A Quantitative Characterization of Weighted Kripke Structures in Temporal Logic

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Abstract. We extend the usual notion of Kripke Structures with a weighted transition relation, and generalize the usual Boolean satisfaction relation of CTL to a map which assigns to states and temporal formulae a real-valued distance describing the degree of satisfaction. We describe a general approach to obtaining quantitative interpretations for a generic extension of the CTL syntax, and show that, for one such interpretation, the logic is both *adequate* and *expressive* with respect to quantitative bisimulation.

1 Introduction

We present a general approach to quantitative analysis and approximate characterizations of *weighted Kripke strucures* (WKS) using formulae expressed using a weighted extension of CTL (WCTL). The theory presented here is an extension of a general framwork for quantitative analysis of reactive systems presented in [5].

The goal of [5] was to set the scene for a generic approach to simulation-based analysis, measuring the degree with which one system may simulate another. Developing this paradigm, the current objective is to extend the analysis to verification of *specifications in temporal logic.* Thus we introduce here a quantitative semantics for WCTL which lifts the usual Boolean satisfaction relation of the logic to a function mapping formulae and states into $\mathbb{R}_{\geq 0} \cup \{\infty\}$, and we show that with this semantics, WCTL is both *adequate* and *expressive* with respect to one of the quantitative bisimulation relations introduced in [5].

Using logics for analysis of concurrent and reactive systems is a well-established approach [1], but the standard qualitative techniques are arguably insufficient when reasoning about *quantitative* aspects. Indeed, it can be argued that in a setting where system models and properties include both discrete and continuous, *i.e.* quantitative, information, *e.g.* real-time or probabilistic systems, a quantitative approach is necessary.

The notion of quantitative analysis is closely related to *robustness*, *i.e.* the tolerance for estimation errors and imprecision in order to provide more realistic analysis for real-world applications. Existing work on quantitative logics comparable to ours includes [3] which presents an interpretation with relaxed timing constraints for *timed CTL* and a discounted notion of quantitative CTL where discounting is applied according to the depth of a subformula.

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Another related work is [2], which presents an alternative approach to quantifying versions of LTL and μ-calculus, giving a mapping from states and formulae to the interval $[0, 1]$, where formulae are interpreted over a notion of quantitative transition systems.

In both [2] and [3], quantitative information is only evaluated for atomic propositions, where as path operators are only used to propagate the values obtained at subformulae. Moreover, the semantic interpretations measure only a point-wise property similar to one also discussed in [5], whereas the semantics in the present work accumulates quantitative information based on the paths used to evaluate formulae.

2 Weighted Kripke Structures and Bisimulation

We present a notion of *weighted Kripke structures* (WKS) and bisimulation based measurements for these. The following definition represents a straight forward extension of Kripke structures with weight functions labelling each transition, which may be interpreted as the cost of taking transitions in the structure. This extension is similar to the one presented in [5] for labelled transition systems, thus the results presented in this paper are transferable to our setting.

Definition 1. *For a finite set $\mathcal{AP}$ of atomic propositions, a* weighted Kripke structure *is a quadruple $M = (S, T, \mathcal{L}, w)$ where*

- *S is a finite set of states,*
- *$T \subseteq S \times S$ is a transition relation*
- *$\mathcal{L}\ : S \to 2^{\mathcal{AP}}$ is the proposition labelling, and*
- *$w\ : T \to \mathbb{R}_{\geq 0}$ assigns weights to transitions.*

We write $s \to s'$ instead of $(s, s') \in R$ and $s \xrightarrow{w} s'$ to indicate $w(s, s') = w$.

A *(weighted) path* in a $M = (S, T, \mathcal{L}, w)$ is a (possibly infinite) sequence $\sigma = ((s_0, w_0), (s_1, w_1), (s_2, w_2) \cdots)$ with $(s_i, w_i) \in S \times \mathbb{R}_{\geq 0}$ and such that $s_i \to s_{i+1}$ and $w_i = w(s_i, s_{i+1})$ for all i. We denote by $\mathsf{P}(s)$ the set of paths in M starting at state s. Given path σ, we write $\sigma(i) = (\sigma(i)_s, \sigma(i)_w)$ for its i-th state-weight pair, and σ^i for the suffix starting at $\sigma(i)$.

Notice that we have restricted ourselves to *finite* weighted Kripke structures here, *i.e.* structures with a finite set of states and finitely many atomic propositions. Our characterization results in Section 4 only hold for such finite structures.

2.1 Quantitative Bisimulation

We extend the standard notion of *strong bisimulation* [4] to distances (formally pseudometrics, see below) over WKS, thereby filling the gap between *unweighted* and *weighted* strong bisimulation defined for WKS as follows:

Definition 2. *Let $(S, T, \mathcal{L}, w)$ be a WKS on a set $\mathcal{AP}$ of atomic propositions. A relation $B \subseteq S \times S$ is*

- *an* unweighted bisimulation *provided that for all $(s,t) \in B$, $\mathcal{L}(s) = \mathcal{L}(t)$ and*
  if $s \to s'$, then also $t \to t'$ where $(s', t') \in B$ for some $t' \in S'$,
  if $t \to t'$, also also $s \to s'$ where $(s', t') \in B$ for some $s' \in S$;
- *a* (weighted) bisimulation *provided that for all $(s,t) \in B$, $\mathcal{L}(s) = \mathcal{L}(t)$ and*
  if $s \xrightarrow{c} s'$, then also $t \xrightarrow{c} t'$ and $(s', t') \in B$ for some $t' \in S'$,
  if $t \xrightarrow{c} t'$, then also $s \xrightarrow{c} s'$ and $(s', t') \in B$ for some $s' \in S$.

We write $s \overset{u}{\sim} t$ if $(s,t) \in B$ for some unweighted bisimulation B, and $s \sim t$ if $(s,t) \in B$ for some weighted bisimulation B.

The idea is that, in order to relate structures, we do not always need perfect matching of transition weights, rather it is relevant to know how close weights are matched. Similar to the simulation distances of [5], we call a *bisimulation distance* any pseudometric on the states of a WKS which mediates between unweighted and weighted bisimilarity:

Definition 3. *A* bisimulation distance *on a WKS $(S, T, \mathcal{L}, w)$ is a function $d : S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies the following for all $s_1, s_2, s_3 \in S$:*

- $d(s_1, s_1) = 0$,
- $d(s_1, s_2) + d(s_2, s_3) \geq d(s_1, s_3)$,
- $d(s_1, s_2) = d(s_2, s_1)$,
- $s_1 \sim s_2$ *implies* $d(s_1, s_2) = 0$ *and*
- $d(s_1, s_2) \neq \infty$ *implies* $s_1 \overset{u}{\sim} s_2$

The distance which we shall consider here corresponds to the *accumulated simulation distance* from [5], but we expect that results similar to the ones of this paper also are available for the other distances considered in [5]. Our distance is based on a distance of (infinite) sequences of real numbers, which is appropriate as for (s,t) in $\overset{u}{\sim}$ (or $\sim$), any path $(s, a, s_1, a_1 s_2, \dots) \in \mathsf{P}(s)$ must be matched by an equal-length path $(t, b, t_1, b_1, t_2, \dots) \in \mathsf{P}(t)$ with (s_i, t_i) in $\overset{u}{\sim}$ (respectively $\sim$).

If $a = (a_i)$ and $b = (b_i)$ are sequences representing the weights of such paths, then the following distance measures the *discounted* accumulated sum (in terms of absolute values) of the entries' differences:

$$d_+(a, b) = \sum_i \lambda^i |a_i - b_i| \tag{1}$$

Discounting, with a factor $\lambda \in]0, 1[$, ensures finiteness of such (possibly infinite) sums, by reducing the contribution from each step (difference) exponentially over time. For the remainder of this paper we fix a discounting factor $\lambda \in]0, 1[$.

By extending bisimulation with the d_+ distance, we collect a family of relations $\{\mathcal{R}_\varepsilon \subseteq S \times S\}$ (*i.e.* a map $\mathbb{R}_{\geq 0} \to 2^{S \times S}$) since, due to discounting, for each step the distance between each successor pair may grow:

Definition 4. *A family of relations* $\mathbf{R} = \{\mathcal{R}_\varepsilon \subseteq S \times S \mid \varepsilon > 0\}$ *on a WKS* $(S, T, \mathcal{L}, w)$ *is an* accumulating bisimulation family *provided that for all* $(s,t) \in \mathcal{R}_\varepsilon \in \mathbf{R}$, $\mathcal{L}(s) = \mathcal{L}(t)$ *and*

- *for all* $s \xrightarrow{c} s'$, *also* $t \xrightarrow{d} t'$ *with* $|c - d| \leq \varepsilon$ *for some* $d \in \mathbb{R}_{\geq 0}$ *and* $(s', t') \in \mathcal{R}_{\varepsilon'} \in \mathbf{R}$ *with* $\varepsilon' \leq \frac{\varepsilon - |c-d|}{\lambda}$,
- *for all* $t \xrightarrow{c} t'$, *also* $s \xrightarrow{d} s'$ *with* $|c - d| \leq \varepsilon$ *for some* $d \in \mathbb{R}_{\geq 0}$ *and* $(s', t') \in \mathcal{R}_{\varepsilon'} \in \mathbf{R}$ *with* $\varepsilon' \leq \frac{\varepsilon - |c-d|}{\lambda}$.

We write $s \overset{+}{\sim}_\varepsilon t$ *if* $(s,t) \in \mathcal{R}_\varepsilon \in \mathbf{R}$ *for an accumulating bisimulation family* $\mathbf{R}$.

An accumulating bisimulation family $\mathbf{R}$ gives raise to a bisimulation distance in the sense of Definition 3 by $d(s,t) = \inf\{\varepsilon \mid s \overset{+}{\sim}_\varepsilon t\}$. Observe the following easy facts:

Lemma 1.

1. *For* $\varepsilon \leq \varepsilon'$ *and members* $\mathcal{R}_\varepsilon, \mathcal{R}_{\varepsilon'} \in \mathbf{R}$ *of an accumulating bisimulation family,* $\mathcal{R}_\varepsilon \subseteq \mathcal{R}_{\varepsilon'}$.
2. *Given* $s \overset{+}{\sim}_\varepsilon t$, *then every path* $\sigma = (s_0, w_0, s_1, w_1 s_2, \dots) \in \mathsf{P}(s)$ *has a corresponding path* $\sigma' = (t_0, w_0', t_1, w_1' t_2, \dots) \in \mathsf{P}(t)$ *such that* $\varepsilon = \varepsilon_0$ *and* $s_i \overset{+}{\sim}_{\varepsilon_i} t_i$ *for all* i, *where* $\varepsilon_{i+1} = \frac{\varepsilon_i - |w_i - w_i'|}{\lambda}$.

Note that as we only consider finite WKS, all $\mathcal{R}_\varepsilon$ relations are finite. Also, we shall use the term *correspondence* between paths to denote the second property of the above lemma.

3 Weighted CTL

We now consider a generalization of the well-known CTL formalism to quantities. Our notion of *weighted CTL* (WCTL) is as usual defined in terms of state and path formulae. Notice that our syntactic extensions are restricted to path formulae, which are annotated with real numbers (weights). Satisfaction of a formula by a system is no longer interpreted as a true or false statement, but rather in terms of a real-valued distance. A smaller distance is to mean a closer (better) match of the specified weights in the formula, and 0 denotes the exact match, whereas ∞ indicates an incompatibility between the system and the specified atomic propositions of a formula. Hence in some sense, 0 corresponds to truth and ∞ to falsehood. We will use $[\![\varphi]\!](s) = \varepsilon$ to denote the value $\varepsilon \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ obtained by evaluating φ at state s.

For the remainder of this paper we fix a set $\mathcal{AP}$ of atomic propositions and a WKS $(S, T, \mathcal{L}, w)$. All definitions and results below will be given for the states of one single WKS, but we note that to relate states of different WKS, one can simply form the disjoint union.

Definition 5. *For* $\mathsf{p} \in \mathcal{AP}$*,* Φ *generates the set of state formulae, and* Ψ*, the set of path formulae, annotated by weights* $c \in \mathbb{R}_{\geq 0}$*, according to the following abstract syntax:*

$$\Phi ::= \mathsf{p} \mid \neg\mathsf{p} \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid \mathsf{E}\Psi \mid \mathsf{A}\Psi$$
$$\Psi ::= \mathsf{X}_c\Phi \mid \mathsf{G}_c\Phi \mid \mathsf{F}_c\Phi \mid [\Phi_1 \mathsf{U}_c \Phi_2]$$

The logic WCTL is the set of state formulae, which we denote $\mathcal{L}_w(\mathcal{AP})$ *or simply* $\mathcal{L}_w$*.*

The annotated modalities in the above syntax specify requirements on weights in a WKS. Before discussing these exact requirements, let us consider the usual meaning of the CTL modalities, as well as how these may be generalized to adhere to the type of quantitative analysis considered in the previous section:

Given CTL propositions on the form $M, s \models \mathsf{E}\psi$ and $M, s \models \mathsf{A}\psi$, we may interpret these as infinite *existential*, respectively *universal*, quantifications over paths in M from s satisfying ψ. Similarly, $M, \sigma \models \mathsf{F}\varphi$ and $M, \sigma \models \mathsf{G}\varphi$ may be interpreted as an infinite *disjunction*, respectively *conjunction*, over propositions on the form: $M, s_i \models \varphi$ for $i \geq 0$, where s_i is a state on σ.

Using this observation, we expect that a generic approach to defining quantitative semantics, *i.e.* a function $\mathcal{L}_w \times S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ for WCTL is obtainable. To this end, the standard sup and inf operators are reasonable generalization of E,A,F and G (interpreted as disjunction and conjunction over the standard Boolean domain) to the (complete) lattice $\mathbb{R}_{\geq 0} \cup \{\infty\}$.

Furthermore, this approach requires only modification to the evaluation (*i.e.* semantics) of path formulae. Observe that our semantics below specializes to the usual one in two ways: by mapping a distance $\varepsilon < \infty$ to true and ∞ to false, or by mapping 0 to true and $\varepsilon > 0$ to false.

In the following we present a *discounted accumulating semantics*, designed to match the d_+ distance (1), where weights of transition are accumulated (and discounted). Formally, the semantics of $\varphi \in \mathcal{L}_w$ defines a map from the set of states S to the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$. Given a state formula φ and a state s, an evaluation $[\![\varphi]\!](s) = \varepsilon$ means that s satisfies φ with distance ε. Also, given a path formulae ψ and a path σ, an evaluation $[\![\psi]\!](\sigma) = \varepsilon$ means that ψ holds along σ with distance ε. Conversely, ε describes how close s (or σ) satisfies the specified weights in the formula.

Definition 6. *Let* $\varphi, \varphi_1, \varphi_2$ *be state formulae and* ψ *a path formula. The valuation* $[\![\cdot]\!] : S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ *is defined inductively. For state formulae:*

$$[\![\mathsf{p}]\!](s) = \begin{cases} 0 & \text{if } \mathsf{p} \in \mathcal{L}(s) \\ \infty & \text{otherwise} \end{cases} \qquad [\![\neg\mathsf{p}]\!](s) = \begin{cases} 0 & \text{if } \mathsf{p} \in \mathcal{AP} \setminus \mathcal{L}(s) \\ \infty & \text{otherwise} \end{cases}$$

$$[\![\varphi_1 \vee \varphi_2]\!](s) = \inf\left\{[\![\varphi_1]\!](s), [\![\varphi_2]\!](s)\right\} \qquad [\![\varphi_1 \wedge \varphi_2]\!](s) = \sup\left\{[\![\varphi_1]\!](s), [\![\varphi_2]\!](s)\right\}$$

$$[\![\mathsf{E}\psi]\!](s) = \inf\left\{[\![\psi]\!](\sigma) \mid \sigma \in \mathsf{P}(s)\right\} \qquad [\![\mathsf{A}\psi]\!](s) = \sup\left\{[\![\psi]\!](\sigma) \mid \sigma \in \mathsf{P}(s)\right\}$$

For path formulae:

$$\llbracket \varphi \rrbracket(\sigma) = \llbracket \varphi \rrbracket(\sigma(0)_s)$$

$$\llbracket \mathsf{X}_c \varphi \rrbracket(\sigma) = |c - \sigma(0)_w| + \lambda \llbracket \varphi \rrbracket(\sigma^1)$$

$$\llbracket \mathsf{F}_c \varphi \rrbracket(\sigma) = \inf_k \left(\Big| \sum_{j=0}^{k-1} \lambda^j \sigma(j)_w - c \Big| + \lambda^k \llbracket \varphi \rrbracket(\sigma^k) \right)$$

$$\llbracket \mathsf{G}_c \varphi \rrbracket(\sigma) = \sup_k \left(\Big| \sum_{j=0}^{k-1} \lambda^j \sigma(j)_w - c \Big| + \lambda^k \llbracket \varphi \rrbracket(\sigma^k) \right)$$

$$\llbracket \varphi_1 \mathsf{U}_c \varphi_2 \rrbracket(\sigma) = \inf_k \left(\Big| \sum_{j=0}^{k-1} \lambda^j \llbracket \varphi_1 \rrbracket(\sigma^j) - c \Big| + \lambda^k \llbracket \varphi_2 \rrbracket(\sigma^k) \right)$$

Note again that this interpretation matches the d_+ equation (1). To measure other types of quantitative properties of systems, one may define an alternative semantic valuation for paths.

4 Characterization

In this section we show that WCTL with accumulating semantics is adequate and expressive with respect to accumulating bisimilarity.

4.1 Adequacy

The link between accumulating bisimilarity and our accumulating semantics for WCTL is as follows:

Theorem 1. *For states $s, t \in S$, $s \overset{+}{\sim}_\varepsilon t$ if and only if $\big| \llbracket \varphi \rrbracket(s) - \llbracket \varphi \rrbracket(t) \big| \le \varepsilon$ for all $\varphi \in \mathcal{L}_w$.*

The proof follows from Lemmas 2 and 3 below.

Corollary 1. *For states $s, t \in S$, $s \overset{+}{\sim}_0 t$ if and only if $\llbracket \varphi \rrbracket(s) = \llbracket \varphi \rrbracket(t)$ for all $\varphi \in \mathcal{L}_w$.*

Lemma 2. *Let $s, t \in S$ with $s \overset{+}{\sim}_\varepsilon t$, and let $\sigma_s = (s, u, s_1, u_1, \dots) \in \mathsf{P}(s)$, $\sigma_t = (t, v, t_1, v_1, \dots) \in \mathsf{P}(t)$ be corresponding paths. Then $\big| \llbracket \varphi \rrbracket(s) - \llbracket \varphi \rrbracket(t) \big| \le \varepsilon$ for all state formulae φ, and $\big| \llbracket \varphi \rrbracket(\sigma_s) - \llbracket \varphi \rrbracket(\sigma_t) \big| \le \varepsilon$ for all path formulae φ.*

Proof. We prove the lemma by structural induction in φ. The induction base is clear, as $s \overset{+}{\sim}_\varepsilon t$ implies that $\mathsf{p} \in \mathcal{L}(s)$ if and only if $\mathsf{p} \in \mathcal{L}(t)$, hence $\llbracket \varphi \rrbracket(s) = \llbracket \varphi \rrbracket(t)$ for $\varphi = \mathsf{p}$ or $\varphi = \neg \mathsf{p}$. For the inductive step, we examine each syntactic construction in turn:

1. $\varphi = \varphi_1 \vee \varphi_2$
   There are four cases to consider, corresponding to whether $[\![\varphi_1]\!](s) \leq [\![\varphi_2]\!](s)$ or $[\![\varphi_1]\!](s) > [\![\varphi_2]\!](s)$, and whether $[\![\varphi_1]\!](t) \leq [\![\varphi_2]\!](t)$ or $[\![\varphi_1]\!](t) > [\![\varphi_2]\!](t)$. We show the proof for one of the "mixed" cases; the other three are similar or easier:
   Assume $[\![\varphi_1]\!](s) \leq [\![\varphi_2]\!](s)$ and $[\![\varphi_1]\!](t) > [\![\varphi_2]\!](t)$. Then $[\![\varphi_1 \vee \varphi_2]\!](s) - [\![\varphi_1 \vee \varphi_2]\!](t) = [\![\varphi_1]\!](s) - [\![\varphi_2]\!](t)$, and $[\![\varphi_1]\!](s) - [\![\varphi_1]\!](t) \leq [\![\varphi_1]\!](s) - [\![\varphi_2]\!](t) \leq [\![\varphi_2]\!](s) - [\![\varphi_2]\!](t)$, and by induction hypothesis, $-\varepsilon \leq [\![\varphi_1]\!](s) - [\![\varphi_1]\!](t)$ and $[\![\varphi_2]\!](s) - [\![\varphi_2]\!](t) \leq \varepsilon$.
2. $\varphi = \varphi_1 \wedge \varphi_2$. This is similar to the previous case.
3. $\varphi = \mathsf{E}\varphi_1$
   By definition of $[\![\mathsf{E}\varphi_1]\!]$ there is a path $\sigma \in \mathsf{P}(s)$ for which $[\![\varphi_1]\!](\sigma) = [\![\varphi]\!](s)$. By Lemma 1 there is a corresponding path $\sigma' \in \mathsf{P}(t)$, and from the induction hypothesis we know that $|[\![\varphi_1]\!](\sigma) - [\![\varphi_1]\!](\sigma')| \leq \varepsilon$. Thus $|[\![\varphi]\!](s) - [\![\varphi]\!](t)| \leq \varepsilon$.
4. $\varphi = \mathsf{A}\varphi_1$. This is similar to the previous case.
5. $\varphi = \mathsf{X}_c\varphi_1$
   By definition, $[\![\varphi]\!](\sigma_s) = \lambda[\![\varphi_1]\!](\sigma_s^1) + |c-u|$ and $[\![\varphi]\!](\sigma_t) = \lambda[\![\varphi_1]\!](\sigma_t^1) + |c-v|$ where $\sigma_s = s \xrightarrow{u} \sigma_s^1$ and $\sigma_t = t \xrightarrow{v} \sigma_t^1$. Since $s \overset{+}{\sim}_\varepsilon t$ and σ_s and σ_t correspond, we have $\sigma_s(1) \overset{+}{\sim}_{\varepsilon'} \sigma_t(1)$ with $\varepsilon' \leq \frac{\varepsilon - |u-v|}{\lambda}$, and by induction hypothesis $|[\![\varphi_1]\!](\sigma_t^1) - [\![\varphi_1]\!](\sigma_s^1)| \leq \varepsilon'$. Hence $\big|[\![\varphi]\!](\sigma_s) - [\![\varphi]\!](\sigma_t)\big| \leq \big||c-u| - |c-v|\big| + \lambda\big|[\![\varphi_1]\!](\sigma_s^1) - [\![\varphi_2]\!](\sigma_t^1)\big| \leq |u-v| + \varepsilon - |u-v| = \varepsilon$.
6. $\varphi = \mathsf{F}_c\varphi_1$
   By definition, $[\![\varphi]\!](\sigma_s) = \inf_k \big(\big|\sum_{j=0}^{k-1} \lambda^j \sigma(j)_w - c\big| + \lambda^k [\![\varphi]\!](\sigma^k)\big)$, hence there is a k for which the infimum is obtained. Now as σ_s and σ_t correspond, the infimum for $[\![\varphi]\!](\sigma_t)$ is obtained for the same k. Repeated use of the definition of $\overset{+}{\sim}_\varepsilon$ yields $\sigma_s(k) \overset{+}{\sim}_{\varepsilon'} \sigma_t(k)$ with $\varepsilon' \leq \lambda^{-k}\big(\varepsilon - \sum_{j=0}^{k-1} \lambda^j \big|\sigma_s(j)_w - \sigma_t(j)_w\big|\big)$, and $\big|[\![\varphi]\!](\sigma_s) - [\![\varphi]\!](\sigma_t)\big| \leq \varepsilon$ follows by the triangle inequality as in the previous case.
7. $\varphi = \mathsf{G}_c\varphi_1$. This is similar to the previous case.
8. $\varphi = \varphi_1 \mathsf{U}_c \varphi_2$
   Assume $[\![\varphi]\!](\sigma_s) = \delta$, then by definition there is a k such that $\lambda[\![\varphi_2]\!](\sigma_s^k) = \delta'$ and $\delta = \delta' + \big|\sum_{j=0}^{k-1} \lambda[\![\varphi_1]\!](\sigma_s^j) - c\big|$. Since σ_s and σ_t correspond, so do σ_s^j and σ_t^j for any j. Therefore by induction hypothesis, $\big|[\![\varphi_2]\!](\sigma_s^k) - [\![\varphi_2]\!](\sigma_t^k)\big| \leq \varepsilon$ and $\big|[\![\varphi_1]\!](\sigma_s^j) - [\![\varphi_1]\!](\sigma_t^j)\big| \leq \varepsilon$ for all $0 \leq j \leq k$. Again we can apply the triangle inequality to arrive at $\big|[\![\varphi]\!](\sigma_s) - [\![\varphi]\!](\sigma_t)\big| \leq \varepsilon$.

Lemma 3. *Let $s, t \in S$ and assume that $\big|[\![\varphi]\!](s) - [\![\varphi]\!](t)\big| \leq \varepsilon$ for all state formulae $\varphi \in \mathcal{L}_w$. Then $s \overset{+}{\sim}_\varepsilon t$.*

Proof. This follows directly from Theorem 2, but one can also observe that the family $\mathbf{R} = \{\mathcal{R}_\varepsilon\}$ defined by

$$\mathcal{R}_\varepsilon = \big\{(s,t) \mid s,t \in S, \forall \varphi \in \mathcal{L}_w : \big|[\![\varphi]\!](s) - [\![\varphi]\!](t)\big| \leq \varepsilon\big\}$$

is indeed an accumulating bisimulation in terms of Definition 4.

4.2 Expressivity

We show that WCTL with accumulating semantics is expressive with respect to accumulating bisimulation in the following sense:

Theorem 2. *For each $s \in S$ and every $\gamma \in \mathbb{R}_+$, there exists a state formula $\varphi_\gamma^s \in \mathcal{L}_w$ which characterizes s up to accumulating bisimulation and up to γ, i.e. such that for all $s' \in S$, $s \overset{+}{\sim}_\varepsilon s'$ if and only if $[\![\varphi_\gamma^s]\!](s') \in [\varepsilon - \gamma, \varepsilon + \gamma]$ for all γ.*

Proof. We define characteristic formulae of unfoldings, as follows: For each $s \in S$ and $n \in \mathbb{N}$, denote $\mathcal{L}(s) = \{\mathsf{p}_1, \dots, \mathsf{p}_k\}$ and $\mathcal{AP} \setminus \mathcal{L}(s) = \{\mathsf{q}_1, \dots, \mathsf{q}_\ell\}$ and let $\varphi(s, n)$ be the WCTL formula defined inductively as follows:

$$\varphi(s, 0) = (\mathsf{p}_1 \wedge \dots \wedge \mathsf{p}_k) \wedge (\neg \mathsf{q}_1 \wedge \dots \wedge \neg \mathsf{q}_\ell)$$
$$\varphi(s, n+1) = \bigwedge_{s \xrightarrow{w} s'} \mathsf{EX}_w \varphi(s', n) \;\wedge \bigwedge_{w : s \xrightarrow{w} s'} \mathsf{AX}_w \Big(\bigvee_{s \xrightarrow{w} s'} \varphi(s', n) \Big) \wedge \varphi(s, 0)$$

It is easy to see that $[\![\varphi(s, n)]\!](s) = 0$ for all n.

To complete the proof, one observes that for each $\gamma > 0$, there is $n(\gamma) \in \mathbb{N}$ such that $\varphi(s, n(\gamma))$ can play the role of φ_γ^s in the theorem. Intuitively this is due to discounting: The further the unfolding in $\varphi(s, n)$, the higher are the weights discounted, hence from some $n(\gamma)$ on, maximum weight difference is below γ.

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