the points we compute basic neighborhoods.

Definition 2 (axioms of computable separation). *For $i \in \{0, 1, 2, 3\}$ define conditions $\mathrm{CT_i}$ as follows.*

CT_0 : *The multi-function t_0 is (δ, δ, ν)-computable where t_0 maps each $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that*

$$(x \in U \text{ and } y \notin U) \text{ or } (x \notin U \text{ and } y \in U). \tag{1}$$

CT_1 : *The multi-function t_1 is (δ, δ, ν)-computable, where t_1 maps each $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that $x \in U$ and $y \notin U$.*

CT_2 : *The multi-function t_2 is $(\delta, \delta, [\nu, \nu])$-computable, where t_2 maps each $(x, y) \in X^2$ such that $x \neq y$ to some $(U, V) \in \beta^2$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$.*

CT_3 : *The multi-function t_3 is $(\delta, \psi^-, [\nu, \theta])$-computable, where t_3 maps each (x, A) such that $x \in X$, $A \subseteq X$ closed, and $x \notin A$ to some $(U, V) \in \beta \times \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $A \subseteq V$.*

Obviously, $\mathrm{CT_i}$ implies $\mathrm{T_i}$. We introduce some further computable T_i-conditions.

Definition 3 (further axioms of computable separation).
WCT_0 : *There is an r.e. set* $H\subseteq\mathrm{dom}(\nu)\times\mathrm{dom}(\nu)$ *such that*

$$(\forall x,y,\ x\neq y)(\exists(u,v)\in H)(x\in\nu(u)\ \wedge\ y\in\nu(v))\quad \textit{and} \tag{2}$$

$$(\forall(u,v)\in H)\begin{cases}\quad\nu(u)\cap\nu(v)=\emptyset\\ \vee\ (\exists x)\,\nu(u)=\{x\}\subseteq\nu(v)\\ \vee\ (\exists y)\,\nu(v)=\{y\}\subseteq\nu(u)\,.\end{cases} \tag{3}$$

SCT_0 : *The multi-function* t_0^s *is* $(\delta,\delta,[\nu_{\mathbb{N}},\nu])$*-computable where* t_0^s *maps each* $(x,y)\in X^2$ *such that* $x\neq y$ *to some* $(k,U)\in\mathbb{N}\times\beta$ *such that* $(k=1,\ x\in U$ *and* $y\notin U)$ *or* $(k=2,\ x\notin U$ *and* $y\in U)$*.*

CT_0' : *There is an r.e. set* $H\subseteq\mathrm{dom}(\nu_{\mathbb{N}})\times\mathrm{dom}(\nu)\times\mathrm{dom}(\nu)$ *such that*

$$(\forall x,y,\ x\neq y)(\exists(w,u,v)\in H)(x\in\nu(u)\ \wedge\ y\in\nu(v))\quad \textit{and} \tag{4}$$

$$(\forall(w,u,v)\in H)\begin{cases}\quad\nu(u)\cap\nu(v)=\emptyset\\ \vee\ \nu_{\mathbb{N}}(w)=1\wedge(\exists x)\,\nu(u)=\{x\}\subseteq\nu(v)\\ \vee\ \nu_{\mathbb{N}}(w)=2\wedge(\exists y)\,\nu(v)=\{y\}\subseteq\nu(u)\,.\end{cases} \tag{5}$$

CT_1' : *There is an r.e. set* $H\in\Sigma^*\times\Sigma^*$ *such that*

$$(\forall x,y,\ x\neq y)(\exists(u,v)\in H)(x\in\nu(u)\wedge y\in\nu(v))\quad \textit{and} \tag{6}$$

$$(\forall(u,v)\in H)\begin{cases}\quad\nu(u)\cap\nu(v)=\emptyset\\ \vee\ (\exists x)\,\nu(u)=\{x\}\subseteq\nu(v)\,.\end{cases} \tag{7}$$

CT_2' : *There is an r.e. set* $H\in\Sigma^*\times\Sigma^*$ *such that*

$$(\forall x,y,\ x\neq y)(\exists(u,v)\in H)(x\in\nu(u)\wedge y\in\nu(v))\quad \textit{and} \tag{8}$$

$$(\forall(u,v)\in H)\begin{cases}\quad\nu(u)\cap\nu(v)=\emptyset\\ \vee\ (\exists x)\,\nu(u)=\{x\}=\nu(v)\,.\end{cases} \tag{9}$$

SCT_2 : *There is an r.e. set* $H\in\Sigma^*\times\Sigma^*$ *such that*

$$(\forall x,y,\ x\neq y)(\exists(u,v)\in H)(x\in\nu(u)\wedge y\in\nu(v))\quad \textit{and} \tag{10}$$

$$(\forall(u,v)\in H)\ \nu(u)\cap\nu(v)=\emptyset\,. \tag{11}$$

CT_3' : *The multi-function* t_3' *is* $(\delta,\nu,[\nu,\psi^-])$*-computable where* t_3' *maps each* $(x,W)\in X\times\beta$ *such that* $x\in W$ *to some* (U,B) *such that* $U\in\beta$*,* $B\subseteq X$ *is closed and* $x\in U\subseteq B\subseteq W$*.*

WCT_3 : *The multi-function* t_3^w *is* (δ,ν,ν)*-computable where* t_3^w *maps each* $(x,W)\in X\times\beta$ *such that* $x\in W$ *to some* U *such that* $U\in\beta$ *and* $x\in U\subseteq\overline{U}\subseteq W$*.*

SCT_3*: There are an r.e. set* $R\subseteq\mathrm{dom}(\nu)\times\mathrm{dom}(\nu)$ *and a computable function* $r:\subseteq\Sigma^*\times\Sigma^*\to\Sigma^\omega$ *such that for all* $u,w\in\mathrm{dom}(\nu)$*,*

$$\nu(w)=\bigcup\{\nu(u)\mid(u,w)\in R\}\,, \tag{12}$$

$$(u,w)\in R\Longrightarrow\ \nu(u)\subseteq\psi^-\circ r(u,w)\subseteq\nu(w)\,. \tag{13}$$

CT'_0, CT'_1 and CT'_2 are versions of CT_0, CT_1 and CT_2, respectively, where base sets are used instead of points (see Theorem 1 below). Similarly, SCT_3 is a pointless version of CT'_3. In contrast to CT_0, in SCT_0 the separating function gives immediate information about the direction of the separation. Also in CT'_0 some information about the direction of the separation is included while no such information is given in its weak version WCT_0. The strong version SCT_2 results from CT'_2 by excluding the case $(\exists x)\,\nu(u) = \{x\} = \nu(v)$. Notice that SCT_2 results also from WCT_0, CT'_0 and CT'_1 by excluding the corresponding cases. The following examples illustrate the definitions. Further examples can be found in Section 4.

Example 1. 1. Consider the computable real line $\mathbf{R} := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ such that $\tau_{\mathbb{R}}$ is the real line topology and ν is a canonical notation of the set of all open intervals with rational endpoints. $\mathbf{R}$ is SCT_3 (easy proof).

2. (T_0 but not WCT_0) Consider the computable lower real line $\mathbf{R}_< := (\mathbb{R}, \tau_<, \beta_<, \nu_<)$, defined by $\nu_<(w) := (\nu_{\mathbb{Q}}; \infty)$, which is T_0 but not T_1. Suppose $\mathbf{R}_<$ is WCT_0. Since for any two base elements U, V, U is not a singleton and $U \cap V \neq \emptyset$, $H = \emptyset$ by (3). But $H \neq \emptyset$ by (2).
3. (T_1 but not T_2 or WCT_0) Let $\mathbf{X} = (\mathbb{N}, \tau, \beta, \nu)$ such that $\tau = \beta$ is the set of cofinite subsets of $\mathbb{N}$ and ν is a canonical notation of ν. Then $\mathbf{X}$ is a computable topological space. It is T_1 since singletons $\{x\}$ are closed. Suppose $\mathbf{X}$ is WCT_0. Since the intersection of base elements cannot be empty and singletons are not open the set H in (3) must be empty. But then (2) cannot be true. The space is not T_2 since the intersection of any two non-empty open set is not empty.

By the next lemma the above computable separation axioms are robust, that is, they do not depend on the notation ν of the base explicitly but only on the computability concept on the points induced by it. Call the computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ equivalent, iff $\delta \equiv \widetilde{\delta}$ [7, Definition 21 and Theorem 22].

Lemma 1. *1. For $i \in \{0,1,2,3\}$ let $\overline{\mathrm{CT_i}}$ be the condition obtained from $\mathrm{CT_i}$ and let $\overline{\mathrm{SCT}_0}$ be the condition obtained from SCT_0 by replacing β and ν by τ and θ, respectively. Then $\overline{\mathrm{CT_i}} \iff \mathrm{CT_i}$ and $\overline{\mathrm{SCT}_0} \iff \mathrm{SCT}_0$.*
2. Let $\widetilde{\mathbf{X}} = (X, \tau, \widetilde{\beta}, \widetilde{\nu})$ be a computable topological space equivalent to $\mathbf{X} = (X, \tau, \beta, \nu)$. Then each separation axiom from Definitions 2 and 3 for $\mathbf{X}$ is equivalent to the corresponding axiom for $\widetilde{\mathbf{X}}$.

The proofs are straightforward. In particular, apply [7, Theorem 22] by which "equivalence" is equivalent to $(\nu \leq \widetilde{\theta}$ and $\widetilde{\nu} \leq \theta)$.

3 Implications

In this section we prove the implications between the separation properties, in the next section we give counterexamples for the proper ones.

Theorem 1.

1. $\mathrm{SCT}_3 \Longrightarrow \mathrm{CT}_3 \Longrightarrow \mathrm{SCT}_2 \Longrightarrow \mathrm{CT}_2 \Longrightarrow \mathrm{CT}_0 \Longrightarrow \mathrm{WCT}_0$,
2. $CT_3 \iff CT_3' \Longrightarrow \mathrm{WCT}_3$,
3. $CT_2 \iff CT_2' \iff CT_1 \iff CT_1'$,
4. $CT_0 \iff SCT_0 \iff CT_0'$,

The proofs of $SCT_0 \Longrightarrow CT_0'$ and $\mathrm{CT}_3' \Longrightarrow \mathrm{SCT}_2$ need some care. They are based on the observation that a realizing machine needs only finitely many steps for finding an appropriate base element for the result. We omit the details (approximately 2 pages).

Surprisingly, computable T_1-spaces are exactly computable T_2. We add some further interesting results. Let "D" be the axiom stating that the topological space is discrete.

Theorem 2. *For computable topological spaces,*

1. *if* $\{x\}$ *is not open for all* $x \in X$ *then* $\mathrm{WCT}_0 \Longrightarrow \mathrm{SCT}_2$,
2. SCT_2 *if* T_2 *and* $\{(u,v) \mid \nu(u) \cap \nu(v) = \emptyset\}$ *is r.e.,*
3. $\mathrm{SCT}_2 \iff (x \neq y$ *is* (δ,δ)*-r.e.),*
4. $\mathrm{CT}_3 \Longrightarrow \mathrm{SCT}_3$ *if the set* $\{w \in \Sigma^* \mid \nu(w) \neq \emptyset\}$ *is r.e.*
5. $\mathrm{D} \Longrightarrow \mathrm{WCT}_3$

We include only the proof of 4. For the terminology see [7].

Proof: Since finite intersection is computable, there is a computable function g such that $\bigcap \nu^{\mathrm{fs}}(w) = \theta \circ g(w)$. Therefore, the set $\{w \in \Sigma^* \mid \bigcap \nu^{\mathrm{fs}}(w) \neq \emptyset\}$ is r.e. There is a machine M such that f_M realizes the multi-function t_3'. If $x = \delta(p) \in \nu(w)$ then for some $u_1 \in \mathrm{dom}(\nu)$ and $q \in \mathrm{dom}(\psi^-)$, $f_M(p,w) = \langle u_1, q\rangle = \iota(u_1)q$ such that

$$x \in \nu(u_1) \subseteq \psi^-(q) \subseteq \nu(w)\,. \tag{14}$$

For computing $\iota(u_1)$ some prefix $u_0 \in \mathrm{dom}(\nu^{\mathrm{fs}}) \cap \Sigma^*11$ of p suffices. Since $\delta(p) \in \nu(w)$ we may assume $w \ll u_0$. Since $x \in \delta[u_0 11 \Sigma^\omega] = \bigcap \nu^{\mathrm{fs}}(u_0)$, $\bigcap \nu^{\mathrm{fs}}(u_0) \neq \emptyset$. We will compute $\bigcap \nu^{\mathrm{fs}}(u_0) \cap \nu(u_1)$ as a union $\bigcup \{\nu(u) \mid u \in L\}$ of base sets and add all these (u,w) to R.

There is a machine N that works on input (u,w) as follows:
(S1) If $u, w \in \mathrm{dom}(\nu)$, $\nu(u) \neq \emptyset$ and $\nu(w) \neq \emptyset$ then
(S2) N searches for words $u_0 \in \mathrm{dom}(\nu^{\mathrm{fs}}) \cap \Sigma^*11$ and $u_1 \in \mathrm{dom}(\nu)$ such that $w \ll u_0$, M on input $(u_0 1^\omega, w)$ writes $\iota(u_1)$ in at most $|u_0|$ steps and $u \ll g(u_0 \iota(u_1))$,
(S3) and then writes all words $\iota(v)$ for which there are words u_2, u_3 such that $u_0 u_2 \in \mathrm{dom}(\nu^{\mathrm{fs}})$, $\bigcap \nu^{\mathrm{fs}}(u_0 u_2) \neq \emptyset$, the machine M on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1) u_3$ in at most $|u_0 u_2|$ steps and $v \ll 11 u_3$. (In order to guarantee an infinite output, N writes 11 from time to time.)
(S4) If (1) is false or the search in (2) is not successful then N computes forever without writing. Let $r := f_N$ and $R := \mathrm{dom}(f_N)$. Then $R \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ and R is r.e. We must prove correctness.

We show (12): Suppose $x = \delta(p) \in \nu(w)$. Then for some u_1, q, $f_M(p, w) = \iota(u_1)q$, hence for some prefix $u_0 \sqsubseteq p$ such that $w \ll u_0$ and $u_0 \in \Sigma^*11$ (since we my assume that p has the subword 11 infinitely often), M on input $(u_0 1^\omega, w)$ writes $\iota(u_1)$ in at most $|u_0|$ steps. Since $x \in \bigcap \nu^{\rm fs}(u_0)$ and $x \in \nu(u_1)$ by (14), $x \in \theta \circ g(u_0 \iota(u_1))$, hence $x \in \nu(u)$ for some $u \ll g(u_0 \iota(u_1))$. Therefore, there is some u such that $x \in \nu(u)$ and the machine N on input (u, w) will find some words such that (S2) is true. Therefore $x \in \nu(u)$ for some $(u, w) \in R$, hence "$\supseteq$" is true in (12).

On the other hand, suppose $(u, w) \in R$ and $x \in \nu(u)$ for some x. Then on input (u, w) the machine N finds words u_0, u_1 such that the conditions in (S2) above are true. Since $u \ll g(u_0 \iota(u_1))$ and $w \ll u_0$, $x \in \nu(u) \subseteq \bigcap \nu^{\rm fs}(u_0) \subseteq \nu(w)$. Therefore, "$\subseteq$" is true in (12).

For showing (13) suppose $(u, w) \in R$ and $x \in \nu(u)$ for some x again. Then on input (u, w) the machine N finds words u_0, u_1 such that the conditions in (S2) above are true. Since $x \in \bigcap \nu^{\rm fs}(u_0)$, $x = \delta(u_0 p')$ for some $p' \in \Sigma^\omega$. Since $x \in \nu(w)$, $f_M(u_0 p', w) = \langle u_1, q \rangle = \iota(u_1)q$ for some $q \in \Sigma^\omega$ such that (14). Suppose $v \ll q$. Then for some u_2, u_3 such that $u_0 u_2 \in {\rm dom}(\bigcap \nu^{\rm fs})$, the machine M on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1) u_3$ in at most $|u_0 u_2|$ steps and $v \ll \iota(u_1) u_3$, therefore, $v \ll r(u, w)$. By (14),

$$\nu(w)^c \subseteq \theta(q) = \bigcup \{\nu(v) \mid v \ll q\} \subseteq \bigcup \{\nu(v) \mid v \in r(u, w)\} = \theta \circ r(u, w).$$

This proves $\psi^- \circ r(u, w) \subseteq \nu(w)$ in(13).

Finally let v be some word such that $\iota(v)$ is listed by the machine N on input (u, w), that is, $v \ll r(u, w)$. Then there are words u_2, u_3 such that $\bigcap \nu^{\rm fs}(u_0 u_2) \neq \emptyset$, the machine M on input $(u_0 u_2 1^\omega, w)$ writes $\iota(u_1) u_3$ in at most $|u_0 u_2|$ steps and $v \ll 11 u_3$. Since $\bigcap \nu^{\rm fs}(u_0 u_2) \neq \emptyset$ and $w \ll u_0$, there is some p' such that $\delta(u_0 u_2 p') \in \nu(w)$ and $f_M(u_0 u_2 p', w) = \iota(u_1) u_3 q'$ for some q'. By (14) $\nu(u_1) \cap \theta(u_3 q') = \emptyset$. Since $\nu(u) \subseteq \nu(u_1)$ (by $u \ll g(u_0 \iota(u_1))$ in (S2)) and $\nu(v) \subseteq \theta(u_3 q')$ (since $v \ll u_3$), $\nu(u) \cap \nu(v) = \emptyset$.

Since this is true for all $v \ll r(u, w)$, $\nu(u) \cap \theta \circ r(u, w) = \emptyset$, hence $\nu(u) \subseteq \psi^- \circ r(u, w)$.

Therefore, we have also proved (13). □

4 Counterexamples

A topological space is discrete iff every singleton $\{x\}$ is open iff every subset $B \subseteq X$ is open. A discrete space is T_i for $i = 0, \ldots, 4$. Let "D" be the axiom stating that the topological space is discrete. Counterexamples show that the implications in Theorem 1.1 are proper. Since this is an extended abstract we include only two of them.

Theorem 3. *For computable topological spaces,*

$\mathrm{T}_0 \not\Longrightarrow \mathrm{WCT}_0$	*by Example 1.2;*
$\mathrm{T}_1 \not\Longrightarrow \mathrm{WCT}_0$	*by Example 1.3;*
$\mathrm{D} \not\Longrightarrow \mathrm{WCT}_0$	*by Example 2;*
$\mathrm{D} + \mathrm{WCT}_0 \not\Longrightarrow \mathrm{CT}_0$	*by Example 3;*
$\mathrm{D} + \mathrm{CT}_0 \not\Longrightarrow \mathrm{CT}_1$	*by Example 4;*
$\mathrm{D} + \mathrm{CT}_2 \not\Longrightarrow \mathrm{SCT}_2$	*by Example 5;*
$\mathrm{WCT}_3 + \mathrm{CT}_2 \not\Longrightarrow \mathrm{SCT}_2$	*by Example 5;*
$\mathrm{T}_4 + \mathrm{SCT}_2 \not\Longrightarrow \mathrm{WCT}_3$	*by Example 7;*
$\mathrm{SCT}_2 \not\Longrightarrow \mathrm{T}_3$	*by Example 6;*
$\mathrm{CT}_3 \not\Longrightarrow \mathrm{SCT}_3$	*by Example 8.*

In the following examples let $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$, ..., $(e_i)_{i\in\mathbb{N}}$ be injective families with pairwise disjoint ranges and let $\{0,1,\dots,7\}\subseteq\Sigma$.

Example 2. (D but not WCT_0) Omitted. □

Example 3. ($D + WCT_0$ but not CT_0) Let $A\subseteq\mathbb{N}$ be some non-r.e. set. Let $X := \{a_i, b_i \mid i \in \mathbb{N}\}$ and let τ be the discrete topology on X. Below we will define sets $B, C, D\subseteq\mathbb{N}$ such that $\{A, B, C, D\}$ is a partition of $\mathbb{N}$. Define a notation ν of a basis β of the topology as follows.

	0^i1	0^i2	0^i3	0^i12	0^i13	0^i23
$i \in A \cup D$	$\{a_i\}$	$\{b_i\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$i \in B$	$\{a_i\}$	$\{a_i, b_i\}$	$\{b_i\}$	$\{a_i\}$	$\emptyset$	$\{b_i\}$
$i \in C$	$\{a_i, b_i\}$	$\{b_i\}$	$\{a_i\}$	$\{b_i\}$	$\{a_i\}$	$\emptyset$

Since $\nu(0^ik)\cap\nu(0^im) = \nu(0^ikm)$, $\nu(u)\cap\nu(v) = \nu\circ g(u,v)$ for some computable function g. Therefore $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space. Let $H := \{(0^ik, 0^jl) \mid i, j \in \mathbb{N};\ k, l \in \{1,2\};\ (i \neq j \vee k \neq l\}$. Then H satisfies (2) and (3) for the space $\mathbf{X}$. Therefore, $\mathbf{X}$ is a WCT_0-space.

We show that $\mathbf{X}$ is not SCT_0.

Let $l, r \in \Sigma^*$ such that $\nu_{\mathbb{N}}(l) = 1$ and $\nu_{\mathbb{N}}(r) = 2$. We assume w.l.o.g. that $\nu_{\mathbb{N}}$ is injective. For $i \in \mathbb{N}$ let

$$S_i := \{\langle l, 0^i1\rangle,\ \langle r, 0^i3\rangle,\ \langle l, 0^i12\rangle,\ \langle r, 0^i23\rangle\},$$
$$T_i := \{\langle r, 0^i2\rangle,\ \langle l, 0^i3\rangle,\ \langle r, 0^i12\rangle,\ \langle l, 0^i13\rangle\}.$$

Suppose, the function $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ realizes the separation function t_0^s for $\mathbf{X}$. If $\delta(p) = a_i$ and $\delta(q) = b_i$ then

$$f(p,q) \in \begin{cases} S_i \text{ if } & i \in B \\ T_i \text{ if } & i \in C \end{cases} \tag{15}$$

since $\nu(u)$ must be either $\{a_i\}$ or $\{b_i\}$ if $f(p,q) = \langle w, u\rangle$. Notice that $S_i \cap T_i = \emptyset$.

For all $i \in \mathbb{N}$ define $p_i, q_i \in \Sigma^\omega$ by $p_i := \iota(0^i1)\iota(0^i1)\iota(0^i1)\ldots$ and $q_i := \iota(0^i2)\iota(0^i2)\iota(0^i2)\ldots$. Let F be the set of all computable functions $f :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $f(p_i, q_i)$ exists for all $i \in A$. Consider $f \in F$. Then $f' : i \mapsto f(p_i, q_i)$ is computable such that $A \subseteq \mathrm{dom}(f')$. Since A is not r.e. and $\mathrm{dom}(f')$ is r.e., $\mathrm{dom}(f') \setminus A$ is infinite. Since F is countable, there is a bijective function $g : E \to F$ for some $E \subseteq \mathbb{N}$ such that $i \in \mathrm{dom}(g_i') \setminus A$ for all $i \in E$ ($g_i := g(i)$). Then $A \cap E = \emptyset$.

For each $i \in E$ we put i to B or C in such a way that g_i does not realize the separating function t_0^s for SCT_0.

$$B := \{i \in E \mid g_i(p_i, q_i) \notin S_i\},$$
$$C := \{i \in E \mid g_i(p_i, q_i) \in S_i\},$$

and $D := \mathbb{N} \setminus (A \cup B \cup C)$. Since $A \cap E = \emptyset$, $E = B \cup C$ and $B \cap C = \emptyset$, $\{A, B, C, D\}$ is a partition of $\mathbb{N}$.

Suppose some computable function f realizes t_0^s. Since for $i \in A, \delta(p_i) = a_i$ and $\delta(q_i) = b_i$, $f(p_i, q_i)$ exists for all $i \in A$, hence $f = g_i$ for some $i \in E$.

If $i \in B$ then $g_i(p_i, q_i) \notin S_i$, hence by (15) the function g_i does not realize t_0^s. If $i \in C$ then $g_i(p_i, q_i) \in S_i$, hence not in T_i since $S_i \cap T_i = \emptyset$. By (15) the function g_i does not realize t_0^s.

From this contradiction we conclude that $\mathbf{X}$ is not SCT_0. By Theorem 1 $\mathbf{X}$ is not CT_0. □

Example 4. (D and CT_0 but not CT_1) Omitted. □

Example 5. (D and CT_2 but not SCT_2) Let $A \subseteq \mathbb{N}$ be an r.e. set with non-r.e. complement. Define a notation ν by
$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{a_i\}$ for $i \in A$,
$\nu(0^i1) := \{a_i\}, \nu(0^i2) := \{b_i\}$ for $i \notin A$
for all $i \in \mathbb{N}$. Then ν is a notation of a base β of a topology (the discrete topology) τ on a subset $X \subseteq \mathbb{N}$ such that $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space.

The space $\mathbf{X}$ is T_i for $i = 0, \ldots, 4$ since it is discrete. It is CT_2 but not SCT_2: The set $H := \{(0^ik, 0^jl) \mid i, j \in \mathbb{N},\ k, l \in \{1, 2\}\}$ satisfies CT_2'. Therefore, the space is CT_2. Suppose SCT_2. Let H be the r.e. set for SCT_2. By (10), $i \notin A \Longrightarrow (0^i1, 0^i2) \in H$ and by (11), $i \in A \Longrightarrow (0^i1, 0^i2) \notin H$. Since H is r.e., the complement of A must be r.e. (contradiction). Notice that $x \neq y$ is not (δ, δ)-r.e., see Theorem 2.3. It can be shown easily that $\mathbf{X}$ is WCT_3. □

Example 6. (SCT_2 but not T_3) Omitted. □

Example 7. (T_4 and SCT_2 but not WCT_3) Omitted. □

Example 8. (CT_3 but not SCT_3) Define a notation I of the open rational intervals by $I\langle u, v\rangle := (\nu_{\mathbb{Q}}(u); \nu_{\mathbb{Q}}(v)) \subseteq \mathbb{R}$. Let $\mathbb{R}_c \subseteq \mathbb{R}$ be the set of (ρ-) computable real numbers. There is a computable function $g : \Sigma^* \to \Sigma^*$ such that $\mathbb{R}_c \subseteq \bigcup_{i \in \mathbb{N}} I \circ g(0^i)$ and $\sum_{i \in \mathbb{N}} \mathrm{length}(I \circ g(0^i)) < 1$ [6, Theorem 4.2.8]. Let $z := \inf\{a \in \mathbb{Q} \mid [a; 1] \subseteq \bigcup_{i \in \mathbb{N}} I \circ g(0^i)\}$. Then $0 < z < 1$, z is $\rho_>$-computable and not ρ-computable, hence not $\rho_<$-computable [6]. Furthermore for all k, $z \notin I \circ g(0^k)$.

Let $X := \mathbb{R}_c \cup \{z\}$. Define a notation ν of subsets of X by $\nu(0v) := I(v) \cap X$ and $\nu(1v) := I(v) \cap (-\infty; z) \cap X$ ($v \in \mathrm{dom}(I)$).Then $\beta := \mathrm{range}(\nu)$ is a base of a topology τ such that $\mathbf{X} := (X, \tau, \beta, \nu)$ is a computable topological space. Notice that for $x < z$, $z \in \mathrm{cls}_X((x; z) \cap X)$. Let δ be the inner representation for the points of $\mathbf{X}$.

Proposition 1: The multi-function $h : x \rightrightarrows a$ mapping each $x \in X$ such that $x < z$ to some $a \in \mathbb{Q}$ such that $x < a < z$ is $(\delta, \nu_{\mathbb{Q}})$-computable.

Proof 1: If $x < z$ and $x \in I \circ g(0^k)$, then $\sup I \circ g(0^k) < z$, since $z \not\leq \inf I \circ g(0^k)$ (since $x < z$), $z \notin I \circ g(0^k)$ and $z \neq \sup I \circ g(0^k)$ (since $z \notin \mathbb{Q}$). There is a machine M that on input p searches for some $k \in \mathbb{N}$ such that $0g(0^k) \ll p$ and writes some u such that $\nu_{\mathbb{Q}}(u) = \sup I \circ g(0^k)$. Let $\delta(p) = x < z$. Since $x \in \mathbb{R}_c$, there is some k such that $x \in I \circ g(0^k)$, hence $0g(0^k) \ll p$. We obtain $\nu_{\mathbb{Q}} \circ f_M(p) < z$. Therefore, the multi-function h is $(\delta, \nu_{\mathbb{Q}})$-computable.

Proposition 2: The multi-function $f : (x, U) \rightrightarrows V$ mapping each $(x, (a; b)) \in X \times \mathrm{range}(I)$ such that $x \in (a; b)$ to some $(c; d) \in \mathrm{range}(I)$ such that $x \in (c; d) \subseteq [c; d] \subseteq (a; b)$ is (δ, I, I)-computable.

Proof 2: Every δ-name of x lists arbitrarily short rational intervals containing x. Search for a sufficiently short interval $(c; d)$.

We show that t_3' from Definition 3 is computable. Suppose $x \in W \in \beta$. If $W = \nu(0w) = I(w) \cap X$ for some w then $W' := I(w)$. If $W = \nu(1w) = I(w) \cap (-\infty; z) \cap X$ for some w then by means of h find some $e \in \mathbb{Q}$ such that $x < e < z$ and let $W' := I(w) \cap (-\infty; e)$. Then $x \in W' \cap X \subseteq W$. By means of f from x and $(a; b) := W'$ find $(c; d) \in \mathrm{range}(I)$ such that $x \in (c; d) \subseteq [c; d] \subseteq (a; b)$. Then $x \in (c; d) \cap X \subseteq [c; d] \cap x \subseteq W$.

From a, b, c and d some u and q can be computed such that $\nu(u) = (c; d) \cap X$ and $\psi^-(q) = [c; d] \cap X$. Then $x \in \nu(u) \subseteq \psi^-(q) \subseteq W$. Therefore, t_3' is $(\delta, \nu, [\nu, \psi^-])$-computable.

Suppose, $\mathbf{X}$ is SCT_3. Let R be the r.e. set for SCT_3 from Definition 3. There is some w such that $\nu(w) = (0; z) \cap X$. Suppose $(u, w) \in R$. Then $\nu(u) \subseteq \nu(w)$, hence for some $a, b \in \mathbb{Q}$ such that $a < b < z$, $\nu(u) = (a; b) \cap X$ or $\nu(u) = (a; z) \cap X$. If $\nu(u) = (a; z) \cap X$, then $z \in \mathrm{cls}_X(\nu(u))$, but $\mathrm{cls}_X(\nu(u)) \subseteq \nu(w) = (0; z)$ by SCT_3, hence $z \in \nu(w) = (0; z)$ (contradiction). Therefore, $\sup \nu(u) = (a; b)$ for some rational numbers a, b such that $a < b < z$.

The function $U \mapsto \sup U$ for all $U = (a; x) \in \beta$ such that $x < z$ is $(\nu, \nu_{\mathbb{Q}})$-computable. Since R is r.e., the number $y := \sup\{\sup \nu(u) \mid (u, w) \in R\}$ is $\rho_<$-computable such that $y \leq z$. Since $(0; z) = \nu(w) = \bigcup_{(u,w) \in R} \nu(u)$, for every $x < z$ there is some $(u, w) \in R$ such that $x < \sup \nu(u)$. Therefore, $y = z$, hence z is $\rho_<$-computable. Contradiction. Therefore, $\mathbf{X}$ is not SCT_3. Notice that $U \neq \emptyset$ is not ν-r.e. □

Further results can be obtained in combination with the positive results from Theorem 1. Figure 1 visualizes the interplay between the computable versions of T_i for $i = 0, 1, 2, 3$ from Definitions 2 and 3 we have proved. "$A \longrightarrow B$" means $A \Longrightarrow B$, "$A \not\longrightarrow B$" means that we have constructed a computable topological space for which $A \wedge \neg B$, and $A \overset{C}{\not\longrightarrow} B$" means that we have constructed a computable topological space for which $(A\wedge C)\wedge\neg B$. Remember that $\mathrm{SCT}_0 \iff \mathrm{CT}_0 \iff \mathrm{CT}'_0$, $\mathrm{CT}_1 \iff \mathrm{CT}'_1 \iff \mathrm{CT}_2 \iff \mathrm{CT}'_2$ and $\mathrm{CT}_3 \iff \mathrm{CT}'_3$.

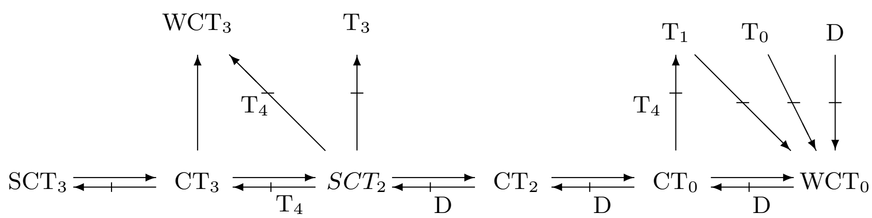

Fig. 1. The relation between computable T_0-, T_1-, T_2- and T_3-separation.

5 Further Results

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ and $B \subseteq X$ the subspace $\mathbf{X}_B = (B, \tau_B, \beta_B, \nu_B)$ of $\mathbf{X}$ to B is the computable topological space defined by $\mathrm{dom}(\nu_B) := \mathrm{dom}(\nu)$, $\nu_B(w) := \nu(w) \cap B$. The separation axioms from Definitions 2 and 3 are invariant under restriction to subspaces.

Theorem 4. *If a computable topological space satisfies some separation axiom from Definitions 2 and 3 then each subspace satisfies this axiom.*

Proof: Straightforward. □

The product of two T_i-spaces is a T_i-space for $i = 0, 1, 2, 3$. This is no longer true for some of the computable separation axioms. By definition for the product $\mathbf{X}_1 \times \mathbf{X}_2 = \overline{\mathbf{X}} = (X_1 \times X_2, \overline{\tau}, \overline{\beta}, \overline{\nu})$ of two computable topological spaces $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X_2, \tau_2, \beta_2, \nu_2)$, $\overline{\nu}\langle u_1, u_2\rangle = \nu_1(u_1) \times \nu_2(u_2)$.

Example 9. The space $\mathbf{X}$ from Example 5 is CT_2 but not SCT_2. Let $\mathbf{R}$ be the computable real line from Example 1.1. We show that the product $\mathbf{X} \times \mathbf{R}$ is not WCT_0. Suppose, $\mathbf{X} \times \mathbf{R}$ is WCT_0. Since every base element of $\mathbf{X} \times \mathbf{R}$ has the form $\nu(u) \times (a; b)$ ($a, b \in \mathbb{Q}$, $a < b$) no singleton $\{(x, y)\}$ ($x \in X$, $y \in \mathbb{R}$) is open. By Theorem 2.1, $\mathbf{X} \times \mathbf{R}$ is SCT_2. By Theorem 1 the relation $(x, x') \neq (y, y')$ is $([\delta, \rho], [\delta, \rho])$-r.e. where δ is the inner representation of the points of $\mathbf{X}$. There is a machine M that halts on input $(\langle p, p'\rangle, \langle q, p'\rangle)$ for $p, q \in \mathrm{dom}(\delta)$ and $p' \in \mathrm{dom}(\rho)$ iff $\delta(p) \neq \delta(q)$. There is a computable element $p' \in \mathrm{dom}(\rho)$. Therefore, there is a machine N that halts on input (p, q) iff $\delta(p) \neq \delta(q)$, hence $x \neq y$ is (δ, δ)-r.e. By Theorem 1, $\mathbf{X}$ must be SCT_2. But $\mathbf{X}$ is not SCT_2.

Theorem 5. *1. The SCT_2-, WCT_3-, CT_3- and SCT_3-spaces are closed under product.*
2. The WCT_0-, CT_0- and CT_2-spaces are not closed under product.

Proof: 1. Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are SCT_2. By Theorem 1, $x_i \neq y_i$ is (δ_i, δ_i)-r.e. for $i = 1, 2$, hence $(x_1, x_2) \neq (y_1, y_2)$ is $([\delta_1, \delta_2], [\delta_1, \delta_2])$-r.e., hence again by Theorem 1, $\mathbf{X}_1 \times \mathbf{X}_2$ is SCT_2.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are WCT_3. Let $(x_1, x_2) \in W_1 \times W_2$. From x_i and W_i we can find $U_i \in \beta_i$ such that $x_i \in U_i \subseteq \overline{U}_i \subseteq W_i$ (for $i = 1, 2$). Then $(x_1, x_2) \in U_1 \times U_2 \subseteq \overline{U_1 \times U_2} = \overline{U}_1 \times \overline{U}_2 \subseteq W_1 \times W_2$.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are CT_3'. We consider computability w.r.t. ν_i, δ_i, ψ_i^-, $\overline{\nu}$, $\overline{\delta}$ and $\overline{\psi}^-$. Suppose $(x_1, x_2) \in (W_1, W_2) \in \beta_1 \times \beta_2$. From $((x_1, x_2), (W_1, W_2))$ we can compute x_1, x_2, W_1 and W_2. Using t_3' for $\mathbf{X_1}$ and $\mathbf{X_2}$ we can compute (U_i, B_i) such hat $U_i \in \beta_i$ $B_i \subseteq X_i$ is closed and $x_i \in U_i \subseteq B_i \subseteq W_i$ $(i = 1, 2)$. Observe that $(x_1, x_2) \in U_1 \times U_2 \subseteq B_1 \times B_2 \subseteq W_1 \times W_2$. Form (U_1, B_1) and (U_2, B_2) we can compute $((u_1, u_2), (B_1, B_2))$.

Suppose, $\mathbf{X}_1$ and $\mathbf{X}_2$ are SCT_3. For $\mathbf{X_i}$ $(i = 1, 2)$ let R_i be the r.e. set and let r_i be the computable function for SCT_3 from Definition 3. There is a computable function h such that $\psi_1^-(p_1) \times \psi_2^-(p_2) = \overline{\psi}^- \langle p_1, p_2 \rangle$. Let

$$\overline{R} := \{(\langle u_1, u_1 \rangle, \langle w_1, w_2 \rangle) \mid (u_1, w_1) \in R_1 \wedge (u_2, w_2) \in R_2\},$$
$$\overline{r}(\langle u_1, u_1 \rangle, \langle w_1, w_2 \rangle) := h(r_1(u_1, w_1), r_2(u_2, w_2)).$$

A straightforward calculation shows that $\overline{R}$ is the r.e. set and $\overline{r}$ be the computable function for SCT_3 from Definition 3 for the product $\mathbf{X}_1 \times \mathbf{X}_2$.

2. In Example 9, the spaces $\mathbf{X}$ and $\mathbf{R}$ are CT_2, CT_0 and WCT_0. Their product $\mathbf{X} \times \mathbf{R}$, however, is not WCT_0, hence not CT_0 and not CT_2. □

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