

# Complex Semidefinite Programming and Max- $k$ -Cut\*

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## Abstract

In a second seminal paper on the application of semidefinite programming to graph partitioning problems, Goemans and Williamson showed how to formulate and round a *complex semidefinite program* to give what is to date still the best-known approximation guarantee of .836008 for MAX-3-CUT [5]. (This approximation ratio was also achieved independently by De Klerk et al. [2].) Goemans and Williamson left open the problem of how to apply their techniques to MAX- $k$ -CUT for general  $k$ . They point out that it does not seem straightforward or even possible to formulate a good quality complex semidefinite program for the general MAX- $k$ -CUT problem, which presents a barrier for the further application of their techniques.

We present a simple rounding algorithm for the standard semidefinite programming relaxation of MAX- $k$ -CUT and show that it is equivalent to the rounding of Goemans and Williamson in the case of MAX-3-CUT. This allows us to transfer the elegant analysis of Goemans and Williamson for MAX-3-CUT to MAX- $k$ -CUT. For  $k \geq 4$ , the resulting approximation ratios are about .01 worse than the best known guarantees. Finally, we present a generalization of our rounding algorithm and conjecture (based on computational observations) that it matches the best-known guarantees of De Klerk et al.

**1998 ACM Subject Classification** F.2 Analysis of Algorithms and Problem Complexity

**Keywords and phrases** Graph Partitioning, Max- $k$ -Cut, Semidefinite Programming

**Digital Object Identifier** 10.4230/OASICS.SOSA.2018.13

## 1 Introduction

In the MAX- $k$ -CUT problem, we are given an undirected graph,  $G = (V, E)$ , with non-negative edge weights. Our objective is to divide the vertices into at most  $k$  disjoint sets, for some given positive integer  $k$ , so as to maximize the weight of the edges whose endpoints lie in different sets. When  $k = 2$ , this problem is known simply as the MAX-CUT problem. The approximation guarantee of  $1 - 1/k$  can be achieved for all  $k$  by placing each vertex uniformly at random in one of  $k$  sets. For all values of  $k \geq 2$ , this simple algorithm yielded the best-known approximation ratio until 1994. In that year, Goemans and Williamson gave a .87856-approximation algorithm for the MAX-CUT problem based on semidefinite programming (SDP), thereby introducing this method as a successful new technique for designing approximation algorithms [4].

Frieze and Jerrum subsequently developed an algorithm for the MAX- $k$ -CUT problem that can be viewed as a generalization of Goemans and Williamson’s algorithm for MAX-CUT in the sense that it is same algorithm when  $k = 2$  [3]. Although the rounding algorithm of Frieze and Jerrum is arguably simple and natural, the analysis is quite involved. Their

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\* Supported in part by LabEx PERSYVAL-Lab (ANR-11-LABX-0025).



approximation ratios improved upon the previously best-known guarantees of  $1 - 1/k$  for  $k \geq 3$  and are shown in Table 1. A few years later, Andersson, Engebretsen and Håstad also used semidefinite programming to design an algorithm for the more general problem of MAX-E2-LIN MOD  $k$ , in which the input is a set of 2-variable equations or inequations mod  $k$  (e.g.  $x - y \equiv c \pmod{k}$ ) and the objective is to assign an integer from the range  $[0, k - 1]$  to each variable so that the maximum number of equations are satisfied [1]. They proved that the approximation guarantee of their algorithm is at least  $f(k)$  more than that of the simple randomized algorithm, where  $f(k)$  is a (small) linear function of  $k$ . In the special case of MAX- $k$ -CUT, they showed that the performance ratio of their algorithm is no better than that of Frieze and Jerrum. Although they did not show the equivalence of these two algorithms, they stated that numerical evidence suggested that the two algorithms have the same approximation ratio. Shortly thereafter, De Klerk, Pasechnik and Warners presented an algorithm for MAX- $k$ -CUT with improved approximation guarantees for all  $k \geq 3$ , shown in Table 1. Additionally, they showed that their algorithm has the same worst-case performance guarantee as that of Frieze and Jerrum [2].

Around the same time, Goemans and Williamson independently presented another algorithm for MAX-3-CUT based on *complex semidefinite programming* (CSDP) [5]. For this problem, they improved the best-known approximation guarantee of .832718 due to Frieze and Jerrum to .836008, the same approximation ratio proven by De Klerk, Pasechnik and Warners. Goemans and Williamson showed that their algorithm is equivalent to that of Andersson, Engebretsen and Håstad and to that of Frieze and Jerrum (and therefore to that of De Klerk, Pasechnik and Warners) in the case of MAX-3-CUT [5]. However, they argued that their decision to use complex semidefinite programming and, specifically, their choice to represent each vertex by a single complex vector resulted in “cleaner models, algorithms, and analysis than the equivalent models using standard semidefinite programming.”

One issue noted by Goemans and Williamson with respect to their elegant new model was that it is not clear how to apply their techniques to MAX- $k$ -CUT for  $k \geq 4$ . Their approach seemed to be tailored specifically to the MAX-3-CUT problem. This is because one cannot model, say, the MAX-4-CUT problem directly using a complex semidefinite program. This limitation is discussed in Section 8 of [5]. In fact, as they point out, a direct attempt to model MAX- $k$ -CUT with a complex semidefinite program would only result in a  $(1 - 1/k)$ -approximation for  $k \geq 4$ . De Klerk et al. also state that there is no obvious way to extend the approach based on CSDP to MAX- $k$ -CUT for  $k > 3$ . (See page 269 in [2].)

## 1.1 Our Contribution

In this paper, we make the following contributions.

1. We present a simple rounding algorithm based on the standard semidefinite programming relaxation of MAX- $k$ -CUT and show that it can be analyzed using the tools from [5].
  - For  $k = 3$ , this results in an implementation of the Goemans-Williamson algorithm that avoids complex semidefinite programming.
  - For  $k \geq 4$ , the resulting approximation ratios are slightly worse than the best-known guarantees.
2. We present a simple generalization of this rounding algorithm and conjecture that it yields the best-known approximation ratios.

Thus, the main contribution of this paper is to show that, despite its limited modeling power, we can still apply the tools from complex semidefinite programming developed by Goemans and Williamson to MAX- $k$ -CUT. In fact, we obtain the following worst-case

■ **Table 1** Approximation guarantees for MAX- $k$ -CUT.

$k$	[4]	[3]	[5]	[2]	This paper
$k = 2$	.878956	-	-	-	-
$k = 3$	-	.832718	.836008	.836008	-
$k = 4$	-	.850304	-	.857487	.846478
$k = 5$	-	.874243	-	.876610	.862440
$k = 10$	-	.926642	-	.926788	.915885

approximation guarantee for the MAX- $k$ -CUT problem for all  $k$ , which is the same bound they achieve for  $k = 3$ :

$$\phi_k = \frac{k-1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2 \left( \left( \frac{1}{k-1} \right) \cos \left( \frac{2\pi}{k} \right) \right) - \arccos^2 \left( \frac{1}{k-1} \right) \right]. \quad (1)$$

We note that for  $k \geq 4$ , the approximation ratio  $\phi_k$  is about .01 worse than the approximation ratio proved by Frieze and Jerrum. See Table 1 for a comparison. However, given the technical difficulty of Frieze and Jerrum’s analysis, we believe that it is beneficial to present an alternative algorithm and analysis that yields a similar approximation guarantee. Moreover, we wish to take a closer look at the techniques used by Goemans and Williamson for MAX-3-CUT since these tools have not been widely applied in the area of approximation algorithms, in sharp contrast to the tools used to solve the MAX-CUT problem. In fact, we are aware of only two papers that use the main tools of [5]: The first is for a generalization of the MAX-3-CUT problem [6] and the second is for an optimization problem in which the variables are to be assigned complex vectors [8].

While Goemans and Williamson’s framework of complex semidefinite programming does result in an elegant formulation and analysis for MAX-3-CUT, it also to some extent obscures the geometric structure that is apparent when one views the same algorithm from the viewpoint of standard semidefinite programming. Specifically, in the latter framework, their complex semidefinite program is equivalent to modeling each vertex with a 2-dimensional circle or disc of vectors. In our opinion, their main technical contribution is a formula for the exact distribution of the difference of the angles resulting when a normal vector is projected onto two of these discs that are correlated in a particular way. (See Lemma 8 in [5].) Thus, while the limitation in modeling MAX- $k$ -CUT with complex semidefinite programming comes from the fact that we cannot model the general problem with these 2-dimensional discs, we can circumvent this barrier in the following way. We construct 2-dimensional discs using the vectors obtained from a solution to the standard semidefinite program. We then show that a pair of these 2-dimensional discs (i.e. one disc for each vertex) are correlated in the same way as those produced in the case of MAX-3-CUT. Then we can apply and analyze the same algorithm used for MAX-3-CUT.

In some cases, e.g. MAX-3-CUT, using the distribution of the angle between two elements is stronger than using the expected angle, which is what is used for MAX-CUT. It therefore seems that this tool has unexplored potential applications for other optimization problems, for which it may also be possible to overcome the modeling limitations of complex semidefinite programming in a similar manner as we do here. On a high level, the idea of constructing the “complex” vectors from a solution to a standard semidefinite program was used for a circular arrangement problem [7].

Finally, we remark that the approach used in Section 4 to create a disc from a vector is reminiscent of Zwick’s method of outward rotations in which he combines hyperplane

rounding and independent random assignment [9]. For each unit vector  $v_i$  from an SDP solution, he computes a disc in the plane spanned by  $v_i$  and  $u_i$ , where the  $u_i$ 's form a set of pairwise orthogonal vectors that are also orthogonal to the  $v_i$ 's, and chooses a new vector from this disc based on a predetermined angle. Thus, the goal is to rotate each vector  $v_i$  to obtain a new set of unit vectors, which are then given as input to a now standard rounding algorithm, such as random-hyperplane rounding. In contrast, our goal is to use the actual disc in the rounding, as done originally by Goemans and Williamson in the case of MAX-3-CUT.

## 1.2 Organization

We give some background on the (standard) semidefinite programming relaxation used by Frieze and Jerrum and discuss their algorithm for MAX- $k$ -CUT in Section 2. In Section 3, we present Goemans and Williamson's algorithm for MAX-3-CUT from the viewpoint of standard semidefinite programming. In Section 4, we show how to create a 2-dimensional disc for each vertex given a solution to the standard semidefinite program for MAX- $k$ -CUT. We do not wish to formally prove the relationship between these discs and the complex vectors. Thus, in Section 5, we simply prove that if two discs are correlated in a specified way, then the distribution of the angle is equivalent to a distribution already computed exactly by Goemans and Williamson in [5]. We can then easily prove that the 2-dimensional discs we create for the vertices have the required pairwise correlation. This results in a closed form approximation ratio for general  $k$ , Theorem 6.

## 2 Frieze and Jerrum's Algorithm

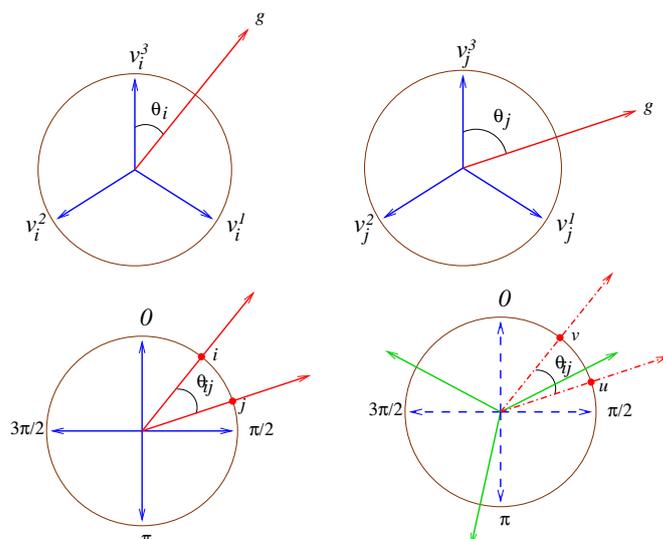
Consider the following integer program for MAX- $k$ -CUT:

$$\begin{aligned} & \max \sum_{ij \in E} (1 - v_i \cdot v_j) \frac{k-1}{k} \\ & v_i \cdot v_i = 1, \quad \forall i \in V, \\ & v_i \in \Sigma_k, \quad \forall i \in V. \end{aligned} \tag{P}$$

Here,  $\Sigma_k$  are the vertices of the equilateral simplex, where each vertex is represented by a  $k$ -dimensional vector, and each pair of vectors corresponding to a pair of vertices has dot product  $-1/(k-1)$ . If we relax the dimension of the vectors, we obtain the following semidefinite relaxation, where  $n = |V|$ :

$$\begin{aligned} & \max \sum_{ij \in E} (1 - v_i \cdot v_j) \frac{k-1}{k} \\ & v_i \cdot v_i = 1, \quad \forall i \in V, \\ & v_i \cdot v_j \geq -\frac{1}{k-1}, \quad \forall i, j \in V, \\ & v_i \in \mathbb{R}^n, \quad \forall i \in V. \end{aligned} \tag{Q}$$

Frieze and Jerrum used this semidefinite relaxation to obtain an algorithm for the MAX- $k$ -CUT problem [3]. Specifically, they proposed the following rounding algorithm: Choose  $k$  random vectors,  $g_1, g_2, \dots, g_k \in \mathbb{R}^n$ , with each entry of each vector chosen from the normal distribution  $\mathcal{N}(0, 1)$ . For each vertex  $i \in V$ , consider the  $k$  dot products of vector  $v_i$  with each of the  $k$  random vectors,  $v_i \cdot g_1, v_i \cdot g_2, \dots, v_i \cdot g_k$ . One of these dot products is maximum. Assign the vertex the label of the random vector with which it has the maximum dot product.



■ **Figure 1** Three vectors  $v_i^1, v_i^2$  and  $v_i^3$  lie on a 2-dimensional plane corresponding to vertex  $i$ . The vector  $g$  is projected onto the disc for element  $i$  to obtain  $\theta_i$ . Angle  $\theta_{ij}$  is the difference between angles  $\theta_i$  and  $\theta_j$ .

In other words, if  $v_i \cdot g_h = \max_{\ell=1}^k \{v_i \cdot g_\ell\}$ , then vertex  $i$  is assigned to cluster  $h$ . Frieze and Jerrum were able to prove a lower bound on the approximation guarantee of this algorithm for every  $k$ . See Table 1 for some of these ratios.

### 3 Goemans-Williamson Algorithm for Max-3-Cut

Goemans and Williamson gave an algorithm for MAX-3-CUT in which they first modeled the problem as a complex semidefinite program, i.e. each element is represented by a complex vector. It is not too difficult to see that these complex vectors are equivalent to 2-dimensional discs or sets of unit vectors. For example, here is an equivalent semidefinite program for MAX-3-CUT. The input is an undirected graph  $G = (V, E)$  with non-negative edge weights  $\{w_{ij}\}$ .

$$\max \sum_{ij \in E} w_{ij} (1 - v_i^1 \cdot v_j^1) \frac{2}{3} \tag{2}$$

$$v_i^a \cdot v_i^b = -1/2, \quad \forall i \in V, a \neq b \in [3], \tag{3}$$

$$v_i^a \cdot v_j^b = v_i^{a+c} \cdot v_j^{b+c}, \quad \forall i, j \in V, a, b, c \in [3], \tag{4}$$

$$v_i^a \cdot v_j^b \geq -1/2, \quad \forall i, j \in V, a, b \in [3], \tag{5}$$

$$v_i^a \cdot v_i^a = 1, \quad \forall i \in V, a \in [3], \tag{6}$$

$$v_i^a \in \mathbb{R}^{3n}, \quad \forall i \in V, a \in [3]. \tag{7}$$

Consider a set of  $3n$  unit vectors forming a solution to this semidefinite program. Note that for a fixed vertex  $i \in V$ , the vectors  $v_i^1, v_i^2$  and  $v_i^3$  are in the same two dimensional plane, since they are constrained to be pairwise  $120^\circ$  apart. In an “integer” solution for this semidefinite program, all these discs would be constrained to be in the same 2-dimensional space and each angle of rotation of the discs would be constrained to be  $0, 2\pi/3$  or  $4\pi/3$ , where each angle would correspond to a partition. In a solution to the above relaxation, these discs are no longer constrained to be in 2 dimensions.

In the rounding algorithm of Goemans and Williamson, we first pick a vector  $g \in \mathbb{R}^{3n}$  such that each entry is chosen according to the normal distribution  $\mathcal{N}(0, 1)$ . Then for each vertex  $i \in V$ , we project this vector  $g$  onto its corresponding disc. This gives an angle  $\theta_i$  in the range  $[0, 2\pi)$  for each element  $i$ . (Note that without loss of generality, we can assume that  $\theta_i$  is the angle in the clockwise direction between the projection of  $g$  and the vector  $v_i^3$ .) We can envision the angles  $\{\theta_i\}$  for each  $i \in V$  embedded onto the same disc. Then we randomly partition this disc into three equal pieces, each of length  $2\pi/3$ , i.e. we choose an angle  $\psi \in [0, 2\pi)$  and let the three angles of partition be  $\psi, \psi + 2\pi/3$  and  $\psi + 4\pi/3$ . These three pieces correspond to the three sets in the partition.

The angle  $\theta_{ij}$  is the angle  $\theta_j - \theta_i$  modulo  $2\pi$ . The probability that an edge  $ij$  is cut in this partitioning scheme is equal to  $3\theta_{ij}/2\pi$  if  $\theta_{ij} < 2\pi/3$  and 1 otherwise. In expectation, the angle  $\theta_{ij}$  is equal to  $\arccos(v_i^1 \cdot v_j^1)$ . (This can be shown using the techniques in [4]. See Lemma 3 in [7].) But using the expected angle is not sufficient to obtain an approximation guarantee better than  $2/3$ ; If angle  $\theta_{ij}$  is  $2\pi/3$  in expectation, then one third of the time it could be zero (not cut) and two thirds of the time it could be  $\pi$  (cut). However, it contributes one to the objective function. The exact probability  $\Pr[\text{edge } ij \text{ is cut}]$  that edge  $ij$  is cut is:

$$\sum_{\gamma=0}^{2\pi/3} \Pr[\theta_{ij} = \theta] \times \frac{\theta}{2\pi/3} + \sum_{\gamma=2\pi/3}^{4\pi/3} \Pr[\theta_{ij} = \theta] + \sum_{\theta=4\pi/3}^{2\pi} \Pr[\theta_{ij} = \theta] \times \frac{2\pi - \theta}{2\pi/3}.$$

Therefore, we must compute  $\Pr[\theta_{ij} = \theta]$  for all  $\theta \in [0, 2\pi)$ . One of the main technical contributions of Goemans and Williamson [5] is that they compute the exact probability that  $\theta_{ij} < \delta$  for all  $\delta \in [0, 2\pi)$ . This can be found in Lemma 8 [5]. This enables them to compute the probability that an edge is cut, resulting in their approximation guarantee.

#### 4 Algorithm for Max- $k$ -Cut

As previously mentioned, we cannot model MAX- $k$ -CUT as an integer program directly using 2-dimensional discs as we do for MAX-3-CUT, because any rotation corresponding to an angle of at least  $2\pi/k$  should contribute one to the objective function. Note that in the case of MAX-3-CUT, there are two possible non-zero rotations in an integer solution:  $2\pi/3$  and  $4\pi/3$  and both of the contribute the same amount (one) to the objective function. Since it seems impossible to penalize all angles greater than  $2\pi/k$  at the same cost, it seems similarly impossible to model the problem directly with a complex semidefinite program.

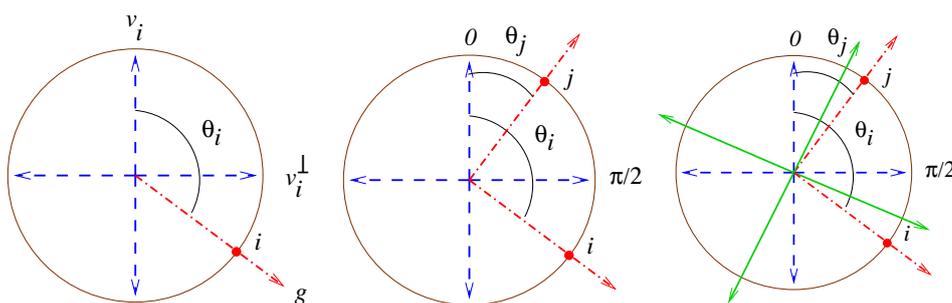
We now present our approach for rounding the semidefinite programming relaxation ( $Q$ ) for MAX- $k$ -CUT. After solving the semidefinite program, we obtain a set of vectors  $\{v_i\}$  corresponding to each vertex  $i \in V$ . We can assume these vectors to be in dimension  $n$ . Let  $\mathbf{0}$  represent the vector with  $n$  zeros. For each vertex  $i \in V$ , we construct the following two orthogonal vectors:

$$v_i := (v_i, \mathbf{0}), \quad v_i^\perp := (\mathbf{0}, v_i). \quad (8)$$

Each vertex  $i \in V$  now corresponds to a 2-dimensional disc spanned by vectors  $v_i$  and  $v_i^\perp$ . Specifically, this 2-dimensional disc consists of the (continuous) set of vectors defined for  $\phi \in [0, 2\pi)$ :

$$v_i(\phi) = v_i \cos \phi + v_i^\perp \sin \phi. \quad (9)$$

Now that we have constructed a 2-dimensional disc for each element, we can use the same rounding scheme due to Goemans and Williamson described in the previous section: First,



■ **Figure 2** A 2-dimensional plane for vertex  $i$  spanning  $v_i$  and  $v_i^\perp$ . After projecting  $\theta_i$  and  $\theta_j$  onto the same disc, we partition the disc into  $k = 4$  equal sized pieces.

we choose a vector  $g \in \mathbb{R}^{2n}$  in which each coordinate is randomly chosen according to the normal distribution  $\mathcal{N}(0, 1)$ . For each  $i \in V$ , we project this vector  $g$  onto the disc  $\{v_i(\phi)\}$ , which results in an angle  $\theta_i$ , where:

$$g \cdot v_i(\theta_i) = \max_{0 \leq \phi < 2\pi} g \cdot v_i(\phi).$$

Note that we do not have to compute infinitely many dot products, since, for example, if  $g \cdot v_i$ ,  $g \cdot v_i^\perp \geq 0$ , then:

$$\theta_i = \arctan\left(\frac{g \cdot v_i^\perp}{g \cdot v_i}\right),$$

and the three other cases depending on the sign of  $g \cdot v_i$  and  $g \cdot v_i^\perp$  can be handled accordingly.

After we find an angle  $\theta_i$  for each  $i \in V$ , we can assign each element to a position corresponding to its angle  $\theta_i$  on a single disc and divide this disc (randomly) into  $k$  equal sections of size  $2\pi/k$ . Specifically, choose a random angle  $\psi$  and use the partition  $\psi + \frac{c \cdot 2\pi}{k}$  for all integers  $c \in [0, k)$ , where angles are taken modulo  $2\pi$ . These are the  $k$  partitions of the vertices in the  $k$ -cut.

## 5 Analysis

We prove that the distribution of the angle  $\theta_{ij}$  is the same as Lemma 8 of [5]. This implies that we can use the analysis that Goemans and Williamson use for MAX-3-CUT to obtain an analogous approximation ratio for MAX- $k$ -CUT.

► **Lemma 1.** *Given two sets of vectors  $x_i = \{x_i(\phi)\}$  and  $x_j = \{x_j(\phi)\}$  defined on  $\phi \in [0, 2\pi)$ , where*

$$\begin{aligned} x_i(\phi) &= (\cos \phi, \sin \phi, 0, 0), \\ x_j(\phi) &= (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta \cos \phi, \sin \theta \sin \phi). \end{aligned}$$

Let  $\gamma \in [0, 2\pi)$  denote the angle  $\theta_j - \theta_i$  after the vector  $g \in \mathcal{N}(0, 1)^{2n}$  is projected onto  $x_i$  and  $x_j$ . Then for  $\delta \in [0, 2\pi)$ ,

$$\Pr[0 \leq \gamma < \delta] = \frac{1}{2\pi} \left( \delta + \frac{r \sin \delta}{\sqrt{1 - r^2 \cos^2 \delta}} \arccos(-r \cos \delta) \right). \quad (10)$$

**Proof.** Note that the set of vectors  $x_j$  is 2-dimensional, since the angle between  $x_j(\phi_1)$  and  $x_j(\phi_2)$  for  $\phi_2 > \phi_1$  is  $\phi_2 - \phi_1$ . Thus, the rounding algorithm in Section 4 is well defined.

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Recall that each coordinate of the vector  $g$  is chosen according to the normal distribution  $\mathcal{N}(0, 1)$ . Even though the vector  $g$  has  $2n$  dimensions, we only need to consider the first four,  $g = (g_1, g_2, g_3, g_4)$ . This vector is chosen equivalently to choosing  $\alpha, \beta$  uniformly in  $[0, 2\pi)$  and  $p_1, p_2$  according to the distribution:

$$f(y) = ye^{-y^2/2}.$$

In other words, the vector  $g$  is equivalent to:

$$g = (p_1 \cos \beta, p_1 \sin \beta, p_2 \cos \alpha, p_2 \sin \alpha).$$

Let  $r = \cos \theta$  and let  $s = \sin \theta$ . We will show that the probability that  $\gamma \in [0, \delta)$  for  $\delta \leq \pi$  is:

$$\Pr[0 \leq \gamma < \delta] = \frac{1}{2\pi} \left[ \delta + \int_{\delta}^{\pi} \Pr \left[ \frac{p_2 \cdot s}{\sin \delta} \leq \frac{p_1 \cdot r}{\sin(\alpha - \delta)} \right] d\alpha \right]. \quad (11)$$

Lemma 8 in [5] shows this is equivalent to probability in (10).

First, let us consider the case when  $\theta \in [0, \pi/2]$ , or  $\cos \theta \geq 0$ . Without loss of generality, assume that the projection of  $g$  onto the 2-dimensional disc  $x_i$  occurs at  $\phi = 0$ . Then we can see that

$$x_i(0) \cdot g = p_1.$$

In other words, we can assume that  $\theta_i = 0$ . As previously mentioned,  $\alpha$  is chosen uniformly in the range  $[0, 2\pi)$ . However, if  $\gamma < \delta$ , then  $\alpha < \pi$ . If  $\alpha < \delta$ , then the projection of  $g$  onto  $x_j$ , namely  $\theta_j$  (which equals  $\theta_{ij}$  in this case, because we have assumed that  $\theta_i = 0$ ), is less than  $\delta$ . The probability that  $\gamma \leq \delta$  if  $\alpha \in [\delta, \pi)$  is equal to the probability that:

$$\begin{aligned} \frac{p_2 \cdot s}{\sin \delta} &\leq \frac{p_1 \cdot r}{\sin(\alpha - \delta)} && \iff \\ p_2 \cdot s &\leq \frac{p_1 \cdot r}{\sin(\alpha - \delta)} \cdot \sin \delta. \end{aligned}$$

(See Figure 3 in [5].) If  $\theta \in (\pi/2, \pi)$  and  $r = \cos \theta < 0$ , then the probability that  $\gamma$  is in  $[0, \delta)$  is the probability that  $\gamma$  is in  $[\pi, \pi + \delta)$ , which is  $\delta/(2\pi)$ . And the probability that  $\gamma$  is in  $[\delta, \pi)$  is the probability that  $\gamma$  is in  $[\pi + \delta, 2\pi)$  for  $-r$ . This is:

$$p_2 \cdot s \leq \frac{p_1 \cdot (-r)}{\sin(\alpha - \delta)} \cdot \sin(\pi + \delta). \quad (12)$$

However, since  $\sin(\pi + \delta) = -\sin \delta$ , we have:

$$p_2 \cdot s \leq \frac{p_1 \cdot r}{\sin(\alpha - \delta)} \cdot \sin \delta. \quad (13)$$

Thus for all  $\delta < \pi$ , we have proved the expression in (11). In Lemma 8 of [5], they show that Equation (11) is equivalent to Equation (10) when  $\delta < \pi$ . Then they argue by symmetry that Equation (10) also holds when  $\pi \leq \delta < 2\pi$ .  $\blacktriangleleft$

► **Lemma 2.** Suppose  $v_i \cdot v_j = \cos \theta$  for two unit vectors  $v_i$  and  $v_j$ . Let  $v_i(\phi)$  and  $v_j(\phi)$  be defined as in equation (9). Then, we can assume that:

$$\begin{aligned} v_i(\phi) &= (\cos \phi, \sin \phi, 0, 0), \\ v_j(\phi) &= (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta \cos \phi, \sin \theta \sin \phi). \end{aligned}$$

**Proof.** From the definition (in Equation (9)) of  $v_i(\phi)$ , we can see that:

$$\begin{aligned} v_i(\phi_1) \cdot v_j(\phi_2) &= (v_i \cos \phi_1 + v_i^\perp \sin \phi_1) \cdot (v_j \cos \phi_2 + v_j^\perp \sin \phi_2) \\ &= v_i \cdot v_j \cos \phi_1 \cos \phi_2 + v_i^\perp \cdot v_j^\perp \sin \phi_1 \sin \phi_2 \\ &\quad + v_i \cdot v_j^\perp \cos \phi_1 \sin \phi_2 + v_i^\perp \cdot v_j \sin \phi_1 \cos \phi_2 \\ &= \cos \theta (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2). \end{aligned}$$

Note that  $v_i \cdot v_j^\perp = v_i^\perp \cdot v_j = 0$  since each  $v_i$  vector has  $n$  zeros in the second half of the entries and each  $v_i^\perp$  vector has  $n$  zeros in the first half of the entries. If we compute  $v_i(\phi_1) \cdot v_j(\phi_2)$  using the assumption in the lemma, then we get the same dot product. Thus, the two sets are equivalent.  $\blacktriangleleft$

Since the distribution of the angle is the same, we can use the same analysis of [5] (generalized from 3 to  $k$ ) to prove the following Lemma. Although it is essentially the exact same proof, we include it here for completeness. As in Corollary 9 of [5], we define:

$$g(r, \delta) = \frac{1}{2\pi} \left( \delta + \frac{r \sin \delta}{\sqrt{1 - r^2 \cos^2 \delta}} \arccos(-r \cos \delta) \right).$$

In other words,  $g(r, \delta)$  is the probability that angle  $\theta_{ij}$  obtained by projecting  $g$  onto the two discs  $\{v_i(\phi)\}$  and  $\{v_j(\phi)\}$ , correlated by  $r = v_i \cdot v_j$ , is less than  $\delta$ .

► **Lemma 3.** *Let  $r = v_i \cdot v_j$  and let  $y_i \in [0, 1, 2, \dots, k)$  be the integer assignment of vertex  $i$  to its partition. Then the probability that the equation  $y_i - y_j \equiv c \pmod{k}$  is satisfied is*

$$\begin{aligned} \frac{1}{k} + \frac{k}{8\pi^2} \left[ 2 \arccos^2 \left( -r \cos \left( \frac{2\pi c}{k} \right) \right) - \arccos^2 \left( -r \cos \left( \frac{2\pi(c+1)}{k} \right) \right) \right. \\ \left. - \arccos^2 \left( -r \cos \left( \frac{2\pi(c-1)}{k} \right) \right) \right]. \end{aligned}$$

**Proof.**  $\Pr[y_i - y_j \equiv c \pmod{k} \text{ satisfied}]$

$$\begin{aligned} &= \frac{k}{2\pi} \int_0^{\frac{k}{2\pi}} \Pr_\gamma \left[ \frac{2\pi c}{k} - \tau \leq \gamma < \frac{2\pi(c+1)}{k} - \tau \right] d\tau \\ &= \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} \left( g \left( \tau, \frac{2\pi(c+1)}{k} - \tau \right) - g \left( \tau, \frac{2\pi c}{k} - \tau \right) \right) d\tau \\ &= \frac{k}{2\pi} \int_{\frac{2\pi c}{k}}^{\frac{2\pi(c+1)}{k}} g(r, \nu) d\nu - \frac{k}{2\pi} \int_{\frac{2\pi(c-1)}{k}}^{\frac{2\pi c}{k}} g(r, \nu) d\nu \end{aligned}$$

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$$\begin{aligned}
&= \frac{k}{2\pi} \frac{1}{2\pi} \left( \int_{\frac{2\pi c}{k}}^{\frac{2\pi(c+1)}{k}} \nu d\nu - \left[ \frac{1}{2} \arccos^2(-r \cos \nu) \right]_{\frac{2\pi c}{k}}^{\frac{2\pi(c+1)}{k}} \right. \\
&\quad \left. - \int_{\frac{2\pi(c-1)}{k}}^{\frac{2\pi c}{k}} \nu d\nu + \left[ \frac{1}{2} \arccos^2(-r \cos \nu) \right]_{\frac{2\pi(c-1)}{k}}^{\frac{2\pi c}{k}} \right) \\
&= \frac{k}{8\pi^2} \left[ \left( \frac{2\pi(c+1)}{k} \right)^2 + \left( \frac{2\pi(c-1)}{k} \right)^2 - 2 \left( \frac{2\pi c}{k} \right)^2 \right] \\
&\quad + \frac{k}{8\pi^2} \left[ 2 \arccos^2 \left( -r \cos \left( \frac{2\pi c}{k} \right) \right) \right. \\
&\quad \left. - \arccos^2 \left( -r \cos \left( \frac{2\pi(c+1)}{k} \right) \right) - \arccos^2 \left( -r \cos \left( \frac{2\pi(c-1)}{k} \right) \right) \right] \\
&= \frac{1}{k} + \frac{k}{8\pi^2} \left[ 2 \arccos^2 \left( -r \cos \left( \frac{2\pi c}{k} \right) \right) \right. \\
&\quad \left. - \arccos^2 \left( -r \cos \left( \frac{2\pi(c+1)}{k} \right) \right) - \arccos^2 \left( -r \cos \left( \frac{2\pi(c-1)}{k} \right) \right) \right]. \quad \blacktriangleleft
\end{aligned}$$

► **Lemma 4.** Let  $r = v_i \cdot v_j$ . The probability that edge  $ij$  is not cut by our algorithm is:

$$\frac{1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2(-r) - \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) \right].$$

**Proof.** In the case of MAX- $k$ -CUT, we set  $c = 0$ . By Lemma 3, we have the probability that edge  $ij$  is not cut is:

$$\begin{aligned}
&\frac{1}{k} + \frac{k}{8\pi^2} \left[ 2 \arccos^2(-r) - \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) - \arccos^2 \left( -r \cos \left( -\frac{2\pi}{k} \right) \right) \right] \\
&= \frac{1}{k} + \frac{k}{8\pi^2} \left[ 2 \arccos^2(-r) - 2 \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) \right] \\
&= \frac{1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2(-r) - \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) \right]. \quad \blacktriangleleft
\end{aligned}$$

► **Lemma 5.** Let  $r = v_i \cdot v_j$ . The probability that edge  $ij$  is cut by our algorithm is:

$$\frac{k-1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2 \left( -r \cdot \cos \left( \frac{2\pi}{k} \right) \right) - \arccos^2(-r) \right]. \quad (14)$$

**Proof.** By Lemma 4 and the previously stated assumption that  $r = v_i \cdot v_j = \cos(\theta_{ij})$ , we have:

$$\begin{aligned}
&1 - \left[ \frac{1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2(-r) - \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) \right] \right] \\
&= \frac{k-1}{k} - \frac{k}{4\pi^2} \left[ \arccos^2(-r) - \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) \right] \\
&= \frac{k-1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2 \left( -r \cos \left( \frac{2\pi}{k} \right) \right) - \arccos^2(-r) \right]. \quad \blacktriangleleft
\end{aligned}$$

► **Theorem 6.** The worst case approximation ratio of our algorithm for MAX- $k$ -CUT is:

$$\phi_k = \frac{k-1}{k} + \frac{k}{4\pi^2} \left[ \arccos^2 \left( \left( \frac{1}{k-1} \right) \cos \left( \frac{2\pi}{k} \right) \right) - \arccos^2 \left( \frac{1}{k-1} \right) \right].$$

**Proof.** As a function of  $r$  in the range  $[1, -1/(k-1)]$ , the expression in Equation 14 is minimized when  $r = -1/(k-1)$ . Thus, if we do an edge-by-edge analysis, the worst case approximation ratio is obtained when  $v_i \cdot v_j = -1/(k-1)$  for all edges  $ij \in E$ .  $\blacktriangleleft$

## 6 Another Rounding Algorithm

The algorithm presented in Section 4 can be restated as the following rounding scheme. Let  $w_1, w_2$  and  $w_3$  denote vectors in  $\mathbb{R}^2$  with pairwise dot product  $-1/2$ . In other words,  $w_1, w_2$  and  $w_3$  are the vertices of the simplex  $\Sigma_3$ . Now take two random gaussians  $g_1, g_2 \in \mathbb{R}^n$  and set  $x_i = g_1 \cdot v_i, y_i = g_2 \cdot v_i$ . To assign the vertex  $i$  to one of the three partitions, we simply assign it to  $j$  such that  $w_j \cdot (x_i, y_i)$  is maximized.

We can generalize this approach by choosing  $k - 1$  random gaussians,  $g_1, \dots, g_{k-1}$ . For each vertex  $i$ , we obtain the vector  $(g_1 \cdot v_i, g_2 \cdot v_i, \dots, g_{k-1} \cdot v_i)$  in  $\mathbb{R}^{k-1}$ . This vector is assigned to the closest vertex of  $\Sigma_k$ . Computationally, this rounding scheme seems to yield approximation ratios that match those of De Klerk et al.

**Acknowledgements.** Thanks to Moses Charikar, Anupam Gupta, R. Ravi and Madhur Tulsiani for helpful discussions and comments on the presentation.

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