Semiring Provenance in the Infinite

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Abstract

Semiring provenance evaluates database queries or logical statements not just by true or false but by values in some commutative semiring. This permits to track which combinations of atomic facts are responsible for the truth of a statement, and to derive further information, for instance concerning costs, confidence scores, number of proof trees, or access levels to protected data. The focus of this approach, proposed and developed to a large extent by Val Tannen and his collaborators, has first been on (positive) database query languages, but has later been extended, again in collaboration with Val, to a systematic semiring semantics for first-order logic (and other logical systems), as well as to a method for the strategy analysis of games.

So far, semiring provenance has been studied for finite structures. To extend the semiring provenance approach for first-order logic to infinite domains, the semirings need to be equipped with addition and multiplication operators over infinite collections of values. This needs solid algebraic foundations, and we study here the necessary and desirable properties of semirings with infinitary operations to provide a well-defined and informative provenance analysis over infinite domains. We show that, with suitable definitions for such infinitary semiring, large parts of the theory of semiring provenance can be succesfully generalised to infinite structures.

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1 Introduction

Semiring provenance was proposed in 2007 by Val Tannen in the seminal paper [7], together with Todd Green and Grigoris Karvounarakis. It is based on the idea to annotate the atomic facts in a database by values in some commutative semiring, and to propagate these values through a database query, keeping track whether information is used alternatively or jointly. This approach has been successfully applied to many variants of database queries, including conjunctive queries, positive relational algebra, datalog, nested relations, XML, SQL-aggregates, graph databases (see, e.g., the surveys [8, 3]). Depending on the chosen semiring, provenance valuations give practical information about a query, beyond its truth or falsity, for instance concerning the *confidence* that we may have in its truth, the *cost* of its evaluation, the number of successful evaluation strategies, and so on. Beyond such provenance evaluations in specific *application semirings*, more precise information is obtained by evaluations in *provenance semirings* of polynomials or formal power series, which permit us to *track* which atomic facts are used (and how often) to compute the answer to the query.

While semiring provenance had for a long time been restricted to negation-free query languages, a new approach for dealing with negation has been proposed in 2017 by Grädel and Tannen [4], based on transformations into negation normal form, quotient semirings

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of polynomials with dual indeterminates, and a close relationship to semiring valuations of games. In particular, this provides a semiring provenance analysis for full first-order logic (over finite domains). Further such a provenance analysis can be applied to many other logics and query languages with negation, and also permits a reverse provenance analysis, i.e., finding models that satisfy various properties under given provenance tracking assumptions, with potential applications to explaining missing query answers or failures of integrity constraints, and to using these explanations for computing repairs. An updated exposition of this approach can be found in [6].

If we investigate semiring provenance, beyond applications to finite data, as a general semiring-based semantics for first-order logic (and other logical systems), the question arises whether this semantics also makes sense over infinite domains, and what properties of the underlying semirings are needed to make such an extension possible and meaningful. This is the question that we want to study in this paper.

The obvious problem is the interpretation of quantifiers. A semiring interpretation π, over the universe A, assigns to a formula $\psi(\bar{a})$ a value $\pi[\![\psi(\bar{a})]\!]$ in some commutative semiring $\mathcal{S}$. This is defined by induction on ψ, and for the quantifiers, we have that

$$\pi[\![\exists x\varphi(x,\bar{b})]\!] := \sum_{a\in A}\pi[\![\varphi(a,\bar{b})]\!] \qquad \text{and} \qquad \pi[\![\forall x\varphi(x,\bar{b})]\!] := \prod_{a\in A}\pi[\![\varphi(a,\bar{b})]\!],$$

so for infinite universes, we need to equip the semirings with infinitary addition and multiplication operations, with suitable algebraic properties.

In some cases this is completely straightforward and unproblematic, for instance for finite min-max semirings or, more generally, for semirings induced by some complete lattice (with suprema and infima as semiring operations). There are other semirings, for instance the natural semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$, which do not admit infinitary operations, but which can be easily completed to one that does so, such as $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ where there is an obvious natural definition for infinitary addition and multiplication. But such extensions are not always obvious, for instance for semirings of polynomials. Further there are important semirings, such as the tropical semiring $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ where the definition of the infinitary operations (here infimum and infinitary sum) is obvious, but it is not clear whether all relevant algebraic properties of the semiring operations also hold for their infinitary versions. In the case of the tropical semiring, we shall see that most of the basic algebraic properties do generalise to the infinite, with the exception of the distributive law which, in its strong form, does only hold on countable domains, but not on uncountable ones. Of course this poses the questions, what algebraic properties of infinitary semiring operations are actually needed for a well-defined and meaningful semiring semantics. We shall systematically study necessary and desirable algebraic properties of such infinitary operations and, on this basis, propose a definition of infinitary semirings. We will discuss examples of such semirings, focussing on one side on the case of absorptive infinitary semirings and, on the other side, define extensions of the polynomial semiring $\mathbb{N}[X]$ to a semiring of generalised power series, for which we can establish a universality property, similar to the one of $\mathbb{N}[X]$ in the finite case.

Using these infinitary semirings, we shall discuss semiring provenance for first-order logic on possibly infinite structures and show that a large part of the theory developed for the finite case does indeed carry over to infinite domains. In particular, we establish that the Sum-of-Proof-Trees-Theorem, saying that the semiring valuation of a first-order sentence coincides with the sum of the valuations of its proof trees also holds on infinite domains, provided that the underlying infinitary semiring satisfies an appropriate distributivity property.

2 Commutative Semirings

▶ **Definition 1** (Semiring). *A commutative semiring is an algebraic structure* $\mathcal{S} = (S, +, \cdot, 0, 1)$ *with* $0 \neq 1$*, such that* $(S, +, 0)$ *and* $(S, \cdot, 1)$ *are commutative monoids,* $\cdot$ *distributes over* $+$*, and* $0 \cdot s = s \cdot 0 = 0$*.*

In this paper, we only consider commutative semirings and simply refer to them as *semirings*. A semiring is *naturally ordered* (by addition) if $s \leq t :\Leftrightarrow \exists r(s + r = t)$ defines a partial order. Notice that $\leq$ is always reflexive and transitive, so a semiring is naturally ordered if, and only if $\leq$ is antisymmetric, i.e. $r \leq s$ and $s \leq r$ only hold for $s = r$. In particular, this excludes rings.

A semiring $\mathcal{S}$ is *idempotent* if $s + s = s$ for each $s \in S$ and *multiplicatively idempotent* if $s \cdot s = s$ for all $s \in S$. If both properties are satisfied, we say that $\mathcal{S}$ is fully idempotent. Finally, $\mathcal{S}$ is *absorptive* if $s + st = s$ for all $s, t \in S$ or, equivalently, if multiplication is decreasing in $\mathcal{S}$, i.e. $st \leq s$ for all $s, t \in S$. Every absorptive semiring is idempotent, and every idempotent semiring is naturally ordered.

Application semirings. There are many applications which can be modelled by semirings and provide useful practical information about the evaluation of a formula.

- The Boolean semiring $\mathbb{B} = (\mathbb{B}, \vee, \wedge, \bot, \top)$ is the standard habitat of logical truth.
- A totally ordered set $(S, \leq)$ with least element s and greatest element t induces the *min-max semiring* $(S, \max, \min, s, t)$. Specific important examples are the Boolean semiring, the fuzzy semiring $\mathbb{F} = ([0, 1], \max, \min, 0, 1)$, and the *access control semiring*, also called the *security semiring* [2].
- A more general class (than min-max semirings) is the class of *lattice semirings* $(S, \sqcup, \sqcap, s, t)$ induced by a bounded distributive lattice $(S, \leq)$. Clearly, lattice semirings are fully idempotent.
- The *tropical semiring* $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ is used to annotate atomic facts with a cost for accessing them and to compute minimal costs for verifying a logical statement. It is not fully idempotent but absorptive.
- The *Viterbi semiring* $\mathbb{V} = ([0, 1]_\mathbb{R}, \max, \cdot, 0, 1)$, which is in fact isomorphic to $\mathbb{T}$ via $y \mapsto -\ln y$ can be used for reasoning about confidence.
- An alternative semiring for this is the *Łukasiewicz semiring* $\mathbb{L} = ([0, 1]_\mathbb{R}, \max, \odot, 0, 1)$, where multiplication is given by $s \odot t = \max(s + t - 1, 0)$. It is isomorphic to the semiring of doubt $\mathbb{D} = ([0, 1]_\mathbb{R}, \min, \oplus, 1, 0)$ with $s \oplus t = \min(s + t, 1)$. Both $\mathbb{L}$ and $\mathbb{D}$ are absorptive semirings.
- The *natural semiring* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ is used to count the number of proof trees or evaluation strategies that estabish the truth of a sentence. It is also important for bag semantics in databases.

Provenance semirings. Provenance semirings of polynomials provide information on which combinations of literals imply the truth of a formula. The universal provenance semiring over a finite set X is the semiring $\mathbb{N}[X]$ of multivariate polynomials with indeterminates from X and coefficients from $\mathbb{N}$. Other provenance semirings are obtained, for example, as quotient semirings of $\mathbb{N}[X]$ induced by congruences for idempotence and absorption. The resulting provenance values are less informative but their computation is more efficient.

- By dropping coefficients from $\mathbb{N}[X]$, we get the free idempotent semiring $\mathbb{B}[X]$ whose elements are (in one-to-one correspondence with) finite sets of monomials with coefficient 1. It is the quotient induced by $x + x \sim x$.

- If, in addition, exponents are dropped, we obtain the Why-semiring $\mathbb{W}(X)$ of finite sums of monomials with coefficient 1 that are linear in each indeterminate. In this semiring, addition is idempotent but multiplication is not.
- The free absorptive semiring $\mathbb{S}(X)$ consists of $0, 1$ and all antichains of monomials with respect to the absorption order $\succcurlyeq$. A monomial m_1 absorbs m_2, denoted $m_1 \succcurlyeq m_2$, if it has smaller exponents, i.e. $m_2 = m \cdot m_1$ for some monomial m. It is the quotient of $\mathbb{N}[X]$ induced by $x + xy \sim x$.
- Finally, $(\mathrm{PosBool}(X), \vee, \wedge, \bot, \top)$ is the semiring whose elements are classes of equivalent (in the usual sense) positive Boolean expressions with Boolean variables from X. Its elements are in bijection with the positive Boolean expressions in irredundant disjunctive normal form. This is the lattice semiring freely generated by the set X. It arises from $\mathbb{S}(X)$ by dropping exponents.

For treating logical formalisms with fixed-point constructions, such as Datalog or LFP, provenance semirings with more general objects than polynomials are needed (see [1, 7]. Examples include the semirings of formal power series (with possibly infinite sums of monomials) such as $\mathbb{N}^\infty[\![X]\!]$ and the semirings $\mathbb{S}^\infty(X)$ of generalised absorptive polynomials (admitting infinite exponents). Further, all these provenance semirings can be equipped with *dual indeterminates* for treating negation, see [6].

3 Semirings with Infinitary Operations

3.1 Basic Properties of an Infinitary Operation

We first treat the two semiring operations, addition and multiplication, separately, and then look at their connections. The properties of the individual operations are discussed in terms of addition, but apply to multiplication analogously.

So let $\mathcal{S} = (S, +, 0)$ be a commutative monoid which we want to expand by an infinitary operation $\sum$ that maps every sequence $(s_i)_{i \in I}$ (over an arbitrary index set I) to a value $\sum_{i \in I} s_i \in S$. The infinitary sum should be compatible with the finite sum and respect the basic algebraic properties of the monoid. We thus have the following requirements.

Partition invariance (infinite associativity): For each partition $(I_j)_{j \in J}$ of I we have

$$\sum_{i \in I} s_i = \sum_{j \in J} \sum_{i \in I_j} s_i.$$

Bijection invariance (infinite commutativity): For every bijection $\sigma \colon J \to I$

$$\sum_{i \in I} s_i = \sum_{j \in J} s_{\sigma(j)}.$$

Compatibility with the finite: For each finite index set $I = \{i_0, \dots, i_n\}$

$$\sum_{i \in I} s_i = s_{i_0} + \dots + s_{i_n}.$$

Partition invariance is actually a very strong property which, in particular, implies bijection invariance. Indeed, consider the partition $(I_j)_{j \in J}$ of I into singleton sets $I_j = \{\sigma(j)\}$. Then partition invariance (together with compatibility with finite sums) implies that $\sum_{i \in I} s_i = \sum_{j \in J} \sum_{i \in I_j} s_i = \sum_{j \in J} s_{\sigma(j)}$. Bijection invariance also justifies that we consider operations over index *sets* rather than, for instance, transfinite sequences.

Most natural infinitary operations on monoids satisfy these properties. Nevertheless there are quite simple constructions that violate, for instance, infinite associativity, even for certain naturally ordered monoids with an infinite sum defined as the supremum of its finite subsums.

▶ **Example 2.** Let $S = \mathbb{N} \cup \{\omega, \omega'\}$ with the (commutative) addition that extends the natural addition on $\mathbb{N}$ by $n + \omega = \omega$ for $n \in \mathbb{N}$ and $\omega + \omega = \omega + \omega' = \omega' + \omega' = \omega'$. Defining $\sum_{i \in I} s_i = \sup\{\sum_{i \in I_0} s_i \ : \ I_0 \subseteq^{\text{fin}} I\}$ we have a summation operator where, for the sequence $(s_n)_{n \in \mathbb{N}}$ with $s_0 = \omega$ and $s_n = 1$ for $n > 0$, we have that the finite sum takes are all the values $n \leq \omega$, and hence

$$\sum_{n \in \mathbb{N}} s_n = \sup\{n : n \leq \omega\} = \omega \quad \text{but} \quad s_0 + \sum_{n \geq 1} s_n = s_0 + \sup\{n : n < \omega\} = \omega + \omega = \omega',$$

so partition invariance fails.

3.2 Compactness and its Consequences

However, these three requirements do not suffice to avoid "pathological" definitions with undesirable behaviour. Consider, for instance, the monoid $(\mathbb{N} \cup \{\infty\}, +, 0)$ with the infinitary sum defined by $\sum_{i \in I} s_i = \infty$ for all infinite I (and satisfying compatibility with $+$ for finite index sets). This violates, for instance, the following two natural properties.

Neutrality: $\sum$ *respects the neutral element* if $\sum_{i \in I} s_i = \sum_{i \in I, s_i \neq 0} s_i$.

Idempotence: $\sum$ *respects idempotent elements* if for all $s \in S$ such that $s + s = s$, also $\sum_{i \in I} s = s$ for every index set $I \neq \varnothing$.

To guarantee these, and other, desirable properties, we propose *compactness* properties which essentially say that if the infinitary operation takes different values on two sequences $(s_i)_{i \in I}$ and $(t_j)_{j \in J}$ then this is already witnessed by finite subsets, in the sense that some finite subsequence of one takes a value that is not assumed by any finite subsequence of the other. More formally:

Compactness: The operator $\sum$ is *compact* if for all $(s_i)_{i \in I}$ and $(t_j)_{j \in J}$ we have that

$$\sum_{i \in I} s_i = \sum_{j \in J} t_j \quad \text{whenever} \quad \Big\{\sum_{i \in I_0} s_i : I_0 \subseteq^{\text{fin}} I\Big\} = \Big\{\sum_{j \in J_0} t_j : J_0 \subseteq^{\text{fin}} J\Big\}.$$

Strong compactness: The operator $\sum$ is *strongly compact* if for all $(s_i)_{i \in I}$ and $(t_j)_{j \in J}$ and all s, t we have that

$$s + \sum_{i \in I} s_i = t + \sum_{j \in J} t_j \quad \text{whenever} \quad \Big\{s + \sum_{i \in I_0} s_i : I_0 \subseteq^{\text{fin}} I\Big\} = \Big\{t + \sum_{j \in J_0} t_j : J_0 \subseteq^{\text{fin}} J\Big\}.$$

▶ **Lemma 3.**

1. *Every compact operator respects idempotent elements.*
2. *If a partition invariant operator respects idempotent elements then it also respects the neutral element.*
3. *If $\sum$ is partition invariant and respects idempotent elements, then there exists, for every $s \in \mathcal{S}$, a unique element $\infty \cdot s := \sum_{i \in I} s$ for every infinite I.*
4. *If $\sum$ is strongly compact, and $s + p = s$ then $s + \sum_{i \in I} p = s$ for every index set I.*

Proof. (1) If $s+s=s$ then for each index set I, the values $\sum_{i\in I_0} s$ for finite $I_0 \subseteq I$ are s and 0. Taking $J=\{0,1\}$ and $t_0=0$ and $t_1=s$ we also have s and 0 as values for subsequences. Thus, by compactness, $\sum_{i\in I} s = t_0+t_1 = 0+s=s$.

For (2) we note that since 0 is an idempotent element, we have that $\sum_{j\in J} 0 = 0$ for all index sets J. It follows by partition invariance that

$$\sum_{i\in I} s_i = \sum_{\substack{i\in I\\ s_i\neq 0}} s_i + \sum_{\substack{i\in I\\ s_i=0}} 0 = \sum_{\substack{i\in I\\ s_i\neq 0}} s_i.$$

For (3) we first observe that the bijection invariance of an infinitary summation operator implies that $\sum_{i\in I} s = \sum_{j\in J} s$ for every $s\in\mathcal{S}$ and all index sets I, J of the same cardinality. Hence there exists, for every $s\in\mathcal{S}$ and every infinite cardinal κ, a unique element $\kappa\cdot s = \sum_{i\in I} s$ for every index set I of cardinality κ. Obviously, $\omega\cdot s$ is idempotent, and we can decompose any index set of size κ into a partition of sets of size ω. By partition invariance, and the respect of idempotent elements, it follows that $\kappa\cdot s = \omega\cdot s =: \infty\cdot s$. To prove (4), we note that $s+p=s$ implies that $s+\sum_{i\in I_0} p = s$ for all finite I_0. With $J=\varnothing$ and $t=s$, strong compactness implies that $s+\sum_{i\in I} p = s$ for every index set I. ◀

But compactness also has consequences for finite sums. Recall that a finite monoid $\mathcal{S}=(S,+,0)$ is aperiodic if for every $s\in\mathcal{S}$ there exists some $n\in\mathbb{N}$ such that $(n+1)s = ns$.

▶ **Lemma 4.** *The compactness property, even just for finite sums, in a finite monoid $\mathcal{S}=(S,+,0)$ implies that $\mathcal{S}$ must be aperiodic.*

Proof. If $\mathcal{S}$ is not aperiodic then there exist $s\in S$ and some minimal $n\in\mathbb{N}$ such $ns\neq (n+1)s = ks$ for some $k<n$. But then the sums $\sum_{i=1}^{n} s$ and $\sum_{i=1}^{n+1} s$ have different values although they have the same sets of values for subsums, namely $\{0, s, 2s, \dots, ns\}$, which contradicts compactness. ◀

We further notice that the existence of an infinitary operation with (some of) these properties can also have implications for the purely finitary properties of the monoid $\mathcal{S}=(S,+,0)$. Recall that $\mathcal{S}$ is $+$-positive if $s+t=0$ only holds for $s=t=0$. Further, $(S,+,0)$ is naturally ordered, if $s\le t :\Leftrightarrow \exists r(s+r=t)$ is a partial order. Since $\le$ is always reflexive and transitive this is the case if, and only if, $\le$ is antisymmetric, i.e. $s\le t$ and $t\le s$ imply that $s=t$. Obviously, a naturally ordered monoid is $+$-positive, but the converse is not true.

▶ **Lemma 5.** *If $\mathcal{S}=(S,+,0)$ admits a partition invariant infinitary sum that respects the neutral element, then $\mathcal{S}$ is $+$-positive. If this sum is strongly compact then $\mathcal{S}$ is naturally ordered.*

Proof. Suppose that $s+t=0$. Since $\sum$ is partition invariant and respects the neutral element we have that $0=\sum_{i\in\mathbb{N}}(s+t) = s+\sum_{i\in\mathbb{N}}(t+s) = s+0 = s$. For the second claim, suppose that $s+r=t$ and $t+q=s$. For $p=r+q$ we thus have that $s+p=s$ and $t+p=t$. We have to prove that $s=t$. By Lemma 3, strong compactness implies that $s+\sum_{i\in\mathbb{N}} p = s$ and $t+\sum_{i\in\mathbb{N}} p = t$. But then, partition invariance implies that

$$s = s+\sum_{i\in\mathbb{N}} p = s+\sum_{i\in\mathbb{N}}(r+q) = (s+r)+\sum_{i\in\mathbb{N}}(q+r) = t+\sum_{i\in\mathbb{N}} p = t. \quad ◀$$

▶ **Lemma 6.** *There is a monoid $\mathcal{S}=(S,+,0)$ that admits a partition-invariant and compact infinitary sum such that $\mathcal{S}$ is* not *naturally ordered and the sum therefore violates strong compactness. In particular, compactness does not imply strong compactness.*

Proof sketch. Consider the monoid $M := (\mathbb{N}^4 \cup \{\infty\}, +, 0)$ with $a + \infty = \infty$ for all a where the infinitary sum is defined by $\sum_{i \in I} s_i = \infty$ if there are infinitely many nonzero summands $s_i \neq 0$, and otherwise, $\sum_{i \in I} s_i$ corresponds to the usual finite sum of all (finitely many) nonzero summands. Clearly, M is naturally ordered by the usual component-wise partial order on $\mathbb{N}^4$ with an adjoined top element ∞. Moreover, the infinitary sum is both partition-invariant and bijection-invariant.

To construct $\mathcal{S}$, we set $a := (1,0,0,0)$, $b := (0,1,0,0)$, $c := (0,0,1,0)$ and $d := (0,0,0,1)$, and we identify $a + c \sim b$ and $b + d \sim a$. Let $\sim$ be the minimal congruence relation on M that satisfies this and set $\mathcal{S} := M/{\sim}$. It can be shown that such a congruence relation exists and that it satisfies $a \not\sim b$ and is compatible with the infinitary sum on M. Moreover, the infinitary sum on $\mathcal{S}$ inherits partition-invariance from the infinitary sum on M.

Further, it is possible to show that the infinitary sum on $\mathcal{S}$ is still compact. However, $\mathcal{S}$ is not naturally ordered, since $a \not\sim b$ implies $[a]_\sim \neq [b]_\sim$, but by definition, we have $[a]_\sim \leq [b]_\sim$ and $[b]_\sim \leq [a]_\sim$ due to $[a]_\sim + [c]_\sim = [a + c]_\sim = [b]_\sim$ and $[b]_\sim + [d]_\sim = [b + d]_\sim = [a]_\sim$. With Lemma 5, it follows that the infinitary sum on $\mathcal{S}$ cannot be strongly compact. ◀

Although compactness and strong compactness of an infinitary operation are powerful and convenient properties, there is the problem that they are not always easy to verify, and that there are relevant semirings where multiplication is not compact, as for instance $\mathbb{N}^\infty[\![X^\infty]\!]$, an infinitary extension of $\mathbb{N}[X]$ to be defined in Section 5, which does not even respect idempotent elements. We therefore will work with the following simpler property, which by Lemma 3 is implied by compactness, is easier to establish, and suffices for our proofs.

Unique infinite powers: $\sum$ *has unique infinite powers* if for every $s \in \mathcal{S}$, there exists a unique element $\infty \cdot s$ with $\infty \cdot s := \sum_{i \in I} s$ for every infinite I.

3.3 Distributivity

The requirements that relate the two algebraic operations in a commutative semiring are the distributive law $s(r + t) = sr + st$ and the fact that the neutral element of addition is multiplicatively annihilating, i.e. $0 \cdot s = 0$ for all s. If an infinitary product $\prod$ is partition invariant and compatible with finite products, then it follows immediately that $\prod_{i \in I} s_i = 0$ whenever $s_i = 0$ for some $i \in I$. The generalisation of the distributive law to infinitary operations is more complicated and comes in a weak and a strong variant:

Weak distributivity: For each index set I and all s

$$s \cdot \sum_{i \in I} s_i = \sum_{i \in I} (s \cdot s_i).$$

Strong distributivity: For every index set I and every collection $(J_i)_{i \in I}$ of index sets

$$\prod_{i \in I} \sum_{j \in J_i} s_j = \sum_{f \in F} \prod_{i \in I} s_{f(i)},$$

where F is the set of all choice functions $f \colon I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$ for all $i \in I$.

For finite index sets I, strong distributivity is implied by weak distributivity via a rather straightforward induction. For infinite index sets the situation is more complicated.

We first observe that strong distributivity holds for the completion $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ of the natural semiring $(\mathbb{N}, +, \cdot, 0, 1)$ (with the natural extensions of finite addition and multiplication from $\mathbb{N}$ to finite and infinitary addition and multiplication on $\mathbb{N}^\infty$).

▶ **Proposition 7.** *The semiring $\mathbb{N}^\infty$ satisfies strong distributivity.*

Proof. Given an expression $\prod_{i\in I}\sum_{j\in J_i} s_j$ over $\mathbb{N}^\infty$, we argue by case distinction.

If $\prod_{i\in I}\sum_{j\in J_i} s_j = 0$, then there is some $i \in I$ such that $s_j = 0$ for all $j \in J_i$. But then every choice function $f \in F$ has the property that $s_{f(i)} = 0$ which implies that also $\sum_{f\in F}\prod_{i\in I} s_{f(i)} = 0$.

If $\prod_{i\in I}\sum_{j\in J_i} s_j \neq 0$, then there exists, for each $i \in I$, some $j \in J_i$ such that $s_j \neq 0$. Hence there is a choice function $f \in F$ such that $\prod_{i\in I} s_{f(i)} \neq 0$.

We next discuss the cases where $\prod_{i\in I}\sum_{j\in J_i} s_j = \infty$. Assume that $s_k = \infty$ for some $k \in J_i$, so $\sum_{j\in J_i} s_j = \infty$. For the choice function that selects $k \in J_i$, and nonzero elements in the other index sets, we have $\prod_{i\in I} s_{f(i)} = \infty$, and hence $\sum_{f\in F}\prod_{i\in I} s_{f(i)} = \infty$. Another possible case with $\sum_{j\in J_i} s_j = \infty$ appears when there are infinitely many $j \in J_i$ with $s_j \neq 0$. Then there are infinitely many choice functions $f \in F$ such that $s_{f(i)} \neq 0$ for all i, so again $\sum_{f\in F}\prod_{i\in I} s_{f(i)} = \infty$. Finally it may be the case that $\prod_{i\in I}\sum_{j\in J_i} s_j = \infty$ because there are infinitely many $i \in I$, such that $\sum_{j\in J_i} s_j > 1$. We distinguish two possibilities. Either there are infinitely many i, for which there exists some $j \in I_j$ with $s_j \geq 2$. Selecting these elements, and (by the argument above) non-zero values for the other indices gives us a choice function $f \in F$ such that $\prod_{i\in I} s_{f(i)} = \infty$. The other possibility is that there exist infinitely many $i \in I$ with at least two indices $j \in I_j$ with $s_j = 1$. But this implies that there are not just one, but infinitely many choice functions $f \in F$ such that $s_{f(i)} \neq 0$ for all i, so $\sum_{f\in F}\prod_{i\in I} s_{f(i)} = \infty$.

Suppose finally that $\prod_{i\in I}\sum_{j\in J_i} s_j = n$ where $1 \leq n < \infty$. Then there exists a finite index set $I_0 \subset I$, such that $\prod_{i\in I}\sum_{j\in J_i} s_j = \prod_{i\in I_0}\sum_{j\in J_i} s_i$ and that $\sum_{j\in J_i} s_j = 1$ for all $i \in I \setminus I_0$. This implies that in each such J_i there is precisely one j such that $s_j = 1$, and that $s_k = 0$ for $k \neq j$. Let F' be the subset of those choice functions in F that select for each $i \in I \setminus I_0$ the unique $j \in J_i$ with $s_j = 1$. Notice that $\prod_{i\in I} s_{f(i)} = 0$ for all $f \in F \setminus F'$. Further, let F_0 be the set of choice functions on I_0. Each $f \in F_0$ uniquely extends to a choice function in $f' \in F'$, with $\prod_{i\in I_0} s_{f(i)} = \prod_{i\in I} s_{f'(i)}$. We further note that each sum $\sum_{j\in J_i} s_j$ cannot exceed n, so it can only have finitely many non-zero entries and can be written as a finite sum. By (finite) distributivity, we have

$$\sum_{f\in F}\prod_{i\in I} s_{f(i)} = \sum_{f'\in F'}\prod_{i\in I} s_{f'(i)} + \sum_{f\in F\setminus F'}\prod_{i\in I} s_{f(i)} = \sum_{f\in F_0}\prod_{i\in I_0} s_{f(i)} = \prod_{i\in I_0}\sum_{j\in J_i} s_j = n. \quad ◀$$

It is easy to verify that $\mathbb{N}^\infty$ also satisfies all other properties mentioned above, including (strong) compactness. The same holds for other semirings that are obtained by completing a semiring without infinitary operations by an element ∞ to which the appropriate infinite sums and infinite products evaluate. This includes, for instance, the polynomial semirings $\mathbb{N}[X] \cup \{\infty\}$ and $\mathbb{B}[X] \cup \{\infty\}$, for finite sets X of indeterminates.

But there are also important semirings for which strong distributivity depends on the cardinality of the index sets that we consider. We illustrate this for the tropical semiring $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ (whose infinitary operations are, of course, infimum and the natural infinitary sum). Weak distributivity holds for arbitrary index sets. However, this is not the case for strong distributivity.

▶ **Proposition 8.** *In the tropical semiring, strong distributivity holds for countable index sets, but fails for uncountable ones.*

Proof. We first prove the failure of strong distributivity for uncountable index sets. Let $J_i = \omega$ for all i in some uncountable index set I, and let $(s_j)_{j\in\omega}$ be any sequence of positive real numbers that converges to 0. Strong distributivity would mean that

$$\sum_{i\in I}\inf_{j\in\omega} s_j = \inf_{f\in F}\sum_{i\in I} s_{f(i)}$$

where F ranges over all choice functions $f\colon I \to \omega$. However, the left side is 0 since the infimum of $(s_j)_{j\in\omega}$ is 0. But every choice function $f\colon I \to \omega$ must hit some $n \in \omega$ infinitely often so for every f we have that $\sum_{i\in I} s_{f(i)} \geq \infty \cdot s_n = \infty$. Hence the right side evaluates to ∞, and strong distributivity fails.

Next we observe that, for all index sets I and $(J_i)_{i\in I}$,

$$\sum_{i\in I} \inf_{j\in J_i} s_j \leq \inf_{f\in F} \sum_{i\in I} s_{f(i)}.$$

Indeed, for all $f \in F$ we clearly have that $\inf_{j\in J_i} s_j \leq s_{f(i)}$. Summation is monotone, so it follows that $\sum_{i\in I} \inf_{j\in J_i} s_j \leq \sum_{i\in I} s_{f(i)}$ and since this holds for all $f \in F$ it also holds for the infimum.

Finally, it remains to show that for *countable* I and all $(J_i)_{i\in I}$, we have that

$$\sum_{i\in I} \inf_{j\in J_i} s_j \geq \inf_{f\in F} \sum_{i\in I} s_{f(i)}.$$

We know that this holds for finite I, so we assume now that I is countably infinite. Without loss of generality we can take $I = \omega$, and we set $q_i := \inf_{j\in I_j} s_j$ and $q := \sum_{i\in\omega} q_i$. We have to show that $\inf_{f\in F} \sum_{i\in\omega} s_{f(i)} \leq q$. If $q = \infty$, there is nothing to prove. Otherwise q and hence also all q_i are finite. Fix any $\varepsilon > 0$. Since $\inf_{j\in J_i} s_j = q_i$ we can find, for every $i < \omega$, some $j \in J_i$ such that $s_j \leq q_i + \varepsilon 2^{-(i+1)}$. Let f_ε be a choice function that maps each $i \in \omega$ to some $j \in J_i$ with this property. We then have that

$$\sum_{i\in\omega} s_{f_\varepsilon(i)} \leq \sum_{i\in\omega}(q_i + \varepsilon 2^{-(i+1)}) = \sum_{i\in\omega} q_i + \varepsilon \cdot \sum_{i\in\omega} 2^{-(i+1)} = q + \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we conclude that $\inf_{f\in F} \sum_{i\in I} s_{f(i)} \leq q$. ◀

The same proposition, with almost exactly the same proofs, holds also for the Viterbi semiring $\mathbb{V}$, the Łukasiewicz semiring $\mathbb{L}$ and the semiring of doubt $\mathbb{D}$.

3.4 Monotonicity

In a naturally ordered semiring, an important property is that both addition and multiplication are *monotone* in each argument: if $s_1 \leq t_1$ and $s_2 \leq t_2$ then $s_1 + s_2 \leq t_1 + t_2$ and $s_1 s_2 \leq t_1 t_2$. By partition invariance it immediately follows that also the infinitary sum is monotone in each argument.

▶ **Lemma 9.** *If $s_i \leq t_i$ for all $i \in I$, then $\sum_{i\in I} s_i \leq \sum_{i\in I} t_i$.*

Proof. For each $i \in I$, we have that $t_i = s_i + \delta_i$ for some element $\delta_i \in S$. Hence $\sum_{i\in I} t_i = \sum_{i\in I}(s_i + \delta_i) = \sum_{i\in I} s_i + \sum_{i\in I} \delta_i$. ◀

Monotonicity of multiplication is implied by the distributive law: if $t_1 = s_1 + \delta_1$ and $t_2 = s_2 + \delta_2$ then $t_1 t_2 = s_1 s_2 + s_1 \delta_2 + s_2 \delta_1 + \delta_1 \delta_2$, so $s_1 s_2 \leq t_1 t_2$. Monotonicity of infinitary products in each single argument follows by the same argument. If $t_j = s_j + \delta$ for some $j \in I$, and $s_i = t_i$ for all other $i \in I$ then $\prod_{i\in I} t_i = (s_j + \delta) \prod_{i\in I\setminus\{j\}} s_i = \prod_{i\in I} s_i + \delta \prod_{i\in I\setminus\{j\}} s_i$. Assuming strong distributivity, we can apply this argument simultaneously to each factor.

▶ **Lemma 10.** *If $\mathcal{S}$ satisfies strong distributivity, and $s_i \leq t_i$ for all $i \in I$, then $\prod_{i\in I} s_i \leq \prod_{i\in I} t_i$.*

Proof. Again, let $t_i = s_i + \delta_i$ and let F be the set of all choice functions f that map each $i \in I$ to either s_i or δ_i. Further, let $f_0 \in F$ be the function that maps all $i \in I$ to s_i. By strong distributivity, we have that

$$\prod_{i\in I} t_i = \prod_{i\in I}(s_i+\delta_i) = \sum_{f\in F}\prod_{i\in I} f(i) = \prod_{i\in I} f_0(i) + \sum_{f\in F\setminus\{f_0\}}\prod_{i\in I} f(i) = \prod_{i\in I} s_i + \sum_{f\in F\setminus\{f_0\}}\prod_{i\in I} f(i)$$

so $\prod_{i\in I} s_i \le \prod_{i\in I} t_i$. ◀

Given that strong distributivity does not hold in all interesting semirings, we should add monotonicity to our list of desired properties for infinitary semiring operations.

Monotonicity: For each index set I and all families $(s_i)_{i\in I}$ and $(t_i)_{i\in I}$ such that $s_i \le t_i$ (w.r.t. the natural order) we also have that

$$\sum_{i\in I} s_i \le \sum_{i\in I} t_i \qquad \text{and} \qquad \prod_{i\in I} s_i \le \prod_{i\in I} t_i.$$

The requirement of monotonicity for the infinitary sum is redundant since it is implied by partition invariance, but monotonicity of infinitary products does not seem to follow from weaker properties than strong distributivity.

► **Example 11.** Let $S = \mathbb{N} \cup \{\infty\}$ with the natural definition of infinitary sum, but with an infinitary product that evaluates to 0, if there are infinitely many finite factors different from 1. More precisely,

$$\prod_{i\in I} s_i = \begin{cases} \infty & \text{if } s_i = \infty \text{ for some } i \in I \text{ and } s_i \neq 0 \text{ for all } i \in I \\ \prod_{i\in I_0} s_i & \text{for } I_0 \subseteq^{\text{fin}} I \text{ such that } s_i = 1 \text{ for all } i \in I \setminus I_0 \\ 0 & \text{otherwise.} \end{cases}$$

The infinitary product is not monotone since $\prod_{i<\omega} 1 = 1$ but $\prod_{i<\omega} 2 = 0$. Strong distributivity of course also fails, as witnessed by $\prod_{i<\omega}(1+1)$. One can readily verify that the infinitary operations in this semiring satisify all the other properties that we discussed, including strong compactness.

3.5 Infinitary Semirings

We are now ready to propose a definition for semirings with infinitary operations.

► **Definition 12.** *An* infinitary semiring*, also called ∞-semiring, is a commutative, naturally ordered semiring $\mathcal{S} = (S, +, \cdot, 0, 1)$, together with two infinitary operations $\sum$ and $\prod$ that satisfy the following properties:*

- *partition invariance (infinite associativity), and hence also bijection invariance (infinite commutativity),*
- *compatibility with finite addition and multiplication,*
- *neutral elements are respected,*
- *there are unique infinite powers,*
- *weak distributivity, and*
- *monotonicity.*

This should be seen as a working definition that reflects our current state of investigations. There are of course alternative possibilities. For instance we could impose (strong) compactness of infinitary sums and products as a basic property, which would imply that idempotent and neutral elements are respected, and that there exist unique infinite powers. We further note that the requirement that the additive neutral element is respected is in fact redundant, as it is implied by weak distributivity and partition invariance. Indeed, $\sum_{i\in I} 0 = \sum_{i\in I}(0\cdot 0) = 0\cdot\sum_{i\in I} 0 = 0$ and therefore $\sum_{i\in I} s_i = \sum_{i\in I, s_i=0} s_i + \sum_{i\in I, s_i\neq 0} s_i = \sum_{i\in I, s_i\neq 0} s_i$. However, it can be shown that the requirement that the multiplicative neutral element is respected by infinitary products does not follow from the other properties (for a counterexample, take $(\mathbb{N}^\infty,\cdot,1)$ and set all infinite products of non-zero values to ∞). We could also require strong distributivity, which we chose to omit from the definition because we want to include in our study some relevant semirings that (for arbitrary index sets) only satisfy the weak distributive law. Instead, we introduce the following variant.

▶ **Definition 13.** *Let κ be an infinite cardinal. An infinitary semiring is* κ-distributive *if it satisfies strong distributivity for products of cardinality $<\kappa$. That is, it satisfies*

$$\prod_{i\in I}\sum_{j\in J_i} s_j = \sum_{f\in F}\prod_{i\in I} s_{f(i)}$$

for all sets I with $|I|<\kappa$ (and sets J_i of arbitrary cardinality). It is strongly distributive *if it satisfies strong distributivity for all index sets.*

The tropical semiring, the Viterbi semiring, and the Łukasiewicz semiring are examples of ω_1-distributive semirings.

Homomorphisms between ∞-semirings should be compatible with the infinitary operations. We again introduce a weaker variant for a more fine-grained analysis.

▶ **Definition 14** (∞-semiring homomorphisms)**.** *Let κ be an infinite cardinal, and let $\mathcal{S},\mathcal{S}'$ be infinitary semirings. A* semiring κ-homomorphism *$h\colon \mathcal{S}\to\mathcal{S}'$ is a semiring homomorphism such that for all sequences $(s_i)_{i\in I}$ in $\mathcal{S}$ with $|I|<\kappa$, we have that*

$$h\Big(\sum_{i\in I} s_i\Big) = \sum_{i\in I} h(s_i) \qquad \text{and} \qquad h\Big(\prod_{i\in I} s_i\Big) = \prod_{i\in I} h(s_i).$$

Further h is called an ∞-semiring homomorphism*, if it is a semiring κ-homomorphism for all κ.*

Of course, the term *infinitary semiring* does not imply that the semiring has infinitely many elements. Quite to the contrary:

▶ **Proposition 15.** *Every finite semiring, in which both addition and multiplication induce aperiodic monoids, expands to an ∞-semiring (in which, moreover, both operations are strongly compact).*

Proof. Let $(S,\cdot,1)$ be the multiplicative monoid of the semiring. Since it is aperiodic, there exists, for every $s\in S$, a minimal number n_s such that $s^{n_s+1}=s^{n_s}$ and hence $s^n=s^{n_s}$ for all $n\geq n_s$. We put $s^\infty := s^{n_s}$. We can now define an infinitary product $\prod_{i\in I} s_i$ by reducing it to a finite product. For every $s\in S$, let $m_s := \min(n_s, |\{i\in I : s_i = s\}|)$ and set $\prod_{i\in I} s_i := \prod_{s\in S} s^{m_s}$. The definition of infinitary sums is completely analogous. It is easily verified, that the required properties are inherited from finite addition and multiplication. Since the infinitary operations reduced to the finite ones, strong compactness is also straightforward. ◀

Let us next consider infinite lattice semirings, induced by some partial order $(S, \leq)$.

▶ **Proposition 16.** *An infinite lattice semiring expands to an ∞-semiring if, and only if, the underlying order is a complete lattice in which finite infima distribute over arbitrary suprema.*

Proof. In a complete lattice semiring where finite infima distribute over arbitrary suprema, the desired infinitary operations are suprema and infima, which obviously satisfy all required properties. For the converse implication, it suffices to show that in any expansion to an ∞-semiring, infinitary summation is given by suprema. As required, this implies the existence of arbitrary suprema and thus completeness of the lattice, and the weak distributive law w.r.t. the semiring operations reduces to weak distributivity of the lattice operations. Suppose that there are elements $(x_i)_{i\in I}$ such that $z := \sum_{i\in I} x_i$ is not the supremum of $\{x_i \mid i \in I\} =: X$. By partition invariance, $x_i \sqcup \sum_{j\in I, j\neq i} x_i = z$ for each $i \in I$, so z is an upper bound of X. Thus, there must exist some other upper bound y for X such that $z \not\leq y$. But weak distributivity yields $y \sqcap z = y \sqcap \sum_{i\in I} x_i = \sum_{i\in I}(y \sqcap x_i) = \sum_{i\in I} x_i = z$ and thus $z \leq y$, a contradiction. ◀

There are two main further classes of semirings that we consider. A particularly useful class is the class of infinitary absorptive semirings, studied in the next section. Recall that absorption has the consequence that multiplication is decreasing, which leads to dualities that permit to carry over a number of classical logical properties to semiring semantics. The other relevant class consists of the semirings that extend the natural semiring $\mathbb{N}$ or semirings of polynomials such as $\mathbb{N}[X]$, where multiplication is increasing.

4 Infinitary Absorptive Semirings

Recall that a semiring $\mathcal{S}$ is absorptive if $s + st = s$ for all $s, t \in \mathcal{S}$ or, equivalently, $1 + t = 1$ for all $t \in \mathcal{S}$. Every absorptive semiring is idempotent (i.e., $s + s = s$ for all $s \in \mathcal{S}$) and every idempotent semiring is naturally ordered. In naturally ordered semirings, addition and multiplication are monotone w.r.t. the natural order. Further in absorptive semirings, multiplication is decreasing (i.e., $s \cdot r \leq s$ for all $s, r \in \mathcal{S}$).

We will introduce the notion of infinitary absorptive semirings. These are based on absorptive semirings, with some additional properties that permit to define natural infinitary addition and multiplication operations, based on infima and suprema.

▶ **Definition 17.** *An* infinitary absorptive semiring *is the expansion of an absorptive semiring $\mathcal{S}$ which satisfies the additional properties that*

- *the natural order $(\mathcal{S}, \leq)$ is a complete lattice.*
- *$\mathcal{S}$ is (fully) continuous: for every non-empty chain $C \subseteq S$, the supremum $\bigsqcup C$ and the infimum $\bigsqcap C$ are compatible with addition and multiplication, i.e.*

$$s \circ \bigsqcup C = \bigsqcup (s \circ C) \quad \text{and} \quad s \circ \bigsqcap C = \bigsqcap (s \circ C),$$

where $(s \circ C) := \{s \circ c : c \in C\}$ for every $s \in S$ and $\circ \in \{+, \cdot\}$.

As a consequence, we can define natural infinitary addition and multiplication operations in $\mathcal{S}$, by taking suprema of finite subsums and infima of finite subproducts:

$$\sum_{i\in I} s_i := \bigsqcup_{\substack{I_0 \subseteq I \\ I_0 \text{ finite}}} \Big(\sum_{i\in I_0} s_i\Big) \quad \text{and} \quad \prod_{i\in I} s_i := \bigsqcap_{\substack{I_0 \subseteq I \\ I_0 \text{ finite}}} \Big(\prod_{i\in I_0} s_i\Big).$$

Since addition is idempotent in absorptive semirings, the infinitary addition is in fact the same as the supremum: $\sum_{i \in I} s_i = \bigsqcup_{i \in I} s_i$. However, unless multiplication is also idempotent (so that the semiring is a lattice semiring), infinitary products need not coincide with infima. Indeed, we note that there are infinitary absorptive semirings in which the natural order is a completely distributive lattice (i.e., infima and suprema satisfy a strong distributive law), but strong distributivity does not hold for the infinitary semiring operations defined above. One such example is the Viterbi semiring $\mathbb{V}$: the natural order for $\mathbb{V}$ is just the usual linear order on the real interval $[0,1]$, which is a completely distributive lattice, but we have seen that the strong distributive law fails on $\mathbb{V}$ for uncountable index sets.

Most of the common application semirings mentioned in Sect. 2 are in fact absorptive (with the notable exception of the natural semiring) and permit the expansion to an infinitary absorptive semiring. Among the semirings of polynomials mentioned in Sect. 2, only $\mathbb{S}(X)$ and $\mathrm{PosBool}(X)$ are absorptive semirings whereas $\mathbb{N}[X]$, $\mathbb{B}[X]$, and $\mathbb{W}(X)$ are not.

It remains to show that infinitary absorptive semirings are indeed ∞-semirings, i.e. that they satisfy all the properties required by Definition 12. Since the infinitary properties are based on suprema and infima, this is straightforward in most cases. For the weak distributivity law, this is a direct consequence of the continuity of multiplication. The only property that requires work, and also makes use of continuity of multiplication, is the partition invariance of infinitary products (whereas for infinite sums partition invariance is trivial, because they are just suprema).

▶ **Lemma 18.** *Products in infinitary absorptive semirings are partition invariant.*

Proof. To simplify notation, we define the abbreviation $s(I_0) \coloneqq \prod_{i \in I_0} s_i$ for *finite* index sets $I_0 \subseteq^{\text{fin}} I$. We thus have to prove that

$$\prod_{i \in I} s_i = \bigsqcap_{I_0 \subseteq^{\text{fin}} I} s(I_0) \stackrel{!}{=} \bigsqcap_{J_0 \subseteq^{\text{fin}} J} \Big(\prod_{j \in J_0} \bigsqcap_{H_0 \subseteq^{\text{fin}} I_j} s(H_0) \Big) = \prod_{j \in J} \prod_{i \in I_j} s_i .$$

We prove both directions. First fix a finite set $I_0 \subseteq^{\text{fin}} I$. Since $(I_j)_{j \in J}$ is a partition, there is a finite set $J_0 \subseteq^{\text{fin}} J$ such that $I_0 \subseteq \bigcup_{j \in J_0} I_j$. Moreover, for each $i \in I_j$ we clearly have $s_i \geq \bigsqcap_{H_0 \subseteq^{\text{fin}} I_j} s(H_0)$ by considering $H_0 = \{i\}$. Using absorption (abs) and monotonicity (m), we have

$$s(I_0) = \prod_{i \in I_0} s_i \stackrel{\text{(abs)}}{\geq} \prod_{\substack{i \in I_j \\ j \in J_0}} s_i \stackrel{\text{(m)}}{\geq} \prod_{j \in J_0} \bigsqcap_{H_0 \subseteq^{\text{fin}} I_j} s(H_0) \geq \bigsqcap_{J_0 \subseteq^{\text{fin}} J} \prod_{j \in J_0} \bigsqcap_{H_0 \subseteq^{\text{fin}} I_j} s(H_0)$$

which proves direction "$\geq$".

For the other direction, fix a finite set $J_0 = \{j_1, \dots, j_k\} \subseteq^{\text{fin}} J$. Recall that $s(I_0)$ is a finite product and thus associative (a). Together with continuity of multiplication (c), we get

$$\begin{aligned}
\bigsqcap_{I_0 \subseteq^{\text{fin}} I} s(I_0) &\leq \bigsqcap_{I_0 \subseteq^{\text{fin}} \bigcup_{j \in J_0} I_j} s(I_0) = \bigsqcap_{H_{j_1} \subseteq^{\text{fin}} I_{j_1}} \dots \bigsqcap_{H_{j_k} \subseteq^{\text{fin}} I_{j_k}} s(H_{j_1} \cup \dots \cup H_{j_k}) \\
&\stackrel{\text{(a)}}{=} \bigsqcap_{H_{j_1} \subseteq^{\text{fin}} I_{j_1}} \dots \bigsqcap_{H_{j_k} \subseteq^{\text{fin}} I_{j_k}} (s(H_{j_1}) \cdots s(H_{j_k})) \\
&\stackrel{\text{(c)}}{=} \Big(\bigsqcap_{H_{j_1} \subseteq^{\text{fin}} I_{j_1}} a_{H_{j_1}} \Big) \cdots \Big(\bigsqcap_{H_{j_k} \subseteq^{\text{fin}} I_{j_k}} a_{H_{j_k}} \Big) = \prod_{j \in J_0} \bigsqcap_{H_0 \subseteq^{\text{fin}} I_j} s(H_0),
\end{aligned}$$

which closes the proof. ◀

The natural notion of homomorphisms between infinitary absorptive semirings are the fully-continuous homomorphisms (which are compatible with suprema and infima of chains in the same way as the semiring homomorphisms). Since infinitary operations are defined through suprema and infima[1], they are preserved by fully-continuous homomorphisms.

▶ **Proposition 19.** *Every fully-continuous homomorphism $h\colon \mathcal{S} \to \mathcal{S}'$ between infinitary absorptive semirings is an ∞-semiring homomorphism.*

5 Polynomials and Power Series

The semirings $\mathbb{N}[X]$ of multivariate polynomials with a (finite) set X of indeterminates and coefficients from $\mathbb{N}$ play a fundamental role for the provenance analysis of database queries and first-order sentences. This is due to the fact that $\mathbb{N}[X]$ is the semiring that is freely generated by X and has the universal property that every function $h\colon X \to \mathcal{S}$ into an arbitrary commutative semiring $\mathcal{S}$ uniquely extends to a semiring homomorphism $h\colon \mathbb{N}[X] \to \mathcal{S}$.

The question arises, whether we can extend $\mathbb{N}[X]$ to an infinitary semiring that has corresponding universal properties. We must be able to infinitely often add the same monomial (e.g., $x+x+x+\dots$), add infinitely many different monomials (e.g., $x+x^2+x^3+\dots$) and multiply the same variable infinitely often (e.g., $x \cdot x \cdot x \cdot \dots$). To address these issues, we extend $\mathbb{N}[X]$ by allowing coefficients in $\mathbb{N}^\infty$, using formal power series instead of polynomials, and allowing exponents in $\mathbb{N}^\infty$ (as is done for generalised absorptive polynomials). We thus obtain *semirings of generalised power series over* X, denoted $\mathbb{N}^\infty[\![X^\infty]\!]$. Infinite summation in $\mathbb{N}^\infty[\![X^\infty]\!]$ is straightforward and infinite products can be defined by considering the sum over all possible factorisations of a given monomial. Here are the formal definitions.

▶ **Definition 20.** *Fix a finite set X of indeterminates. A* monomial *is a function $m\colon X \to \mathbb{N}^\infty$ that associates with each indeterminate an exponent. Let M be the set of all monomials.*

A generalised power series *is a function $P\colon M \to \mathbb{N}^\infty$ associating with each monomial its coefficient. We obtain the semiring $\mathbb{N}^\infty[\![X^\infty]\!]$ of generalised power series with the infinitary sum and product defined by*

$$\sum_{i\in I} P_i := \Big(m \mapsto \sum_{i\in I} P_i(m)\Big) \quad \text{and} \quad \prod_{i\in I} P_i = \Big(m \mapsto \sum_{(m_i)_i \in \mathsf{splits}(m)} \prod_{i\in I} P_i(m_i)\Big).$$

Here, $\mathsf{splits}_I(m)$ is the set of sequences $(m_i)_{i\in I}$ of monomials with $m = \prod_{i\in I} m_i$. We may omit the index I if it is clear from the context.

The corresponding definition with an *infinite* set X of indeterminates is not consistent with Definition 12, since it does not have unique powers. For the polynomial $P = \sum_{i<\omega} x_i$, we have that P^ω is different from P^{ω_1}. Therefore, we only use finite sets of indeterminates. We further note that $\mathbb{N}^\infty[\![X^\infty]\!]$, even with a single indeterminate, does not preserve idempotent elements and hence is not compact. Indeed, let $Q := \sum_{i<\omega} \infty \cdot x^i$. Then $Q^2 = Q$ but $Q^\omega = \infty \cdot x^\infty + Q \neq Q$. Nevertheless $\mathbb{N}^\infty[\![X^\infty]\!]$ turns out to be an important ∞-semiring, playing a similar role as $\mathbb{N}[X]$ does in the finite case. To establish that $\mathbb{N}^\infty[\![X^\infty]\!]$ is an ∞-semiring, we begin with somewhat technical observations about infinite powers.

[1] Fully-continuous homomorphisms commute with suprema and infima of *chains* by definition, and thus also with suprema/infima of the *directed sets* in the definition (based on arguments in [9]).

▶ **Lemma 21.** *Let $P \in \mathbb{N}^\infty[\![X^\infty]\!]$ and I an infinite index set. The only possible coefficients of $\prod_{i\in I} P$ are 0, 1, and ∞.*

Proof. Let $Q = \prod_{i\in I} P$, fix a monomial m and assume that $Q(m) \neq 0$. Consider a sequence $(m_i)_{i\in I} \in \mathsf{splits}(m)$ with value $\prod_{i\in I} P(m_i) > 0$. If there is a monomial v that occurs only finitely often (but at least once) in $(m_i)_i$, then by permuting the occurrences of v with other monomials in the sequence, we obtain infinitely many pairwise different sequences in $\mathsf{splits}(m)$ with the same value. Since $Q(m)$ is the sum over all sequences in $\mathsf{splits}(m)$, this implies $Q(m) = \infty$. Similarly, if two different monomials v, v' occur infinitely often in $(m_i)_i$, we can again obtain infinitely many different permutations of the sequence and $Q(m) = \infty$.

So the only possibility for $Q(m) < \infty$ is that all sequences $(m_i)_i$ with $\prod_{i\in I} P(m_i) > 0$ consist of only one monomial, i.e. $m_i = m_j$ for all $i, j \in I$. Assume towards a contradiction that two such sequences exist, say $(v)_{i\in I} \in \mathsf{splits}(m)$ and $(w)_{i\in I} \in \mathsf{splits}(m)$. Since $|\omega| + |I| = |I|$, we can construct a sequence $(u_i)_{i\in I}$ that consists of countably many repetitions of v, and $|I|$ many repetitions of w. All indeterminates occurring in v or w must have exponent ∞ in m, hence also $(u_i)_i \in \mathsf{splits}(m)$. But this sequence uses two monomials infinitely often which implies $Q(m) = \infty$, contradiction.

Hence $Q(m) < \infty$ implies that there is only one sequence $(m_i)_i \in \mathsf{splits}(m)$ with value $\prod_{i\in I} P(m_i) > 0$, and since $m_i = m_j$ for all i, j this value must be either 1 or ∞. ◀

▶ **Lemma 22.** *$\mathbb{N}^\infty[\![X^\infty]\!]$ has unique infinite powers.*

Proof. We recall that $\mathbb{N}^\infty$ satisfies compactness and thus has unique infinite powers. The unique power property for summation in $\mathbb{N}^\infty[\![X^\infty]\!]$ is inherited from $\mathbb{N}^\infty$, as summation is defined pointwise by summing over the coefficients of each monomial.

Multiplication requires more work. Notice that the set M of monomials is countable, since X is finite. Let $P \in \mathbb{N}^\infty[\![X^\infty]\!]$ and I, J two infinite index sets. It suffices to prove that

$$Q := \prod_{i\in I} P \;\leq\; \prod_{j\in J} P =: R$$

due to symmetry. Fix a monomial m and a sequence $(m_i)_{i\in I} \in \mathsf{splits}_I(m)$. Consider the set of monomials $\{m_i \mid i \in I\}$ occurring in this sequence. We partition this set into a set M_0 of monomials that occur only finitely often and a set M_∞ of monomials that occur infinitely often. Let $I_0 \subseteq I$ be the set of indices i with $m_i \in M_0$. Since M_0 is countable and each $m \in M_0$ occurs finitely often, I_0 is countable as well. We further fix an enumeration of the set M_∞ (which is countable as well).

We now construct a sequence $(v_j)_{j\in J} \in \mathsf{splits}_J(m)$ by first constructing a sequence $(w_l)_{l\in L} \in \mathsf{splits}_L(m)$ for some index set L and then applying a bijection $f\colon J \to L$. We define L as follows:

$$L = I_0 \;\dot\cup\; (\omega \times \omega) \;\dot\cup\; J.$$

We next define the elements of the sequence $(w_l)_{l\in L}$. We first need a “padding element” m_∞ (in case $|J| > |I|$). If $M_\infty \neq \varnothing$, we choose an arbitrary but fixed $m_\infty \in M_\infty$. If $M_\infty = \varnothing$, then M_0 contains infinitely many monomials (each of which occurs finitely often in $(m_i)_{i\in I}$). Recall that $m = \prod_{i\in I} m_i$ and consider the indeterminates $X_0 \subseteq X$ with *finite* exponents in m. Since the exponents are finite, there can be only finitely many monomials in M_0 containing an indeterminate in X_0 (recall that X is finite!). Consider the infinitely many monomials in M_0 *not* containing an indeterminate in X_0. Among these, we choose m_∞ so that $P(m_\infty)$ is minimal. Notice that when $P(m_\infty) > 1$, then by minimality there are infinitely many monomials $m' \in M_0$ with $P(m') > 1$, and hence $\prod_{i\in I} P(m_i) = \infty$ (†).

We are now ready to define the sequence $(w_l)_{l\in L}$.

- if $l \in I_0$, we set $w_l = m_l$ (i.e., we copy all finitely often occurring monomials),
- if $l = (n, m) \in \omega\times\omega$, we set w_l to the n-th monomial in M_∞ (i.e., we repeat each monomial in M_∞ infinitely often); if $M_\infty = \varnothing$ we set $w_l = m_\infty$ (or we omit the $(\omega \times \omega)$-part),
- if $l \in J$, we set $w_l = m_\infty$ (this is only for padding so that L has the right cardinality).

By construction, the monomials in $(w_l)_{l\in L}$ match the monomials in $(m_i)_{i\in I}$ in the following sense: each monomial appears either infinitely often in both sequences, or the same finite number of times in both sequences. The only exception is m_∞ in the case $M_\infty = \varnothing$, but in this case we know that m_∞ contains only indeterminates that have exponent ∞ in m. It follows that the products of the two sequences result in the same monomial m, so $(w_l)_{l\in L} \in \mathsf{splits}_L(m)$ as claimed.

It further follows (using compactness of $\mathbb{N}^\infty$) that the values of the sequences are also equal: $\prod_{i\in I} P(m_i) = \prod_{l\in L} P(w_l)$. Again, the case $M_\infty = \varnothing$ needs special attention. If $P(m_\infty) = 0$, then both sequences have value 0 and are equal. If $P(m_\infty) = 1$, then repeating m_∞ does not affect the value of the sequence. If $P(m_\infty) > 1$, then we have $\prod_{l\in L} P(w_l) = \infty$ due to the padding, but in this case also $\prod_{i\in I} P(m_i) = \infty$ by (†) and the equality still holds.

We can finally define the sequence $(v_j)_{j\in J} \in \mathsf{splits}_J(m)$. Notice that $|L| = |J|$ since J is infinite and both I_0 and $\omega \times \omega$ are countable. We thus have a bijection $f\colon J \to L$ and can set $v_j = w_{f(j)}$. Since f is bijective, $(v_j)_{j\in J}$ has the same product and value as $(w_l)_{l\in L}$, and thus also as $(m_i)_{i\in I}$.

We still have to prove that $Q(m) \le R(m)$. (This does not immediately follow from the above argument, since we have to sum over all sequences, but different sequences $(m_i)_{i\in I}$ could be mapped to the same sequence $(v_j)_{j\in J}$.) We proceed by a case distinction using Lemma 21. The case $Q(m) = 0$ is trivial. If $Q(m) = 1$, then the construction of $(v_j)_{j\in J}$ witnesses $R(m) \ge 1$. The only other possibility is $Q(m) = \infty$. If there is a sequence $(m_i)_{i\in I} \in \mathsf{splits}_I(m)$ with value ∞, then by the above construction also $R(m) = \infty$. Otherwise there must be a sequence (in fact infinitely many) $(m_i)_{i\in I} \in \mathsf{splits}_I(m)$ with value $1 < s < \infty$. At least two distinct monomials must occur in $(m_i)_{i\in I}$ (otherwise the value would be 0, 1, or ∞), and, by construction, these monomials must also be contained in the sequence $(v_j)_{j\in J}$. It follows that there are infinitely many pairwise different permutations of $(v_j)_{j\in J}$, and since all of these sequences occur in the summation we have $R(m) = \infty$. ◀

Most of the other requirements for ∞-semirings follow by applying the properties of infinitary operations in $\mathbb{N}^\infty$ to coefficients and exponents.

▶ **Theorem 23.** $\mathbb{N}^\infty[\![X^\infty]\!]$ *is a strongly distributive* ∞*-semiring.*

Proof. It follows directly from the definition that addition and multiplication in $\mathbb{N}^\infty[\![X^\infty]\!]$ are compatible with finite operations and respect neutral elements, and we have already considered infinite powers in Lemma 22. It then suffices to prove partition invariance and strong distributivity, as these imply all remaining properties.

Partition invariance of addition follows immediately from the respective property of $\mathbb{N}^\infty$, as addition is defined by adding coefficients. For multiplication, fix a partition $(I_j)_{j\in J}$ of I. Using strong distributivity of $\mathbb{N}^\infty$, it remains to prove for each monomial m:

$$\begin{aligned}\Big(\prod_{j\in J}\prod_{i\in I_j} P_i\Big)(m) &= \sum_{(m_j)_j\in\mathsf{splits}_J(m)} \prod_{j\in J} \sum_{(v_i)_i\in\mathsf{splits}_{I_j}(m_j)} \prod_{i\in I_j} P_i(v_i)\\ &= \sum_{(m_j)_j\in\mathsf{splits}_J(m)} \sum_{f\in F} \prod_{j\in J}\prod_{i\in I_j} P_i(f(j,i))\\ &\overset{!}{=} \sum_{(u_i)_i\in\mathsf{splits}_I(m)} \prod_{i\in I} P_i(u_i).\end{aligned}$$

Here, F is the set of choice functions that choose $(v_i)_i \in \mathsf{splits}_{I_j}(m_j)$ for each j. To simplify the presentation, we let $f(j,i) = v_i$ for the chosen sequence (i.e., we include the index i as argument).

We prove both directions of the last equality. First let $(u_i)_i \in \mathsf{splits}_I(m)$. Set $m_j = \prod_{i\in I_j} u_i$. Then $(m_j)_j \in \mathsf{splits}_J(m)$ by comparing exponents and using partition invariance of $\mathbb{N}^\infty$. For $i \in I_j$, we define $f(j,i) = u_i$. Since each i occurs in exactly one I_j, we have $\prod_{j\in J}\prod_{i\in I_j} P_i(f(j,i)) = \prod_{i\in I} P_i(u_i)$ by partition invariance of $\mathbb{N}^\infty$. Notice that our construction $(u_i)_i \mapsto ((m_j)_j, f)$ is injective, so direction $\geq$ holds (by partition invariance of addition in $\mathbb{N}^\infty$).

For the other direction, let $(m_j)_j \in \mathsf{splits}_J(m)$ and $f \in F$. For each $i \in I$, pick the unique j with $i \in I_j$ and set $u_i = f(j,i)$. Then (by partition invariance of $\mathbb{N}^\infty$ in each exponent):

$$\prod_{i\in I} u_i = \prod_{j\in J}\prod_{i\in I_j} u_i = \prod_{j\in J}\prod_{i\in I_j} f(j,i) = \prod_{j\in J} m_j = m,$$

so $(u_i)_i \in \mathsf{splits}_I(m)$. Applying the same argument to the coefficients yields

$$\prod_{i\in I} P_i(u_i) = \prod_{j\in J}\prod_{i\in I_j} P_i(u_i) = \prod_{j\in J}\prod_{i\in I_j} P_i(f(j,i)).$$

Again, the mapping $((m_j)_j, f) \mapsto (u_i)_i$ we construct is injective, so direction $\leq$ holds as well.

To prove strong distributivity, let $(I_j)_{j\in J}$ be a partition of I and F be the set of choice functions f with $f(j) \in I_j$. Strong distributivity then follows from strong distributivity and partition invariance of $\mathbb{N}^\infty$:

$$\begin{aligned}\Big(\prod_{j\in J}\sum_{i\in I_j} P_i\Big)(m) &= \sum_{(m_j)_j\in\mathsf{splits}_J(m)} \prod_{j\in J}\sum_{i\in I_j} P_i(m_j) \\ &= \sum_{(m_j)_j\in\mathsf{splits}_J(m)} \sum_{f\in F}\prod_{j\in J} P_{f(j)}(m_j) \\ &= \sum_{f\in F}\sum_{(m_j)_j\in\mathsf{splits}_J(m)} \prod_{j\in J} P_{f(j)}(m_j) \\ &= \Big(\sum_{f\in F}\prod_{j\in J} P_{f(j)}\Big)(m). \end{aligned}$$ ◀

We now establish a kind of universal property of $\mathbb{N}^\infty[\![X^\infty]\!]$. This shows, in particular, that $\mathbb{N}^\infty[\![X^\infty]\!]$ is the free strongly distributive ∞-semiring.

▶ **Theorem 24** (κ-universality)**.** *Let $\mathcal{S}$ be a κ-distributive ∞-semiring. Every mapping $h\colon X \to \mathcal{S}$ extends uniquely to a semiring κ-homomorphism $h\colon \mathbb{N}^\infty[\![X^\infty]\!] \to \mathcal{S}$.*

Proof. For every element $s \in \mathcal{S}$ there exist unique elements $\infty \cdot s = \sum_{i\in I} s$ and $s^\infty = \prod_{i\in I} s$ for all infinite index sets I. Thus, $n \cdot s$ and s^n are well-defined for all $s \in \mathcal{S}$ and $n \in \mathbb{N}^\infty$. We first lift h to monomials m by setting $h(m) := \prod_{x\in X} h(x)^{m(x)}$. By partition invariance, h commutes with (finite and infinitary) products of monomials:

$$h(\prod_{i\in I} m_i) = \prod_{x\in X} h(x)^{\sum_{i\in I} m_i(x)} = \prod_{x\in X}\prod_{i\in I} h(x)^{m_i(x)} = \prod_{i\in I}\prod_{x\in X} h(x)^{m_i(x)} = \prod_{i\in I} h(m_i).$$

We write power series $P \in \mathbb{N}^\infty[\![X^\infty]\!]$ as $P = \sum_{m\in M}(P(m)\cdot m)$. Then h is uniquely defined by $h(P) := \sum_{m\in M}(P(m)\cdot h(m))$. We need to show that h commutes with the (finite and infinitary) semiring operations. Since infinitary operations are compatible with the finite ones, it suffices to prove that h commutes with the infinitary sum and product. This is easy for summation (over index sets of arbitrary cardinality) due to partition invariance:

$$h\Big(\sum_{i\in I} P_i\Big) = \sum_{m\in M}\Big(\sum_{i\in I} P_i(m)\Big)\cdot h(m) = \sum_{m\in M}\sum_{i\in I}(P_i(m)\cdot h(m)) = \sum_{i\in I} h(P_i).$$

For products, we use partition invariance (pi) and strong distributivity (sd) for the cardinality of I. Let F be the set of (unrestricted) functions $f\colon I\to M$. Then,

$$\begin{aligned}\prod_{i\in I} h(P_i) &= \prod_{i\in I}\sum_{m\in M}(P_i(m)\cdot h(m)) \overset{\text{(sd)}}{=} \sum_{f\in F}\prod_{i\in I} P_i(f(i))\cdot h(f(i))\\ &\overset{\text{(pi)}}{=} \sum_{m\in M}\sum_{(m_i)_i\in\mathsf{splits}(m)}\Big(\prod_{i\in I}(P_i(m_i)\cdot h(m_i))\Big)\\ &\overset{\text{(pi)}}{=} \sum_{m\in M}\sum_{(m_i)_i\in\mathsf{splits}(m)}\Big(\prod_{i\in I} P_i(m_i)\cdot\prod_{i\in I} h(m_i)\Big)\\ &= \sum_{m\in M}\sum_{(m_i)_i\in\mathsf{splits}(m)}\Big(\Big(\prod_{i\in I} P_i(m_i)\Big)\cdot h(m)\Big)\\ &\overset{\text{(pi)}}{=} \sum_{m\in M}\Big(\sum_{(m_i)_i\in\mathsf{splits}(m)}\prod_{i\in I} P_i(m_i)\Big)\cdot h(m) = h\Big(\prod_{i\in I} P_i\Big).\end{aligned}$$ ◀

▶ Remark 25. Notice that Theorem 24 no longer holds if we drop the exponent ∞. It is not clear how the infinite power $x\cdot x\cdot x\cdots$ is then defined, but we can argue by case distinction that in any case, the universal property is violated.

- If $\prod_{i<\omega} x = 0$, then $h(\prod_{i<\omega} x) = 0 \neq 1 = \prod_{i<\omega} h(x)$ for $h(x) = 1$ (say in the Viterbi semiring).
- If $P = \prod_{i<\omega} x \neq 0$, then there must a monomial m (with finite exponents) and coefficient $P(m) = n > 0$. For $h(x) = \frac{1}{2}$ into the Viterbi semiring, we then get $h(P) \geq h(n\cdot m) = n\cdot h(m) = h(m) > 0$, but also $\prod_{i<\omega} h(x) = (\frac{1}{2})^\infty = 0$, contradiction.

▶ Remark 26. Observe that $\mathbb{N}^\infty[\![X^\infty]\!]$ is not fully continuous. To see this, consider the decreasing chain $P_i = \sum_{i<j<\omega} x^j$, where the monomial x^i disappears in the i-th step and the infimum is thus 0. Then $x^\infty\cdot\prod_i P_i = x^\infty\cdot 0 = 0$, but $\prod_i(x^\infty\cdot P_i) = \prod_i \infty\cdot x^\infty = \infty\cdot x^\infty$.

Similar constructions of universal infinitary semirings are possible for smaller classes of semirings. For idempotent semirings, the appropriate semiring is $\mathbb{B}[\![X^\infty]\!]$, which is constructed in the same way as $\mathbb{N}^\infty[\![X^\infty]\!]$ but with Boolean coefficients. The above proofs can easily be adapted to show that $\mathbb{B}[\![X^\infty]\!]$ is a strongly distributive ∞-semiring and satisfies κ-universality for all idempotent κ-distributive semirings.

For absorptive semirings, the appropriate choice are generalised absorptive polynomials $\mathbb{S}^\infty(X)$, for a finite set X of indeterminates. These are known to be the freely generated absorptive, fully-continuous semirings (cf. [1]). Since fully-continuous homomorphisms are also ∞-homomorphisms (by Proposition 19), $\mathbb{S}^\infty(X)$ is universal also for all infinitary absorptive semirings (notice that we do not have to assume strong distributivity here). More recently, a version of $\mathbb{S}^\infty(X)$ with infinite indeterminate set X was studied in [11]. The resulting semiring is no longer fully continuous, but it is still κ-universal for infinitary absorptive semirings in the sense of Theorem 24 (i.e., assuming κ-distributivity of the target semiring).

Additionally requiring idempotence of multiplication leads to the class of lattice semirings. For a finite set X of indeterminates, the freely generated lattice semiring, also called PosBool(X), is finite and hence infinitary operations become trivial. For infinite X, one can consider the free completely distributive lattice (see, e.g., [10]) which, in our terminology, is universal for all strongly distributive infinitary lattice semirings.

6 Semiring Provenance for First-Order Logic

For a given finite relational vocabulary τ, we denote by $\mathrm{Lit}_n(\tau)$ the set of literals $R\bar{x}$ and $\neg R\bar{x}$ where $R \in \tau$ and $\bar{x}$ is a tuple of variables from $\{x_1, \dots, x_n\}$. The set $\mathrm{Lit}_A(\tau)$ refers to literals $R\bar{a}$ and $\neg R\bar{a}$ that are instantiated with elements from a universe A.

▶ **Definition 27** ($\mathcal{S}$-interpretation). *Given an ∞-semiring, $\mathcal{S}$, a mapping $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ is an $\mathcal{S}$-interpretation (of vocabulary τ and universe A). We say that $\mathcal{S}$ is model-defining if exactly one of the values $\pi(L)$ and $\pi(\neg L)$ is zero for any pair of complementary literals $L, \neg L \in \mathrm{Lit}_A(\tau)$.*

An $\mathcal{S}$-interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ inductively extends to valuations $\pi[\![\varphi(\bar{a})]\!]$ of instantiated first-order formulae $\varphi(\bar{x})$ in negation normal form. Equalities are interpreted by their truth value, that is $\pi[\![a = a]\!] := 1$ and $\pi[\![a = b]\!] := 0$ for $a \neq b$ (and analogously for inequalities). Based on that, the semantics of disjunction and existential quantifiers is defined via (possibly infinitary) sums, while conjunctions and universal quantifiers are interpreted as (possibly infinitary) products.

$$\pi[\![\psi \vee \vartheta]\!] := \pi[\![\psi]\!] + \pi[\![\vartheta]\!] \qquad \pi[\![\psi \wedge \vartheta]\!] := \pi[\![\psi]\!] \cdot \pi[\![\vartheta]\!]$$
$$\pi[\![\exists x \varphi(x, \bar{b})]\!] := \sum_{a \in A} \pi[\![\varphi(a, \bar{b})]\!] \qquad \pi[\![\forall x \varphi(x, \bar{b})]\!] := \prod_{a \in A} \pi[\![\varphi(a, \bar{b})]\!]$$

Negation is handled via negation normal form (denoted nnf), i.e. for every $\psi \in \mathrm{FO}$ we identify $\pi[\![\neg\psi]\!]$ with $\pi[\![\mathrm{nnf}(\neg\psi)]\!]$. This will allow us to compare valuations $\pi[\![\psi]\!]$ and $\pi[\![\neg\psi]\!]$ in a meaningful way. We now examine, which of the basic properties of first-order provenance, as listed for instance in [6] extend to the infinitary case. We start with the *fundamental property for first-order provenance* which is just the simple fact that semiring valuations are compatible with semiring homomorphisms. This obviously translates to the infinitary case, for homomorphisms that also preserve infinitary sums and products.

▶ **Proposition 28** (Fundamental Property). *Let $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ be an $\mathcal{S}$-interpretation with universe A of cardinality at most κ, and let $h\colon \mathcal{S} \to \mathcal{S}'$ be a semiring κ-homomorphism. Then, $(h \circ \pi)$ is an $\mathcal{S}'$-interpretation and $h(\pi[\![\varphi(\bar{a})]\!]) = (h \circ \pi)[\![\varphi(\bar{a})]\!]$ for all $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ and instantiations $\bar{a} \subseteq A$.*

Naturally ordered semirings are $+$-*positive*, and this trivially extends to infinite sums: $\sum_{i \in I} s_i = 0$ only if $s_i = 0$ for all $i \in I$. Recall that a semiring is *positive* if it is $+$-positive and has no divisors of 0. However, in many relevant positive semirings it my be the case that an infinite product evaluates to 0, although all its factors are positive. Simple examples are products $\prod_{i<\omega} s_i$ with $s_i \leq 1 - \varepsilon$ in the Viterbi semiring. We shall see that this may lead to the sometimes undesirable effect, that the semiring valuation of a true sentence may evaluate to 0.

▶ **Definition 29.** *We call an ∞-semiring ∞-positive if it is positive, and any infinitary product of non-zero elements is also non-zero.*

The characterisation of positive semirings by homomorphisms into the Boolean semiring extends to ∞-positivity.

▶ **Lemma 30.** *An ∞-semiring $\mathcal{S}$ is ∞-positive if, and only if, $\dagger_S : S \to \mathbb{B}$, defined by*

$$\dagger_S(s) = \begin{cases} \top & \text{if } s \neq 0 \\ \bot & \text{if } s = 0 \end{cases}$$

is an ∞-homomorphism.

A model-defining $\mathcal{S}$-interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ defines the unique (classical) structure $\mathfrak{A}_\pi$ with universe A, such that $\mathfrak{A}_\pi \models L$ for a literal $L \in \mathrm{Lit}_A(\tau)$ if, and only if, $\pi(L) \neq 0$. We can identify $\mathfrak{A}_\pi$ with its canonical $\mathbb{B}$-interpretation $\dagger_{\mathcal{S}} \circ \pi$ which maps the true literals to $\top$, and the false ones to $\bot$. (Notice that $\dagger_{\mathcal{S}} \circ \pi$ is well-defined for every semiring, although $\dagger_{\mathcal{S}}$ is an ∞-homomorphism, only for ∞-positive semirings.)

Here is an associated semantical notion. We call $\mathcal{S}$ *truth-preserving* (for first-order logic) if for every model-defining $\mathcal{S}$-interpretation and every sentence $\psi \in \mathrm{FO}(\tau)$ we have that $\pi[\![\psi]\!] \neq 0$ if, and only if $\mathfrak{A}_\pi \models \psi$.

▶ **Proposition 31.** *A ∞-semiring $\mathcal{S}$ is truth-preserving if, and only if, it is ∞-positive.*

Proof. If $\mathcal{S}$ is ∞-positive then $\dagger_{\mathcal{S}}$ is an ∞-homomorphism. Given any model-defining $\mathcal{S}$-interpretation π, then by the fundamental property,

$$\mathfrak{A}_\pi \models \psi \iff (\dagger_{\mathcal{S}} \circ \pi)[\![\psi]\!] = \top \iff \dagger_{\mathcal{S}}(\pi[\![\psi]\!]) = \top \iff \pi[\![\psi]\!] \neq 0.$$

Conversely, assume that $\mathcal{S}$ is an infinitary semiring that is not ∞-positive. Then there exists a non-empty finite or infinite sequence $(s_a)_{a\in A}$ such $s_a \neq 0$ for all $a \in A$, but $\prod_{a\in A} s_a = 0$. We use this to define an $\mathcal{S}$-interpretation with universe A and one unary predicate P such that $\pi(Pa) = s_a$ (and $\pi(\neg Pa) = 0$) for all $a \in A$. The model defined by π is $\mathfrak{A}_\pi = (A, P)$ with $P = A$, and clearly $\mathfrak{A}_\pi \models \forall x Px$. However, $\pi[\![\forall x Px]\!] = \prod_{a\in A} s_a = 0$, so $\mathcal{S}$ is not truth-preserving. ◀

Many interesting semiring interpretations in provenance analysis do not define a single structure but a whole class of structures (with common universe and common vocabulary). In general, we can assume that such interpretations are consistent, in the sense that valuations of complementary literals satisfy certain constraints, although they need not be as strict as those for model-defining interpretations. We examine how such constraints for literals constrain the valuations of arbitrary first-order sentences.

▶ **Proposition 32.** *Let $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ be a $\mathcal{S}$-interpretation.*
- *If for every $L \in \mathrm{Lit}_A(\tau)$ at least one of $\pi(L)$ and $\pi(\neg L)$ is 0 then for any sentence $\psi \in \mathrm{FO}$, at least one of $\pi[\![\psi]\!]$ and $\pi[\![\neg\psi]\!]$ is 0.*
- *If for every $L \in \mathrm{Lit}_A(\tau)$ we have $\pi(L) \cdot \pi(\neg L) = 0$ then for any sentence ψ we have $\pi[\![\psi]\!] \cdot \pi[\![\neg\psi]\!] = 0$.*

Proof. If ψ is not a literal, then there exists a finite or infinite collection of $(\varphi_i)_{i\in I}$ of sentences such that one of the values $\pi[\![\psi]\!]$ and $\pi[\![\neg\psi]\!]$ is the sum $\sum_{i\in I}[\![\varphi_i]\!]$, and the other is the product $\prod_{i\in I} \pi[\![\neg\varphi_i]\!]$. To prove the first claim, assume that $\pi[\![\psi]\!]$ and $\pi[\![\neg\psi]\!]$ are both non-zero. It follows that all values $\pi[\![\neg\varphi_i]\!]$ are non-zero. But by induction hypothesis, this implies that all values $\pi[\![\varphi_i]\!]$, and hence also their sum, must be 0, so we have a contradiction.

For the second claim, assume by induction hypothesis, that $\pi[\![\varphi_i]\!] \cdot \pi[\![\neg\varphi_i]\!] = 0$ for all $i \in I$. With weak distributivity, it then follows that

$$\begin{aligned}\pi[\![\psi]\!] \cdot \pi[\![\neg\psi]\!] &= \sum_{i\in I} \pi[\![\varphi_i]\!] \cdot \prod_{j\in I} \pi[\![\neg\varphi_j]\!] = \sum_{i\in I} \Big(\pi[\![\varphi_i]\!] \cdot \prod_{j\in I} \pi[\![\neg\varphi_j]\!] \Big) \\ &= \sum_{i\in I} \Big(\pi[\![\varphi_i]\!] \cdot \pi[\![\neg\varphi_i]\!] \cdot \prod_{j\in I\setminus\{i\}} \pi[\![\neg\varphi_j]\!] \Big) = 0.\end{aligned}$$ ◀

Proposition 32 holds in arbitrary ∞-semirings and supports a kind of "consistency", with the two kinds coinciding when the semiring has no divisors of 0. A related question concerns the constraint that complementary literals are not both mapped to 0, i.e. they are not both considered false under the same interpretation. We would like to conclude that this constraint as well translates to arbitrary sentences. But this is, in general, true only for ∞-positive ∞-semirings.

▶ **Proposition 33.** *Let $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ be an $\mathcal{S}$-interpretation into a ∞-positive ∞-semiring. If for every $L \in \mathrm{Lit}_A(\tau)$ we have $\pi(L) \neq 0$ or $\pi(\neg L) \neq 0$ (equivalently, $\pi(L) + \pi(\neg L) \neq 0$) then for any sentence ψ we have $\pi[\![\psi]\!] \neq 0$ or $\pi[\![\neg\psi]\!] \neq 0$ (equivalently, $\pi[\![\psi]\!] + \pi[\![\neg\psi]\!] \neq 0$).*

Proof. Towards a contradiction, suppose that $\pi[\![\psi]\!] = \pi[\![\neg\psi]\!] = 0$. As in the proof above, take sentences $(\varphi_i)_{i\in I}$ such that one of the values $\pi[\![\psi]\!]$ and $\pi[\![\neg\psi]\!]$ is the (finite or infinite) sum $\sum_{i\in I}[\![\varphi_i]\!]$, and the other is the (finite or infinite) product $\prod_{i\in I}\pi[\![\neg\varphi_i]\!]$. Since $\mathcal{S}$ is ∞-positive, it follows that $\pi[\![\neg\varphi_i]\!] = 0$ for at least one $i \in I$. By induction hypothesis, $\pi[\![\varphi_i]\!] \neq 0$, which, by $+$-positivity, contradicts the assumption that $\pi[\![\psi]\!] = 0$. ◀

The example given in the proof of Proposition 31 shows that the condition of ∞-positivity is necessary for this proposition.

7 Proof Trees

A fundamental theorem for the provenance analysis of first-order logic says that, for every semiring interpretation (over a finite domain) the valuation of a first-order sentence coincides with the sum of the valuations for its proof trees or, equivalently, the sum of the valuations of the strategies for the verifier in the associated model checking game. In game theoretic terms this has been shown in [5], and in terms of proof trees, this is presented in [6]. The question arises under which conditions this theorem generalises to semiring interpretations over infinite domains. For this purpose, we inspect the proof of the Sum-of-Proof-Trees-Theorem [6, Sect. 3.5] to see what properties of the infinitary semiring operations are needed for extending the proof to infinite domains. We first recall the relevant definitions.

An *evaluation tree* for a sentence $\psi \in \mathrm{FO}(\tau)$ on a (possibly infinite) universe A is a directed tree $\mathcal{T}$ whose nodes are labelled by formulae $\varphi(\bar{a})$, where $\varphi(\bar{x})$ is an occurrence[2] of a subformula in ψ whose free variables $\bar{x}$ are instantiated by a tuple $\bar{a}$ of elements from A, such that the following conditions hold.

- The root of $\mathcal{T}$ is ψ.
- A node $\varphi \vee \vartheta$ has one child which is labelled by either φ or ϑ.
- A node $\varphi \wedge \vartheta$ has two children labelled by φ and ϑ, respectively.
- A node $\exists y\, \varphi(\bar{a}, y)$ has one child labelled $\varphi(\bar{a}, b)$ for some $b \in A$.
- A node $\forall y\, \varphi(\bar{a}, y)$ has for each for all $b \in A$ a child labelled by $\varphi(\bar{a}, b)$ for all $b \in A$.
- The leaves of $\mathcal{T}$ are literals $L \in \mathrm{Lit}_A(\tau)$.

For any literal L, $\#_L(\mathcal{T}) \in \mathbb{N} \cup \{\infty\}$ denotes the number of occurrences of L in $\mathcal{T}$. The valuation of $\mathcal{T}$ for a semiring interpretation $\pi : \mathrm{Lit}_A(\tau) \to \mathcal{S}$ into an infinitary semiring $\mathcal{S}$ is defined as

$$\pi(\mathcal{T}) := \prod_{L\in\mathrm{Lit}_A(\tau)} \pi(L)^{\#_L(\mathcal{T})}.$$

Since $\mathcal{S}$ has unique infinite powers, there is a well-defined value $\pi(L)^\infty \in \mathcal{S}$, hence $\pi(\mathcal{T})$ is well-defined for all $\mathcal{S}$-interpretations π into ∞-semirings.

[2] Notice that we consider different occurrences of the same subformula as separate objects. In particular, a sentence $\varphi \vee \varphi$ has twice as many evaluation trees as φ.

A *proof tree* for π and $\psi \in \mathrm{FO}(\tau)$ is an evaluation tree $\mathcal{T}$ with $\pi(\mathcal{T}) \neq 0$. If π is clear from the context, we write $T(\psi)$ for the set of all proof trees for π and ψ.

▶ **Theorem 34** (Sum of Proof Trees). *Let A be domain of cardinality $< \kappa$, and let $\mathcal{S}$ be a κ-distributive ∞-semiring. For every semiring interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ and every sentence $\psi \in \mathrm{FO}(\tau)$, we have that*

$$\pi[\![\psi]\!] = \sum_{\mathcal{T} \in T(\psi)} \pi(\mathcal{T}).$$

Proof. We proceed by induction on ψ.

- Let ψ be a literal. If $\pi(\psi) = 0$ then ψ has no proof tree, so the sum over the valuations of its proof trees is 0. Otherwise ψ has precisely one proof tree which is the literal itself. In both cases, the desired equality holds trivially.
- Let $\psi = \varphi \vee \vartheta$. A proof tree $\mathcal{T}$ for ψ has the root ψ followed by a proof tree $\mathcal{T}'$ for either φ or for ϑ; clearly $\pi(\mathcal{T}) = \pi(\mathcal{T}')$. Thus

$$\pi[\![\psi]\!] = \pi[\![\varphi]\!] + \pi[\![\vartheta]\!] = \sum_{\mathcal{T}' \in T(\varphi)} \pi(\mathcal{T}') + \sum_{\mathcal{T}' \in T(\vartheta)} \pi(\mathcal{T}') = \sum_{\mathcal{T} \in T(\psi)} \pi(\mathcal{T}).$$

- Let $\psi = \varphi \wedge \vartheta$. A proof tree $\mathcal{T}$ for ψ has the root ψ, attached to which are a proof tree $\mathcal{T}'$ for φ and a proof tree $\mathcal{T}''$ for ϑ. We can thus identify every $\mathcal{T} \in T(\psi)$ with a pair $(\mathcal{T}', \mathcal{T}'') \in T(\varphi) \times T(\vartheta)$, and since $\#_L(\mathcal{T}) = \#_L(\mathcal{T}') + \#_L(\mathcal{T}'')$ for every literal L we have that, $\pi(\mathcal{T}) = \pi(\mathcal{T}')\pi(\mathcal{T}'')$. It follows, by weak distributivity, that

$$\begin{aligned}\pi[\![\psi]\!] = \pi[\![\varphi]\!] \cdot \pi[\![\vartheta]\!] &= \sum_{\mathcal{T}' \in T(\varphi)} \pi(\mathcal{T}') \cdot \sum_{\mathcal{T}'' \in T(\vartheta)} \pi(\mathcal{T}'') \\ &= \sum_{(\mathcal{T}', \mathcal{T}'') \in T(\psi)} \pi(\mathcal{T}')\pi(\mathcal{T}'') = \sum_{\mathcal{T} \in T(\psi)} \pi(\mathcal{T}).\end{aligned}$$

- If $\psi = \exists y\, \varphi(y)$, then a proof tree $\mathcal{T}$ for ψ consists of the the root ψ, attached to which is a proof tree $\mathcal{T}_a$ for $\varphi(a)$, for some $a \in A$. Clearly $\pi(\mathcal{T}) = \pi(\mathcal{T}_a)$. It follows, by partition invariance of the infinitary sum, that

$$\pi[\![\psi]\!] = \sum_{a \in A} \pi[\![\varphi(a)]\!] = \sum_{a \in A} \sum_{\mathcal{T}_a \in T(\varphi(a))} \pi(\mathcal{T}_a) = \sum_{\mathcal{T} \in T(\psi)} \pi(\mathcal{T}).$$

- Let finally $\psi = \forall y\, \varphi(y)$. A proof tree for ψ consists of the the root ψ attached to which are proof trees $\mathcal{T}_a$ for $\varphi(a)$, for all $a \in A$. We can thus identify such a proof tree with a choice function $\mathcal{T}$ that associates with every $a \in A$ a proof tree $\mathcal{T}_a \in T(\varphi(a))$, and thus $T(\psi)$ with the set of such choice functions. Further, for every literal L, we have that $\#_L(\mathcal{T}) = \sum_{a \in A} \#_L(\mathcal{T}_a)$ and therefore $\pi(\mathcal{T}) = \prod_{a \in A} \pi(\mathcal{T}_a)$. It follows, by κ-distributivity for the index set A, that

$$\pi[\![\psi]\!] = \prod_{a \in A} \pi[\![\varphi(a)]\!] = \prod_{a \in A} \sum_{\mathcal{T} \in T(\varphi(a))} \pi(\mathcal{T}) = \sum_{\mathcal{T} \in T(\psi)} \prod_{a \in A} \pi(\mathcal{T}_a)) = \sum_{\mathcal{T} \in T(\psi)} \pi(\mathcal{T}). \quad ◀$$

▶ **Example 35.** To see that κ-distributivity is not only used in the proof, but is indeed necessary for the the Sum-of-Proof-Trees-Theorem, we present an example of a semiring interpretation into the Viterbi semiring $\mathbb{V} = ([0,1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ (a ω_1-distributive ∞-semiring whose infinitary operations are supremum and infinite product) over the uncountable domain

$\mathcal{P}^\infty(\mathbb{N})$ of infinite sets of natural numbers and the vocabulary of one binary relation E. The interpretation $\pi\colon \mathrm{Lit}_{\mathcal{P}^\infty(\mathbb{N})}(\{E\}) \to \mathbb{V}$ is defined by

$$\pi(Eab) := \begin{cases} 0 & \text{if } a \cap b = \varnothing \\ 1 - \frac{1}{2+\min(a\cap b)} & \text{otherwise.} \end{cases}$$

Let $\psi := \forall x \exists y Exy$. If the Sum-of-Proof-Trees-Theorem were true also in this case, we would have that

$$\pi[\![\psi]\!] = \sup_{\mathcal{T}\in T(\psi)} \pi(\mathcal{T}).$$

But this is not the case. On the one side, we have that

$$\pi[\![\psi]\!] = \prod_{a\in\mathcal{P}^\infty(\mathbb{N})} \sup_{b\in\mathcal{P}^\infty(\mathbb{N})} \pi(Eab) = 1.$$

Indeed, for every infinite subset $a \subseteq \mathbb{N}$ and every $n \in \mathbb{N}$ we can take $b := a \cap [n, \infty)$, and we then have that $\pi(Eab) \geq 1 - \frac{1}{2+n}$ which implies that $\sup_{b\in\mathcal{P}^\infty(\mathbb{N})} \pi(Eab) = 1$ for every a.

On the other side, the proof trees in $T(\psi)$ are in one-to one correspondence with the functions $e\colon \mathcal{P}^\infty(\mathbb{N}) \to \mathcal{P}^\infty(\mathbb{N})$ such that $a \cap e(a) \neq \varnothing$ for all a. Let $\mathcal{T}_e$ be the proof tree associated with e. From its root ψ, $\mathcal{T}_e$ branches out to the nodes $\exists y Eay$, for all $a \in \mathcal{P}^\infty(\mathbb{N})$, each of which has a unique child Eab, namely the one with $b = e(a)$. The valuation of $\mathcal{T}_e$ is

$$\pi(\mathcal{T}_e) = \prod_{a\in\mathcal{P}^\infty(\mathbb{N})} \pi(Eae(a)) = \prod_{n\in\mathbb{N}} (1 - \tfrac{1}{2+n})^{\#\{a\colon \min(a\cap e(a))=n\}}.$$

Since there are uncountably many $a \in \mathcal{P}^\infty(\mathbb{N})$ there exist $n \in \mathbb{N}$ such that $\min(a \cap e(a)) = n$ for infinitely (in fact uncountably) many a. Hence $\pi(T_e)$ is an infinite product in which infinitely many factors are $(1 - \frac{1}{2+n})$, hence $\pi(T_e) = 0$. Since this holds for all e, and hence all proof trees for ψ we have that

$$\sup_{T\in\mathcal{T}(\psi)} \pi(\mathcal{T}) = 0 \neq 1 = \pi[\![\psi]\!].$$

Similar examples can be constructed for the semirings $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{D}$.

An interesting application of the Sum-of-Proof-Trees-Theorem concerns interpretations into semirings with dual indeterminates, as proposed in [4] and further studied in [6].

Let $X, \bar{X}$ be two disjoint finite sets of indeterminates together with a one-to-one correspondence $X \leftrightarrow \bar{X}$, and denote by $x \in X$ and $\bar{x} \in \bar{X}$ two elements that are in this correspondence. We shall use X for positive literals $R\bar{a}$ and $\bar{X}$ for negated literals $\neg R\bar{a}$. By convention, if we annotate $R\bar{a}$ with x, then $\bar{x}$ can only be used to annotate $\neg R\bar{a}$, and vice versa. We refer to x and $\bar{x}$ as *complementary variables.*

Analogous to the construction of the semiring $\mathbb{N}[X, \bar{X}]$ of dual-indeterminate polynomials in [4] we can define the semiring of dual-indeterminate power series $\mathbb{N}^\infty[\![X^\infty, \bar{X}^\infty]\!]$ as the quotient of $\mathbb{N}^\infty[\![(X \cup \bar{X})^\infty]\!]$ via the congruence induced by $x \cdot \bar{x} = 0$, or, equivalently, as the semiring of power series with indeterminates in $X \cup \bar{X}$ whose monomials do not contain complementary variables. We can multiply such power series as above, provided that we eliminate the monomials with complementary variables afterwards.

Most of the results and applications exhibited in [6] generalise to this setting. As an example, we mention the information that the Sum-of-Proof-Trees-Theorem delivers for model-compatible interpretations.

▶ **Definition 36.** *A model-compatible* $\mathbb{N}^\infty[\![X^\infty, \bar{X}^\infty]\!]$*-interpretation is a semiring interpretation* $\pi\colon \mathrm{Lit}_A(\tau) \to \mathbb{N}^\infty[\![X^\infty, \bar{X}^\infty]\!]$ *such that for each atom* $R\bar{a}$ *one of the following (mutually exclusive) three properties holds:*

1. $\pi(R\bar{a}) = x$ *and* $\pi(\neg R\bar{a}) = \bar{x}$ *for some* $x \in X$*, or*
2. $\pi(R\bar{a}) = 0$ *and* $\pi(\neg R\bar{a}) = 1$*, or*
3. $\pi(R\bar{a}) = 1$ *and* $\pi(\neg R\bar{a}) = 0$*.*

Contrary to model-defining interpretations, a model-compatible interpretation π defines, in general, not a single structure, but a class of structures compatible with π, consisting of all structures $\mathfrak{A}$ (with universe A and vocabulary τ) such only those literals $L \in \mathrm{Lit}_A(\tau)$ can be true in $\mathfrak{A}$ for which $\pi(L) \in X \cup \bar{X} \cup \{1\}$.

▶ **Corollary 37.** *Let* $\pi\colon \mathrm{Lit}_A(\tau) \to \mathbb{N}^\infty[\![X^\infty, \bar{X}^\infty]\!]$ *be model-compatible and let* ψ *be sentence in* $\mathrm{FO}(\tau)$*. Then the power series* $\pi[\![\psi]\!]$ *describes all proof trees that verify* ψ *using premises from the literals that* π *maps to indeterminates or to 1.*

Specifically, each monomial $c\, x_1^{e_1} \cdots x_k^{e_k}$ *in* $\pi[\![\psi]\!]$ *stands for* c *distinct proof trees that use* e_1 *times the literal annotated by* x_1*, ..., and* e_k *times the literal annotated by* x_k*, where* $x_1, \dots, x_k \in X \cup \bar{X}$*. In particular, when* $\pi[\![\psi]\!] = 0$ *no proof tree exists, and hence there is no model of* ψ *that is compatible with* π*.*

8 Summary and Conclusion

Up to now, semiring provenance has essentially been restricted to finite data, typically to database queries against a finite, possibly annotated, database, to first-order sentences evaluated over a finite domain, or to the strategy analysis for a (possibly infinite) game, played on a finite game graph.

In this paper we have provided foundations for semiring provenance over infinite domains. This required to expand semirings by infinitary sum and product operators, and we have investigated in detail the necessary, or at least desirable, algebraic properties that should hold for these operators. Clearly the infinitary operators must be compatible with finite sums and products. Partition invariance, a natural generalisation of associativity, turned out to be a quite strong property which also implies bijection invariance, the natural generalisation of commutativity. However, we have seen that these basic properties do not suffice to exclude "pathological" operators, and we have investigated a number of other algebraic properties, including (strong) compactness, the respect of neutral and idempotent elements, and the existence of unique powers. We have seen that compactness is a very powerful property, which implies the other ones, but since it is sometimes hard to verify, and does in fact not hold in all interesting semirings, we have decided not to impose it as a necessary requirement in our definition of infinitary semirings. Instead we work with the weaker requirements that neutral elements are respected and that there exist unique infinite powers.

Distributivity and monotonicity are the fundamental algebraic properties that govern the interplay of sums and products in (naturally ordered) semirings. The generalisation of the distributive law from finite to infinitary semirings comes in two variants, a weak one and a strong one. While weak distributivity is unproblematic in the semirings we consider, it turned out that strong distributivity is more delicate. In some important semirings it does not hold for arbitrary index sets but only for countable ones. On the other side, strong distributivity is an important property which, for instance, implies monotonicity and is also used later in the Sum-of-Proof-Trees-Theorem. We decided to omit strong distributivity in our definition of infinitary semirings, and to require only weak distributivity and monotonicity. Instead we have introduced also the variant of a κ-distributive infinitary semiring.

Based on this algebraic analysis, and on the requirements for provenance valuations on infinite structures, we thus have come to a proposal for an appropriate definition of infinitary semirings. We have also discussed the appropriate notions of homomorphisms among infinitary semirings. We have studied how finite semirings and infinite lattice semirings can be expanded to infinitary semirings, and we have seen that the extension $\mathbb{N}^\infty$ of the natural semiring $\mathbb{N}$ is a strongly distributive infinitary semiring. For the class of absorptive semirings, multiplication is decreasing and hence infinitary operations can be defined in a natural way by suprema and infima over finite subsets. Finally we have investigated the generalisation of the universal semiring of multivariate polynomials, $\mathbb{N}[X]$, to an infinitary semiring $\mathbb{N}^\infty[\![X^\infty]\!]$ of generalised power series. We have proved that $\mathbb{N}^\infty[\![X^\infty]\!]$ is indeed universal for strongly distributive infinitary semirings.

In the last two sections, we have shown that, based on these infinitary semirings, the provenance analysis for first-order logic can indeed be extended from finite structures to infinite ones, preserving the basic results of the theory. In particular, we have proved that the Sum-of-Proof-Trees-Theorem, saying that the semiring valuation of a first-order sentence coincides with the sum of the valuations of its proof trees, also holds on domains $< \kappa$, provided that the underlying infinitary semiring is κ-distributive. Further, we have briefly discussed the use of dual indeteminates (for treating negation) which leads to semirings of dual-indeterminate generalised power series. The Sum-of-Proof-Trees-Theorem, applied to a model-compatible interpretation into such a semiring, gives valuations that describe all proof trees of a sentence, with precise information, which of the tracked literals are actually used in a proof tree, and how often.

A limitation of this result is that the semirings of generalised power series have only finitely many indeterminates. This means that although we can deal with infinite structures, we can track inside of these only finitely many literals, and take the truth values of the others for granted. Over an infinite universe, a model-compatible interpretation thus defines a class of structures in which the truth values of all but finitely many literals coincide. In this way, we can thus track finite data embedded into an infinite background structure, but not the collection of all atomic facts in an infinite structure. For the Sum-of-Proof-Trees-Theorem as such, no such restriction applies, so it can be used for provenance valuations in application semirings over arbitrary infinite domains.

To overcome this limitation of $\mathbb{N}^\infty[\![X^\infty]\!]$, we would need universal provenance semirings with infinitely many variables. We have seen that for $\mathbb{N}^\infty[\![X^\infty]\!]$ itself, the extension to infinite sets X would not be consistent with our definition of infinitary semirings, but such extensions seem possible in settings of absorptive provenance semirings such as $\mathbb{S}^\infty(X)$ and $\mathrm{PosBool}(X)$, and perhaps also for Why-semirings. But this will have to be studied elsewhere.

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