

# Column Generation for Bi-Objective Vehicle Routing Problems with a Min-Max Objective

Boadu Mensah Sarpong<sup>1,2</sup>, Christian Artigues<sup>2,3</sup>, and  
Nicolas Jozefowicz<sup>1,2</sup>

1 CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France  
{bmsarpon,artigues,njozefow}@laas.fr

2 Université de Toulouse, INSA, LAAS, F-31400 Toulouse, France

3 Université de Toulouse, LAAS, F-31400 Toulouse, France

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## Abstract

Column generation has been very useful in solving single objective vehicle routing problems (VRPs). Its role in a branch-and-price algorithm is to compute a lower bound which is then used in a branch-and-bound framework to guide the search for integer solutions. In spite of the success of the method, only a few papers treat its application to multi-objective problems and this paper seeks to contribute in this respect. We study how good lower bounds for bi-objective VRPs in which one objective is a min-max function can be computed by column generation. A way to model these problems as well as a strategy to effectively search for columns are presented. We apply the ideas to two VRPs and our results show that strong lower bounds for this class of problems can be obtained in “reasonable” times if columns are intelligently managed. Moreover, the quality of the bounds obtained from the proposed model are significantly better than those obtained from the corresponding “standard” approach.

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## 1 Introduction

Bounds (lower and upper) have been the backbone of methods for solving difficult single objective problems including VRPs. For this reason, it is natural to expect that bounds will also be useful for multi-objective problems. It is, thus, necessary to develop models and strategies for computing good bounds for multi-objective problems. In this paper, we study the use of column generation in computing strong lower and upper bounds for bi-objective VRPs in which one objective is a min-max function. We will use the acronym BOVRPMMO to refer to a problem of this kind.

A BOVRPMMO can be defined by means of a Dantzig-Wolfe decomposition as the selection of a set of columns with minimum total cost such that the maximum value of an attribute associated with the set is minimized. More formally, we consider problems of the



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form:

$$\text{Minimize } \sum_{k \in \Omega} c_k \lambda_k \quad (1)$$

$$\text{Minimize } \Gamma_{\max} \quad (2)$$

$$\text{subject to } \sum_{k \in \Omega} a_{ik} \lambda_k \geq b_i \quad (i \in I), \quad (3)$$

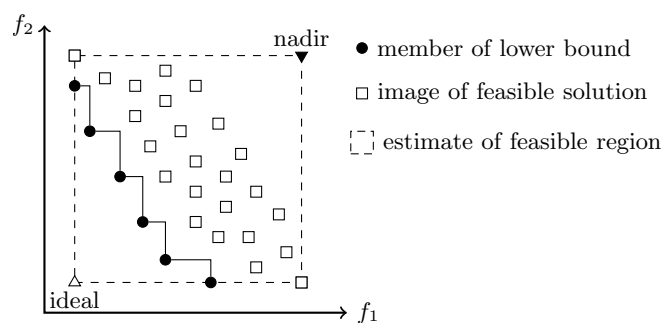
$$\Gamma_{\max} \geq \sigma_k \lambda_k \quad (k \in \Omega), \quad (4)$$

$$\lambda_k \in \{0, 1\} \quad (k \in \Omega), \quad (5)$$

where  $\lambda_k$  and  $\Gamma_{\max}$  are decision variables,  $\Omega$  is the set of all feasible columns whose description depends on the particular problem, and  $I$  is an index set. For each column  $k \in \Omega$ ,  $c_k$  and  $\sigma_k$  are two associated values which we suppose to be integers. We need to select columns with minimum sum of  $c_k$  such that  $\Gamma_{\max} = \max_{k \in \Omega} \{\sigma_k \lambda_k\}$  is also minimized. Bi-objective generalizations of several vehicle routing problems satisfying this condition can be defined. In general, we want to minimize the combined cost of a set of routes such that the value of a property associated with the selected routes (eg. the maximum length of a route, max capacity of a route, etc.) is minimized. We will later present two of such problems namely the bi-objective uncapacitated vehicle routing problem (BOUVRP) and the bi-objective multi-vehicle covering tour problem (BOMCTP).

A BOVRPMO is a special case of a multi-objective combinatorial optimization (MOCO) problem. A general MOCO problem concerns the minimization of a vector of two or more functions  $F(x) = (f_1(x), \dots, f_r(x))$  over a finite domain of feasible solutions  $\mathcal{X}$ . The vector  $x = (x_1, \dots, x_n)$  is the decision variable or solution,  $\mathcal{Y} = F(\mathcal{X})$  corresponds to the images of the feasible solutions in the objective space, and  $y = (y_1, \dots, y_r)$ , where  $y_i = f_i(x)$ , is a point of the objective space. A solution  $x'$  *dominates* another solution  $x''$  if for any index  $i \in \{1, \dots, n\}$ ,  $f_i(x') \leq f_i(x'')$  and there is at least one index  $i \in \{1, \dots, n\}$ , such that  $f_i(x') < f_i(x'')$ . A feasible solution dominated by no other feasible solution is said to be *efficient* or *Pareto optimal* and its image in the objective space is said to be *nondominated*. The set of all efficient solutions is called the *efficient set* (denoted  $\mathcal{X}_E$ ) and the set of all nondominated points is the *nondominated set* (denoted  $\mathcal{Y}_N$ ). Although the meaning of bounds in single objective optimization is well studied and understood, the situation is quite different in the multi-objective case. Ideal and nadir points are well known lower and upper bounds, respectively, of the set  $\mathcal{Y}_N$ . The coordinates of the ideal point are obtained by optimizing each objective function independently of the others, whereas the coordinates of the nadir point correspond to the worse value of each objective function when we consider the set  $\mathcal{X}_E$ . From Figure 1, it can be seen that these points are usually poor bounds since they just estimate the whole region where a member of  $\mathcal{Y}_N$  may lie. In this paper, we will be interested in bounds that can reduce the region where the members of  $\mathcal{Y}_N$  are and thus narrow down the search for nondominated points.

Given that a MOCO problem is a discrete problem, its lower bound can be defined as a finite set of points such that the image of every feasible solution is dominated by at least one of these points [15]. The members of a lower bound set do not necessarily belong to  $\mathcal{Y}$ . An upper bound may also be defined as a finite set of points in  $\mathcal{Y}$  that do not dominate one another. This idea of *bound sets* for bi-objective combinatorial optimization (BOCO) problems has recently been revisited by other authors [3, 6, 14]. They compute strong lower bounds based on a weighted sum scalarization which can then be used in developing exact algorithms for the problems they consider. Although each objective function  $f_i$  of the vector  $F$  can either be a *sum objective* as in (1) or a *min-max objective*, examples given in the



■ **Figure 1** Lower and upper bounds of a bi-objective combinatorial optimization problem.

cited papers consider only the “sum” type of objectives. This in a way justifies the use of a weighted sum method since they can efficiently find supported efficient solutions (those that correspond to points on the convex part of  $\mathcal{Y}_N$ ) by using well known single objective optimization methods. Very good lower bounds for many problems can be defined from the set of supported efficient solutions. The situation is quite different when we consider a combination of a sum and a min-max objective function as in the case of a BOVRPMMO. Indeed, a general linearizing method for the min-max objective destroys the problem structure and so the desirable quality of being able to use known methods for the resulting problem does not necessarily apply [5]. A similar thing happens when we use a standard  $\varepsilon$ -constraint approach although this latter approach can find non-supported solutions which cannot be found by the weighted sum method. Moreover, as shown by results in [6], the quality of the bounds obtained by the weighted sum method for set covering problems are not very good. Nevertheless, the quality of lower bounds produced from set covering based formulations for single objective VRPs are one the best and so we can expect good quality lower bounds from formulations of this type in the multi-objective case too. these reasons, the approach proposed in this paper uses a variant of the  $\varepsilon$ -constraint method applied to a set covering based formulation for the BOVRPMMO.

The main contribution of this work is to define the application of column generation to a BOVRPMMO that does not rely on any “standard” multi-objective technique. In this way, we avoid some drawbacks such as the impossibility to find non-supported solutions by a weighted sum method or the possible loosening of a lower bound by explicitly adding constraints on objectives as it in the case of a standard  $\varepsilon$ -constraint approach. Moreover, if the problem linked to the first objective is a well-studied problem for which an efficient column generation algorithm exists then it is possible to reuse the pricing scheme to solve the bi-objective problems obtained by adding a second objective linked to a property of a selected set of columns. Another contribution is the computation of bound sets by using a scalar method other than the weighted sum method which has been used in published papers (to the best of our knowledge).

The use of column generation to compute bounds for a BOVRPMMO is explained in Section 2. Applications problems are discussed in Section 3. Computational results and conclusions are provided in Sections 4 and 5, respectively.

## 2 Column Generation for a BOVRPMMO

A close examination of formulation (1 – 5) reveals that a BOVRPMMO decomposes naturally into two problems. For any set of feasible columns, the associated value of  $\Gamma_{\max}$  can easily

be computed. We can therefore use a variant of the  $\varepsilon$ -constraint method with one main difference. Instead of explicitly adding a constraint of the form  $\Gamma_{\max} \leq \varepsilon$  to the formulation, we rather drop (4) and use it to redefine the feasibility of a column. Thus, we define a new set of feasible columns  $\bar{\Omega}$  where the feasibility of a column  $k \in \bar{\Omega}$  now depends on its associated value  $\sigma_k$ . Depending whether or not a column  $k \in \bar{\Omega}$  may be associated with more than one value of  $\sigma_k$ , we may have a larger set of feasible columns after the redefinition. The strength of the model is conserved at the expense of having a possibly more difficult problem due to the possible increase in the number feasible columns. The master problem (MP) becomes the following single-objective program:

$$\text{Minimize } \sum_{k \in \bar{\Omega}} c_k \lambda_k \quad (6)$$

$$\text{subject to } \sum_{k \in \bar{\Omega}} a_{ik} \lambda_k \geq b_i \quad (i \in I), \quad (7)$$

$$\lambda_k \in \{0, 1\} \quad (k \in \bar{\Omega}). \quad (8)$$

Before solving MP for a given limit  $\varepsilon$  on the value of  $\Gamma_{\max}$ , we need to set  $\lambda_k = 0$  for all columns  $k \in \bar{\Omega}_k$  having  $\sigma_k > \varepsilon$ . The linear relaxation of MP (ie.  $\lambda_k \geq 0 \forall k \in \bar{\Omega}$ ) is denoted as LMP.

## 2.1 Computing Lower and Upper Bounds

For a BOVRPMMO,  $\Gamma_{\max}$  can only take on a finite number of values. If the complete set of feasible columns  $\bar{\Omega}$  is known, a lower bound can be computed by using a variant of the  $\varepsilon$ -constraint approach as given in Algorithm 1. The algorithm starts with no restriction on the value of  $\sigma_k$  for a column. At each iteration, a linear relaxation of the problem is solved after which the optimal value as well as the value of  $\Gamma_{\max}$  are determined. In the next iteration, the problem is updated to exclude columns  $k$  for which  $\sigma_k$  is greater than  $\Gamma_{\max}$ . This iterative process continues for as long as the problem remains feasible. In practice, the cardinality of  $\bar{\Omega}$  is too large and so a column generation method needs to be used by considering only a subset  $\bar{\Omega}_1$  of  $\bar{\Omega}$ . The restriction of MP to  $\bar{\Omega}_1$  is called the restricted master problem (RMP) and the resulting linear relaxation (ie.  $\lambda_k \geq 0 \forall k \in \bar{\Omega}_1$ ) is denoted LRMP. Let  $\pi_i$  ( $i \in I$ ) be the dual variables associated with LRMP. The subproblem is defined as

$$S(\varepsilon) = \min_{k \in \bar{\Omega} \setminus \bar{\Omega}_1} \left\{ c_k - \sum_{i \in I} \pi_i a_{ik} : \sigma_k \leq \varepsilon \right\}, \quad (9)$$

where  $\varepsilon$  is a maximum allowed value of  $\Gamma_{\max}$  in LRMP. In applying column generation to compute a lower bound, we need to be able to efficiently search for relevant columns and so we explore some strategies in the next subsection.

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### Algorithm 1 Computing a lower bound

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- 1: Set  $lb \leftarrow \emptyset$ .
  - 2: **while** LMP is feasible **do**
  - 3:   Solve LMP. Let  $c^*$  be the optimum, and  $\lambda^*$  be the optimal solution vector.
  - 4:   Compute  $\Gamma_{\max} = \max_{k \in \bar{\Omega}} \sigma_k \lambda_k^*$ .
  - 5:    $lb \leftarrow lb \cup \{(c^*, \Gamma_{\max})\}$ .
  - 6:   Set  $\lambda_k \leftarrow 0$  for all  $k$  such that  $\sigma_k \geq \Gamma_{\max}$ .
  - 7: **end while**
-

After computing a lower bound, a simple way to compute an upper bound is to solve RMP (the integer program) several times by following the idea of Algorithm 1. That is, we consider the RMP with the columns it contains after computing a lower bound and follow Algorithm 1 by replacing LMP with RMP. This is perhaps the simplest column generation heuristic. Although an upper bound is made up of feasible points, they do not necessarily belong to  $\mathcal{Y}_N$  since the RMP may not contain all relevant columns. Nevertheless, if the columns in RMP are relevant for the integer program then we expect that the upper bound produced will be a good approximation of  $\mathcal{Y}_N$ .

## 2.2 Column Search Strategies

A first approach of applying column generation to compute a lower bound of a BOIPMMO follows the idea of a standard  $\varepsilon$ -constraint method. For a fixed value of  $\varepsilon$ , LRMP is solved to optimality by column generation before moving on to another value of  $\varepsilon$ . An iteration of column generation consists in solving LRMP once to obtain a vector of dual values, and then solving the corresponding subproblem to obtain new columns to add to LRMP. The method converges when no new columns are produced from the subproblem. We denote this approach as the “Point-by-Point Search (PPS)”. Although PPS is simple and easy to implement, it takes no advantage of the similarities in the subproblems for the different values of  $\varepsilon$  and so a tremendous amount of computational time may be spent in computing each member of a lower bound.

Using heuristics to generate columns can improve the performance of column generation [4]. These heuristics are used to cheaply generate other relevant columns from those found by a subproblem algorithm. In the bi-objective case, we are interested in heuristics that can take advantage of similarities in the different subproblems solved when computing each point of a lower bound. That is, once the cost of finding a first column has been paid we wish to quickly generate other relevant columns that are relevant for the current subproblem and may also be relevant for other subproblems. A column which has negative reduced cost for a current subproblem, does not necessarily have a negative reduced cost for another subproblem since the associated dual variables do not necessarily have the same values. Nevertheless, it can be expected that two subproblems that are close in terms of objectives, may also be close in terms of the solution of LRMP and therefore close in terms of dual variable values. For this reason, a column generated by a heuristic may also be of negative reduced cost for several other subproblems apart from the current one. In addition, standard algorithms used to solve a subproblem are most times only interested in finding the best columns. This means that many columns having negative reduced costs are left out because the algorithm finds “better” columns. This may be desirable in the single objective case. In the bi-objective case, however, a column which may not be so good for a subproblem may be the best for another subproblem so we are interested in heuristics that can efficiently search for these columns by modifying the ones found by a subproblem algorithm. We denote an approach which incorporate such heuristics as “Improved Point-by-Point Search (IPPS)”. IPPS can be useful as a column generation based heuristic since at each iteration it tries to generate a set of columns that are relevant for several subproblems. IPPS is summarized in Algorithm 2 and the heuristic used in Step 7 obviously depends on the problem at hand.

## 3 Application Problems

In this section, we apply the ideas presented so far to compute lower and upper bounds for two BOVRPMMO. Just as for most VRPs, the subproblem encountered in both examples is

**Algorithm 2** Improved Point-by-Point Search (IPPS)

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1: Set  $\varepsilon \leftarrow \infty$ , and  $lb \leftarrow \emptyset$ .
2: while LRMP is feasible do
3:   Solve LRMP once to obtain a vector of dual values.
4:   Let  $c^*$  be the optimum,  $\lambda^*$  the optimal vector, and compute  $\Gamma_{\max} = \max_{k \in \bar{\Omega}} \sigma_k \lambda_k^*$ .
5:   Solve the subproblem  $S(\varepsilon)$  and let  $\Lambda$  be the set of columns obtained.
6:   if  $|\Lambda| \neq 0$  then
7:     For each column in  $\Lambda$  use heuristics to generate other relevant columns from it.
8:   else
9:      $lb \leftarrow lb \cup \{(c^*, \Gamma_{\max})\}$ .
10:    Set  $\lambda_k \leftarrow 0$  for all  $k$  such that  $\sigma_k \geq \sigma^*$ , and  $\varepsilon \leftarrow \Gamma_{\max} - 1$ .
11:   end if
12: end while

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an elementary shortest path problem with resource constraints (ESPPRC) which we solve by the decremental state space relaxation algorithm (DSSR) [1, 13]. For each considered problem, we discuss the specific implementation of DSSR as well as the heuristic used in implementing IPPS. A complete description of DSSR can be obtained from the two references. We will also be using the same notations introduced earlier and so only their specific meanings for each example will be mentioned.

### 3.1 The Bi-Objective Uncapacitated Vehicle Routing Problem

The bi-objective uncapacitated VRP (BOUVRP) is defined on an undirected graph  $G = (V, E)$  where  $V = \{v_0, \dots, v_n\}$  is a set of nodes and  $E = \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$  is a set of edges. Node  $v_0$  is the depot where all routes should start and end. The other nodes represent  $n$  customers with each having a fixed demand. A distance matrix  $D = (d_{ij})$  which satisfies the triangle inequality is defined on  $E$ . The problem is to design a set of routes with the objectives of minimizing both the total length of all routes and the maximum total demand of customers served by any single route. The demand of each customer is to be met by at least one visiting route and the number of available vehicles as well as the capacity of a vehicle are unlimited.

#### Master Problem and Subproblem

Following the notations used in Section 2, we let  $\Omega$  be the set of all feasible columns. A column  $k \in \Omega$  is a Hamiltonian cycle on a subset of  $V$  which includes the depot. For each column  $k$ ,  $c_k$  is its length and  $\sigma_k$  is the sum of the demands of the customers visited by the route it represents. By redefining the feasibility of a column to take into account the total demand of the customers visited by the route it represents, we obtain a new set of feasible columns  $\bar{\Omega}$ . The master problem (MP) then becomes a capacitated VRP given by:

$$\text{Minimize } \sum_{k \in \bar{\Omega}} c_k \lambda_k \quad (10)$$

$$\text{subject to } \sum_{k \in \bar{\Omega}} a_{ik} \lambda_k \geq 1 \quad (v_i \in V \setminus \{v_0\}), \quad (11)$$

$$\lambda_k \in \{0, 1\} \quad (k \in \bar{\Omega}), \quad (12)$$

where  $a_{ik} = 1$  if the route of column  $k$  visits customer  $v_i$ . As explained in the previous section, the second objective to minimize the total demand of customers served by a single route ( $\Gamma_{\max}$ ) does not appear in the master problem. The subproblem is given by:

$$S(\varepsilon) = \min_{k \in \bar{\Omega} \setminus \Omega_1} \left\{ c_k - \sum_{v_i \in V \setminus \{v_0\}} \pi_i a_{ik} : \sigma_k \leq \varepsilon \right\}, \quad (13)$$

where  $\pi_i$  are dual values and  $\varepsilon$  is a limit imposed on the sum of the demands of customers visited by the same route.

### Subproblem Algorithm

In (13), we are supposed to find feasible routes with negative reduced costs such that the sum of demands of the subset of customers it visits is at most  $\varepsilon$ . The reduced cost of a route is given by its length minus the sum of the profits (dual values) associated to the nodes it visits. The only considered resource when implementing DSSR is the sum of the demands a route visits. This sum cannot be more than  $\varepsilon$ . For two labels  $l_1$  and  $l_2$  arriving at the same node,  $l_1$  dominates  $l_2$  if  $l_1$  has a smaller reduced cost and smaller sum of demands than  $l_2$ . In this situation,  $l_2$  is rejected.

### IPPS Heuristic

Although DSSR is able to return several columns with negative reduced cost, some columns are not generated since they are dominated by other generated ones. Nevertheless, some of these rejected columns have negative reduced costs and may be relevant for a subproblem corresponding to a different value of  $\varepsilon$ . The purpose of the heuristic is to find these kind of routes by using the following idea. Given a column with negative reduced cost, we try to remove a visited node from (or insert a non-visited node in) the route in such a way that we obtain a new route which is still of negative reduced cost but has different values of  $c_k$  and  $\sigma_k$ . Due to the change in the value of  $\sigma_k$ , a new route found in this way can be valid for a different subproblem for which the original route was not valid. This new route may also be of negative reduced cost for a different subproblem.

## 3.2 The Bi-Objective Multi-Vehicle Covering Tour Problem

The bi-objective multi-vehicle covering tour problem (BOMCTP) is an extension of the covering tour problem (CTP) [7]. The CTP consists in designing a route over a subset of locations with the aim of minimizing the length of the route. In addition, each location not visited by the route should lie within a fixed radius of a visited location. The fixed radius is called the *cover distance*. A generic application of the CTP is given in the design of bi-level transportation networks where the aim is to construct a primary route such that all points that are not on it can easily reach it [2]. Other applications are the post box location problem [11] and in the delivery of medical services to villages in developing countries [2, 9]. A bi-objective generalization [10] as well as a multi-vehicle extension [8] of the CTP have been proposed. In the bi-objective version, the cover distance is not fixed in advance but rather induced by the constructed route. It is computed by assigning each non-visited location to the closest visited location and calculating the maximum of these distances. The objectives are to minimize the length of the route as well as the *induced cover distance*. In the multi-vehicle version, the combined length of a set of routes is minimized for a fixed cover distance. The number of locations that a single route can visit is limited by a predetermined constant  $p$ .

The BOMCTP discussed here, is a combination of the bi-objective and the multi-vehicle extensions of the CTP and it is defined on an undirected graph  $G = (V \cup W, E)$ . Set  $V$  represents locations which can be visited by a route whereas the members of  $W$  are to be assigned to visited locations of  $V$ . Node  $v_0 \in V$  is the depot where all routes must start and also end. Set  $E$  consist of edges connecting all pairs of nodes in  $V \cup W$  and a distance matrix  $D = (d_{ij})$  satisfying the triangle inequality is defined on this set. The problem consists in minimizing both the total length of a set of routes constructed over a subset of  $V$  and the induced cover distance.

### Master Problem and Subproblem

Let  $\Omega$  represent the set of all feasible columns. A feasible column  $k \in \Omega$  is defined as a route  $R_k$  which is a Hamiltonian cycle on a subset of  $V$ , includes the depot and visits not more than  $p$  nodes. The length of  $R_k$  is denoted  $c_k$ . For each route  $R_k$ , we choose a subset  $\Psi_k \subseteq W$  of nodes it may cover and define  $\sigma_k$  as the maximum distance between a node of  $\Psi_k$  and the closest node of  $R_k$ . The constant  $a_{ik} = 1$  if  $w_i \in \Psi_k$  and  $a_{ik} = 0$  otherwise. Let  $\Gamma_{\max}$  represent the cover distance induced by a set of routes. Just as before, we define a new set of feasible columns  $\bar{\Omega}$  where the feasibility of a column  $k \in \bar{\Omega}$  depends not just on  $R_k$  but also on  $\sigma_k$ . The master problem (MP) is given by:

$$\text{Minimize } \sum_{k \in \bar{\Omega}} c_k \lambda_k \quad (14)$$

$$\text{subject to } \sum_{k \in \bar{\Omega}} a_{ik} \lambda_k \geq 1 \quad (w_i \in W), \quad (15)$$

$$\lambda_k \in \{0, 1\} \quad (k \in \bar{\Omega}). \quad (16)$$

The function (14) minimizes the total length of the set of routes and (15) ensures that each node of  $W$  is covered by a selected route. The second objective to minimize  $\Gamma_{\max}$  does not appear in the above formulation for the same reasons as before. If  $\pi_i$  are the dual values associated with (15) then the subproblem is

$$S(\varepsilon) = \min_{k \in \bar{\Omega} \setminus \Omega_1} \left\{ c_k - \sum_{w_i \in W} \pi_i a_{ik} : \sigma_k \leq \varepsilon \right\}. \quad (17)$$

### Subproblem Algorithm

Given that  $R_k \subseteq V$  whereas  $\Psi_k \subseteq W$ , we need to construct a route on a subset of  $V$  with the aim of minimizing its length  $c_k$  and also choose a subset of  $W$  with the aim of maximizing the profits ( $\pi_i$ ) associated to its members. The profit associated to a node of  $w_i \in W$  can be collected at most once on any single route even though different nodes of the route may be able to cover  $w_i$ . Two resources are considered in implementing DSSR for this problem. The first is concerned with the number of nodes a route visits which is limited to a maximum of  $p$ . The second resource constraint is that a route may only cover nodes of  $W$  that lie within a radius of  $\varepsilon$  from a node of  $V$  it visits. During the extension of a label, nodes of  $W$  not yet covered by the label but which can be covered are identified and the resulting profit is subtracted from the current reduced cost of the label. Doing so ensures that we obtain the minimum possible reduced cost for each label and this is the goal of the subproblem. Checking whether a label  $l_1$  dominates another label  $l_2$  follows the usual rules when comparing the consumption of resources. When comparing the reduced costs, however, a factor  $F_{12}$  which represents the sum of the profits associated to nodes of  $W$  covered by  $l_1$



but not yet covered by  $l_2$  should be subtracted from the reduced cost of  $l_2$ . This is to ensure that no label that can lead to an optimal solution is eliminated. Similar dominance rules used in dynamic programming algorithms for solving shortest path problems like the one encountered here (called non-additive shortest path problems) are discussed in [12].

### IPPS Heuristic

The subproblem constructs a column  $k$  by taking  $\Psi_k$  to be all the nodes of  $W$  that can be covered by the constructed route  $R_k$ . This helps with the goal of minimizing the reduced cost. We note, however, that  $\Psi_k$  does not necessarily need to include all the nodes of  $W$  that can be covered by  $R_k$ . Indeed,  $\Psi_k$  can be chosen to be any subset of  $W$  such that the sum of the associated profits exceeds the cost  $c_k$ . A column defined in this way is never found by DSSR for the current subproblem since it is dominated by another column defined by the same route, but covers some more nodes of  $W$ . The IPPS heuristic employed here relies on this observation. For each column  $k$  given by  $(R_k, \Psi_k)$  that is found by DSSR, we successively remove the node of  $\Psi_k$  that induces the value of  $\sigma_k$  (i.e. which is farthest from the closest node of  $R_k$ ) in order to create another column  $k'$  with  $c'_k = c_k$  but  $\sigma'_k < \sigma_k$ . We recall that  $\sigma_k = \max\{d_{ij} : v_i \in R_k \text{ and } w_j \in \Psi_k\}$ . A column found in this way can be valid (and possibly have a negative reduced cost) for another subproblem for which the original column returned by DSSR is not valid. This is due to the different value of  $\varepsilon$  each subproblem is based on.

## 4 Computational Results

### Evaluating the Quality of Lower and Upper Bounds

In order to evaluate the quality of the computed lower and upper bound sets, we used a distance based measure ( $\mu_1$ ), and an area based measure ( $\mu_2$ ) which were presented in [6]. Combining an area based measure with a distance based measure gives a better indication of the quality of the bounds. Roughly speaking,  $\mu_1$  represents the worst distance (with respect to the range of objective values) between a point of the upper bound and a point of lower bound closest to it. Also,  $\mu_2$  represents the fraction of the area that is dominated by the lower bound but not by the upper bound. This is, the area where additional points of  $\mathcal{Y}_N$  can be found. If a lower bound and a corresponding upper bound are good then we expect that both  $\mu_1$  and  $\mu_2$  will be small in value. The smaller both values are, the better the quality of the bounds. These two measures complement each other so the quality of the bounds cannot be said to be very good if just one of the measures is small in value but the other is very big. As explained by the author, these measures can be seen to play a role analogous to the optimality gap in single objective optimization. The reader is referred to the relevant paper [6] for further explanation of the measures.

### Experiments

Experiments were conducted to evaluate the quality of lower and upper bounds obtained from the model presented (by redefining the feasibility of a column) with respect to a standard  $\varepsilon$ -constraint model (by explicitly adding constraints on objectives in the master problem). We recall that in the standard  $\varepsilon$ -constraint model, constraints that limit the value of  $\sigma_k$  for a column are directly added to the master problem and nothing special is done in the subproblem. On the other hand, the new model does not explicitly add constraints to the master problem but rather limit the value of  $\sigma_k$  for a column by redefining the meaning

■ **Table 1** Comparison of the quality of bound sets for the BOUVRP.

Instance	Standard				$ lb^* $	PPS			IPPS		
	$ lb $	$ ub $	$\mu_1\%$	$\mu_2\%$		$ ub $	$\mu_1\%$	$\mu_2\%$	$ ub $	$\mu_1\%$	$\mu_2\%$
eil7	2	6	2.60	37.32	6	6	0.00	3.26	6	0.00	3.26
eil13	2	32	2.92	25.22	99	33	0.13	2.72	34	0.10	2.67
eil22	11	36	1.17	11.68	100	41	0.15	2.24	40	0.11	2.11
eil23	2	11	1.48	50.14	130	14	0.30	16.24	17	0.13	15.90
eil30	5	18	9.12	24.30	152	19	1.08	5.32	22	0.98	4.89
eil31	13	44	5.30	10.95	173	41	0.88	2.69	42	0.40	2.65

of a feasible column both in the master problem and subproblem. The principle used to determine the values of  $\varepsilon$  is the same for both models. Given a value for  $\Gamma_{\max}$  in an iteration, we define  $\varepsilon = \Gamma_{\max} - 1$  in the next iteration. We also wanted to compare the quality of the bounds and the computational times of PPS and IPPS. Capacitated VRP instances from the TSPLIB (<http://comopt.ifl.uni-heidelberg.de/software/TSPLIB95/>) having up to 31 nodes were used for the BOUVRP. For the BOMCTP, the Mersenne Twister random number generator (<http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html>) was used to generate instances similar to those described in the literature [7, 8, 10] but which are not publicly available. The node sets were obtained by generating  $|V| + |W|$  points in the  $[0, 100] \times [0, 100]$  square with the depot restricted to lie in  $[25, 75] \times [25, 75]$ . Set  $V$  is taken to be the first  $|V|$  points and set  $W$  is taken as the remaining points. The distance between two points is calculated as the Euclidean distance. Five instances for every combination of  $|V| \in \{40, 50\}$  and  $|W| \in \{2|V|, 3|V|\}$  were generated and values of  $p \in \{5, 8\}$  were tested. The exact instances used for our experiments can be found at [http://homepages.laas.fr/bmsarpon/ctp\\_instances.zip](http://homepages.laas.fr/bmsarpon/ctp_instances.zip). All computer codes were written in C/C++ and the LRMP was solved with ILOG CPLEX 12.4. Tests were run on an Intel(R) Core(TM)2 Duo CPU E7500 @ 2.93GHz computer with a 2 GiB RAM. Summary of results for the BOUVRP are given in Tables 1 and 2 whereas those for the BOMCTP are given in Tables 3 and 4. The values for the BOMCTP are averages over the five instances as explained. The column headings of the tables have the following meaning:  $|lb|$  and  $|ub|$  are the cardinalities of a lower bound and upper bound set, respectively;  $\mu_1\%$  and  $\mu_2\%$  are the values of the quality measures multiplied by 100; *time* is the computational time in cpu seconds; *dssr* is the number of times the sub-problem was solved with DSSR; *cols* is the total number of columns generated. Since PPS and IPPS are based on the same model, the same lower bounds were obtained and this conforms to the theory of column generation which is an exact method for the LMP. The cardinality of the common lower bound is given in the column  $|lb^*|$ .

From the results obtained, we see that the bounds obtained by the model that redefines the feasibility of a column are significantly better than those obtained by a standard  $\varepsilon$ -constraint approach. This is seen by comparing the values of  $\mu_1$  and  $\mu_2$  for PPS and IPPS with their counterparts from “Standard”. For example, in Table 1 a standard  $\varepsilon$ -constraint approach obtained the values ( $\mu_1\% = 9.12, \mu_2\% = 24.30$ ) for the instance eil30 whereas those for PPS and IPPS were ( $\mu_1\% = 1.08, \mu_2\% = 5.32$ ) and ( $\mu_1\% = 0.98, \mu_2\% = 4.89$ ), respectively. A similar thing can be seen in Table 3 for  $p = 8, |V| = 40, |W| = 80$ . The values for a standard  $\varepsilon$ -constraint approach for this instance were ( $\mu_1\% = 8.38, \mu_2\% = 25.86$ ) which are very huge when compared to ( $\mu_1\% = 0.32, \mu_2\% = 3.04$ ) for PPS and ( $\mu_1\% = 0.38, \mu_2\% = 2.46$ ) for IPPS.

■ **Table 2** Comparison of computational times for the BOUVRP.

Instance	Standard			PPS			IPPS		
	time	dssr	cols	time	dssr	cols	time	dssr	cols
eil7	0.1	15	24	0.1	24	29	0.1	13	51
eil13	1.4	60	335	12.3	355	916	8.8	270	1982
eil22	246.3	298	2295	428.7	648	3480	289.6	482	6026
eil23	4216.7	421	3386	7578.5	1238	7946	5979.1	916	12951
eil30	5861.2	541	2976	9746.2	1110	7312	7114.7	828	14922
eil31	2851.0	368	2719	5372.4	957	4514	5027.3	804	7389

■ **Table 3** Comparison of the quality of bound sets for the BOMCTP.

$p$	$ V $	$ W $	Standard				PPS				IPPS		
			$ lb $	$ ub $	$\mu_1\%$	$\mu_2\%$	$ lb^* $	$ ub $	$\mu_1\%$	$\mu_2\%$	$ ub $	$\mu_1\%$	$\mu_2\%$
5	40	80	7	23	6.41	23.22	26	26	1.07	7.69	27	1.01	6.09
5	40	120	8	24	4.67	23.35	27	28	0.91	11.92	29	0.92	10.33
5	50	100	7	28	4.74	21.94	32	33	0.70	9.48	33	0.68	7.85
5	50	150	10	24	4.33	26.44	30	30	1.20	16.67	30	1.20	15.41
8	40	80	7	26	8.38	25.86	27	27	0.32	3.04	27	0.38	2.46
8	40	120	8	28	6.48	21.70	29	30	0.40	4.50	30	0.40	3.70
8	50	100	7	32	6.13	21.83	32	32	0.32	3.83	33	0.31	3.65
8	50	150	9	30	5.00	18.70	30	30	0.31	4.30	30	0.36	3.66

It seems natural that better values for the measures are obtained when the lower and upper bound sets contain more elements. This is probably where a standard  $\varepsilon$ -constraint method falls short since the lower bounds it computes contain very few points in comparison to those computed by the model on which PPS and IPPS are based. Better values of  $\mu_1$  and  $\mu_2$  were obtained for IPPS in comparison to PPS for the tested instances of both the BOUVRP and the BOMCTP. Since the same lower bounds were computed by both approaches for any given instance, we can attribute the better values of the measures for IPPS to the quality of the upper bounds it produces. In terms of computational times, the standard approach is the fastest and this is not so surprising given the number of points it computes and the poor quality of the bounds it produces when compared to the others. When computing the lower bounds, IPPS needs to solve fewer subproblems than PPS (see values for  $dssr$ ) and also generates significantly more columns than (see values for  $cols$ ). The effect is that, the computational times is significantly reduced for IPPS in comparison to PPS. Finally, since the members of an upper bound set correspond to images of feasible solutions, the results mean PPS and IPPS can be used to provide very good approximations of the nondominated set  $\mathcal{Y}_N$  for the class of problems considered.

## 5 Conclusions

This paper discusses the application of column generation to bi-objective VRPs in which one objective is a min-max function. An idea for formulating these problems based on a variant of the  $\varepsilon$ -constraint method is presented. Instead of adding constraints on an objective

■ **Table 4** Comparison of computational times for the BOMCTP.

$p$	$ V $	$ W $	Standard			PPS			IPPS		
			time	dssr	cols	time	dssr	cols	time	dssr	cols
5	40	80	27.5	137	790	49.5	228	1597	40.5	155	2099
5	40	120	38.9	163	1027	126.8	330	2571	94.0	201	3388
5	50	100	68.5	197	1240	205.8	390	3035	153.9	226	3459
5	50	150	42.2	150	875	392.5	486	4053	287.0	247	4054
8	40	80	61.0	217	1498	113.4	302	2283	103.6	218	2961
8	40	120	132.1	299	2205	511.2	481	3949	503.9	293	4663
8	50	100	281.2	326	2398	1343.7	522	4306	1012.5	335	5071
8	50	150	333.0	380	2872	1525.2	672	5799	1186.2	384	6005

in the master problem, we rather redefine the set of feasible columns to take the objective into account. We keep the strength of the model at the expense of a possibly more difficult problem. The advantages of using this model is clearly exhibited from the quality of the bounds obtained from it. We also investigate a strategy to accelerate and improve the column generation method. The proposed ideas are applied to two VRPs and the results obtained indicate that an intelligent management of columns in a multi-objective perspective can yield significant speedups in computing lower and upper bounds. Given that the time needed to compute such quality bounds can be very long, future works are aimed at finding a good compromise between the quality of bounds and the computational time. It will also be interesting to develop branching rules in order to explore the idea of a multi-objective branch-and-price algorithm.

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