Periodic Timetabling: Travel Time vs. Regenerative Energy

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Abstract

While it is important to provide attractive public transportation to the passengers allowing short travel times, it should also be a major concern to reduce the amount of energy used by the public transport system. Electrical trains can regenerate energy when braking, which can be used by a nearby accelerating train. Therefore, apart from the minimization of travel times, the maximization of brake-traction overlaps of nearby trains is an important objective in periodic timetabling. Recently, this has been studied in a model allowing small modifications of a nominal timetable. We investigate the problem of finding periodic timetables that are globally good in both objective functions. We show that the general problem is NP-hard, even restricted to a single transfer station and if only travel time is to be minimized, and give an algorithm with an additive error bound for maximizing the brake-traction overlap on this small network. Moreover, we identify special cases in which the problem is solvable in polynomial time. Finally, we demonstrate the trade-off between the two objective functions in an experimental study.

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1 Introduction

In order to reach the climate goals, it is necessary to strengthen the role of public transportation in passenger transport. However, also the public transport system itself consumes a large amount of energy. Modern electric motors are able to regenerate energy while braking. In the context of rail traffic, the most efficient way to use the regained energy is to transfer it via the catenary to an accelerating train close by. Therefore, it is sensible to schedule train timetables in a way that synchronizes braking and acceleration processes of nearby trains. Such a schedule has two advantages concerning the energy usage. First, it enables a maximum usage of the regenerated energy and, hence, reduces the total amount of energy that needs to be bought by the public transport company. Second, it prevents power peaks that might surcharge the transportation system's power supply.

However, from a passenger perspective, this synchronization of braking and acceleration processes of two trains is the worst possible case as it prevents a passenger transfer from the braking to the accelerating train. Narrowly missing a train leads to frustration of the passengers and long waiting times might cause them to choose the car over public transport.

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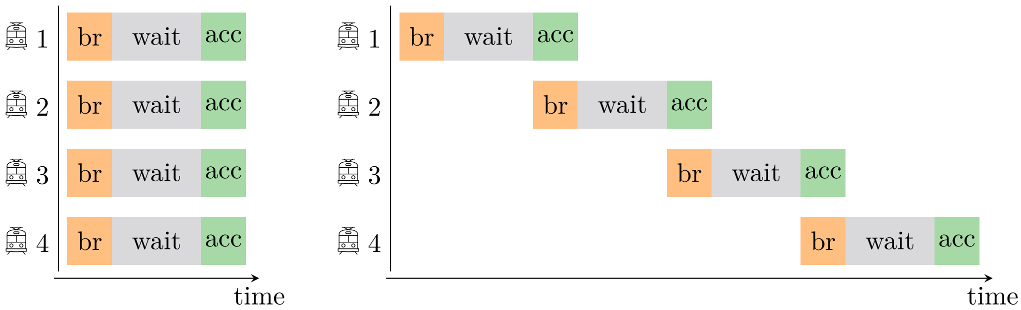


(a) arr/dep around fixed time. **(b)** Simultaneous braking/accelerating of pairs of trains.

Figure 1 Timetable patterns of braking, waiting and acceleration phases of four trains.

As an illustration of this trade-off, we consider an example motivated by Swiss railways. Here, the operated timetable prevents such situations of narrowly missing a train. The trains are scheduled in a regular interval timetable. At each station there is a fixed time. Shortly before this time all trains stopping at the station arrive, and the trains depart shortly after that time, see Figure 1a. This enables short transfer times to all directions. On the other hand, a transfer of regenerative braking energy from one train to another is impossible. For this objective, an efficient timetable would schedule the trains one after another such that the braking and acceleration phases overlap pairwise, see Figure 1b.

While it is beneficial to the environment to use as much of the regenerative energy as possible, it is also of utmost importance to provide attractive public transportation to the passengers. In this paper, we investigate a bicriteria problem with the aims to maximize the brake-traction overlap enabling the usage of regenerative energy and to minimize the passengers' travel times. We study this problem in the periodic version, where all train lines are operated repeatedly with a fixed period time.

Related Work

The task of designing efficient railway timetables has been subject to study at least since 1989 [12]. Traditionally, the literature on timetabling focuses on minimizing the passengers' travel time. An overview can be found in [7]. As mentioned above, the increasing importance of saving energy has sparked significant research efforts towards this goal in the engineering sciences. A complete review of all these works goes far beyond the scope of this paper. Instead, we only mention some particularly important papers and refer to the survey by Scheepmaker, Goverde, and Kroon [11] and the exemplary recent papers [8, 6, 13], which contain more extensive literature reviews.

There are two ways in which the timetable can affect the trains' energy consumption. On the one hand, there is the idea of saving energy by the implementation of energy-efficient driving strategies [5]. These depend on the time scheduled for each driving section; typically longer travel times require less energy. Ghoseiri, Szidarovszky, and Asgharpour [3] considered a multi-objective train scheduling model, combining the objectives of minimizing energy and minimizing travel time, and approximate the Pareto frontier using the ε-constraint method.

On the other hand, the timetable can influence the usage of regenerative energy in train systems. This was first researched by Ramos Pena, Fernández, and Cucala [10], who allow a modification of the dwell times to increase the brake-traction overlap. A more detailed modelling of the energy consumption that combines the driving strategies and the brake-traction overlaps has been studied by Yin, Yang, Tang, Gao, and Ran [17], who devised a

Lagrangian relaxation-based heuristic for this problem. In the other direction, Gupta, Tobin, and Pavel [2] considered a very simplified linear programming model to synchronize the start times of braking phases and the end times of acceleration times.

The bicriteria problem of minimizing the passenger travel times and maximizing the brake-traction overlap has been investigated by Yang, Ning, Li, and Tang [16], who developed a genetic algorithm for it. Moreover, Yang, Liao, Wu, Timmermans, Sun, and Gao [15] apply the NSGA-II algorithm for approximating the Pareto frontier.

All these works considered given aperiodic timetables that can be modified. Only recently, the study of the periodic version of this problem was initiated by Wang, Zhu, and Corman [14]. They assume a given nominal periodic timetable and develop a first model that can be used for local adjustments. On the one hand their aim is to maximize the brake-traction overlap to enable the usage of regenerative energy on a fixed set of synchronized arrival and departure events. On the other hand, they include passenger related objectives such as the minimization of the generalized average travel time of all passengers and the minimization of the maximum increase in individual's generalized travel time. Wang et al. also provide a visualization of the Pareto frontier for these objectives on an instance of Dutch railways.

Our contribution

1. We propose a mixed integer programming (MIP) formulation for the problem of maximizing the brake-traction overlap (PESP-Energy), based on the Periodic Event Scheduling Problem (PESP) (Section 2.1) and including the decision which acceleration and braking processes are synchronized.
2. We extend this MIP formulation to the bicriteria problem that additionally aims at minimizing the passengers' travel time (Section 2.3) and run numerical experiments on a single transfer station. (Section 5)

For our theoretical investigation, we focus on the problem restricted to a single transfer station, for which we derive the following results:

3. We characterize the structure of optimal solutions for the two single-objective problems (Propositions 8 and 9 and Theorem 10).
4. We show that only minimizing the transfer times is already NP-hard for a single transfer station (Theorem 6).
5. Based on a special-form TSP, we obtain a polynomial-time algorithm with an additive performance guarantee (depending on the input parameters) for the energy objective (Theorem 16). We show for some special cases that its solution is optimal (Section 4.4).

2 Including the Brake-Traction Overlap in the Periodic Event Scheduling Problem

In the timetabling problem, we are given a set of lines $l \in \mathcal{L}$, which are given as sequences of served stations $v \in \mathcal{V}$. Every line will be served periodically with the given period of T.

2.1 PESP-Passenger – Minimizing the Travel Times

For the PESP model we are given bounds on the durations of activities (driving, waiting, transfers) as well as weights which correspond to the number of passengers performing each activity. The objective is to minimize the total travel time of all passengers. For this problem, the *event-activity-network* (EAN) $\mathcal{E} = (E, A)$ for given directed lines $\mathcal{L}$ serving stations $v \in \mathcal{V}$ is a directed graph on all arrival and departure events $E = E_{\text{arr}} \,\dot\cup\, E_{\text{dep}}$, given by

$$E_{\text{arr}} \coloneqq \big\{(v, \ell, \text{arr}) \;\big|\; \ell \in \mathcal{L} \text{ arrives at } v \in \mathcal{V}\big\},\; E_{\text{dep}} \coloneqq \big\{(v, \ell, \text{dep}) \;\big|\; \ell \in \mathcal{L} \text{ departs at } v \in \mathcal{V}\big\}.$$

The activities $A \coloneqq A_{\text{drive}} \mathbin{\dot\cup} A_{\text{wait}} \mathbin{\dot\cup} A_{\text{trans}}$ connect the events as follows:

$$\begin{aligned}
A_{\text{drive}} &\coloneqq \big\{((v_1,\ell,\text{dep}),(v_2,\ell,\text{arr})) \in E_{\text{dep}} \times E_{\text{arr}} \,\big|\, \ell \text{ serves } v_2 \text{ directly after } v_1\big\},\\
A_{\text{wait}} &\coloneqq \big\{((v,\ell,\text{arr}),(v,\ell,\text{dep})) \in E_{\text{arr}} \times E_{\text{dep}}\big\},\\
A_{\text{trans}} &\coloneqq \big\{((v,\ell_1,\text{arr}),(v,\ell_2,\text{dep})) \in E_{\text{arr}} \times E_{\text{dep}} \,\big|\, \ell_1 \neq \ell_2\big\}.
\end{aligned}$$

A timetable $\pi\colon E \to \{0,\dots,T-1\}$ assigns a time π_i to each event $i \in E$, meaning that the event takes place at all times from $\pi_i + T\mathbb{Z}$. We can associate the bounds on the driving, transfer and waiting times with the activities: For each activity $a \in A$ let $\Delta_a = [l_a, u_a]$ be the set of allowed durations with $l_a, u_a \in \mathbb{Z}$. Since we only determine the times modulo T, we can ignore multiples of T in the activity durations and therefore assume that $0 \le l_a \le T-1$ and $0 \le u_a - l_a \le T-1$. Then a timetable is *feasible* if the periodic tensions $x_{ij} \coloneqq (\pi_j - \pi_i - l_{ij}) \bmod T + l_{ij}$ lie within the provided bounds for all $ij \in A$. In this paper, we assume the bounds on the transfer arcs a we have $u_a = l_a + T - 1$ and, therefore, the bounds on the transfers do not impose feasibility constraints.

The classical PESP seeks to find a feasible schedule in this network. The PESP-Passenger problem aims to find one with minimal total travel time. Let $w(ij)$ be the total number of passengers performing the activity $ij \in A$. Then we minimize the weighted sum of the periodic tensions (cf. objective (1)) of all activities, yielding a timetable that minimizes the passengers' travel times. This leads to the following mixed integer linear program [7].

$$\begin{array}{llllr}
\text{(PESP-P)} & \min & \displaystyle\sum_{ij\in A} w(ij)x_{ij} & & (1)\\
& \text{subject to} & x_{ij} = \pi_j - \pi_i + p_{ij}T & \forall ij \in A & (2)\\
& & l_{ij} \le x_{ij} \le u_{ij} & \forall ij \in A & (3)\\
& & 0 \le \pi_i \le T-1 & \forall i \in E & (4)\\
& & x_{ij} \in \mathbb{R},\ p_{ij} \in \mathbb{Z} & \forall ij \in A & (5)\\
& & \pi_i \in \mathbb{Z} & \forall i \in E & (6)
\end{array}$$

The variables p_{ij} are called periodic offsets or modulo parameters and are chosen such that the periodic tensions x_{ij} lie within the bounds. This is ensured by constraints (2) and (3). Constraints (4) and (6) ensure that the timetable π takes only values within $\{0,\dots,T-1\}$.

2.2 PESP-Energy – Maximizing the Brake-Traction Overlap

Now we develop an extension of the PESP that allows to maximize our second objective function, the brake-traction overlap. In addition to the standard input, we are given the acceleration and braking times for all departures and arrivals, respectively. Our model, which we term *PESP-Energy*, is also based on an EAN $\mathcal{E} = (E, A)$. The events $E = E_{\text{arr}} \mathbin{\dot\cup} E_{\text{dep}}$ are derived from the set of stations $\mathcal{V}$ and the set of directed lines $\mathcal{L}$ as in PESP-Passenger. However, a different set of activities is considered. Specifically, we now have $A \coloneqq A_{\text{drive}} \mathbin{\dot\cup} A_{\text{wait}} \mathbin{\dot\cup} A_{\text{energy}}$ with A_{drive} and A_{wait} defined as above and

$$A_{\text{energy}} \coloneqq \big\{((v,\ell_1,\text{dep}),(v,\ell_2,\text{arr})) \in E_{\text{dep}} \times E_{\text{arr}}\big\}.$$

Such an energy arc is depicted in red in Figure 2a. The energy activities do not impose any constraints on the feasibility of a timetable, i.e., $\Delta_a = [0, T-1]$ for all $a \in A_{\text{energy}}$. For each arrival event $i \in E_{\text{arr}}$ the time t_i^{br} needed for braking, and the time t_j^{ac} needed for accelerating at each departure event $j \in E_{\text{dep}}$ are given. We assume that $t_j^{\text{ac}} + t_i^{\text{br}} < T$ for any $ji \in A_{\text{energy}}$.

We denote with $t_{ji}^{\min} := \min\{t_j^{\mathrm{ac}}, t_i^{\mathrm{br}}\}$ the minimum and with $t_{ji}^{\max} := \max\{t_j^{\mathrm{ac}}, t_i^{\mathrm{br}}\}$ the maximum of the acceleration and braking times associated with energy arc ji. We consider the periodic intervals of the acceleration and braking phases. By a periodic interval we mean

$$[a,b]_T := \begin{cases} [a \bmod T, b \bmod T] & \text{if } a \bmod T \le b \bmod T, \\ [0, b \bmod T] \cup [a \bmod T, T) & \text{else.} \end{cases}$$

The length of a periodic interval is $\mathrm{length}([a,b]_T) := (b-a) \bmod T$. The periodic interval of the acceleration phase after the departure event j is then $[\pi_j, \pi_j + t_j^{\mathrm{ac}}]_T$ and, analogously, $[\pi_i - t_i^{\mathrm{br}}, \pi_i]_T$ describes the braking phase before the arrival event i. The overlap of the two phases is then determined by the intersection of the periodic intervals. Note that due to the assumption that $t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} < T$, this is again a periodic interval.

▶ **Definition 1** (Brake-Traction Overlap). *For $ji \in A_{\mathrm{energy}}$ we define the brake-traction overlap resulting from a periodic timetable π as* $o_{ji} := \mathrm{length}\big([\pi_j, \pi_j + t_j^{\mathrm{ac}}]_T \cap [\pi_i - t_i^{\mathrm{br}}, \pi_i]_T\big)$.

Clearly, the overlap does not depend on the exact times π_j and π_i but only on their difference, i.e., on the periodic tension x_{ji}. The following lemma gives a formula to compute it, using the function $\mathrm{overlap}_a \colon [0,T) \to \mathbb{R}_{\ge 0}$ depicted in Figure 2b.

▶ **Lemma 2.** *For every $a \in A_{\mathrm{energy}}$ with periodic tension x the brake-traction overlap is*

$$\mathrm{overlap}_a(x) := \max\big\{\min\{x,\, t_a^{\min},\, t_a^{\max} + t_a^{\min} - x\},\ 0\big\}.$$

Proof. Let $a = ji$. There are two cases in which there is an empty intersection $[\pi_j, \pi_j + t_j^{\mathrm{ac}}]_T \cap [\pi_i - t_i^{\mathrm{br}}, \pi_i]_T$. First, the intersection is empty if $\pi_j \le \pi_i$ and $\pi_j + t_j^{\mathrm{ac}} < \pi_i - t_i^{\mathrm{br}}$. This is the case whenever $t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} < \pi_i - \pi_j = (\pi_i - \pi_j) \bmod T = x_{ji}$. The second case in which the intersection is empty is if $\pi_j > \pi_i$ and $\pi_j + t_j^{\mathrm{ac}} < \pi_i + T - t_i^{\mathrm{br}}$. This is true whenever $t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} < \pi_i + T - \pi_j = (\pi_i - \pi_j) \bmod T = x_{ji}$. Hence, we have an empty intersection if and only if $t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} - x_{ji} < 0$. In this case the overlap is $o_{ji} = 0$.

Provided that the intersection is non-empty, we receive the length of the overlap by the minimum of the lengths of the four intervals $[\pi_j, \pi_j + t_j^{\mathrm{ac}}]_T$, $[\pi_i - t_i^{\mathrm{br}}, \pi_i]_T$, $[\pi_j, \pi_i]_T$, $[\pi_i - t_i^{\mathrm{br}}, \pi_j + t_j^{\mathrm{ac}}]_T$. This yields

$$\begin{aligned} o_{ji} &= \min\{t_j^{\mathrm{ac}}, t_i^{\mathrm{br}}, (\pi_i - \pi_j) \bmod T, (t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} - (\pi_i - \pi_j) \bmod T) \bmod T\} \\ &= \min\{t_j^{\mathrm{ac}}, t_i^{\mathrm{br}}, x_{ji}, (t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} - x_{ji}) \bmod T\} \\ &= \min\{t_j^{\mathrm{ac}}, t_i^{\mathrm{br}}, x_{ji}, t_j^{\mathrm{ac}} + t_i^{\mathrm{br}} - x_{ji}\} \\ &= \min\{t_{ji}^{\min}, x_{ji}, t_{ji}^{\min} + t_{ji}^{\max} - x_{ji}\} \ge 0. \end{aligned}$$

The third equation holds by the assumption that we have a non-empty intersection. Therefore,

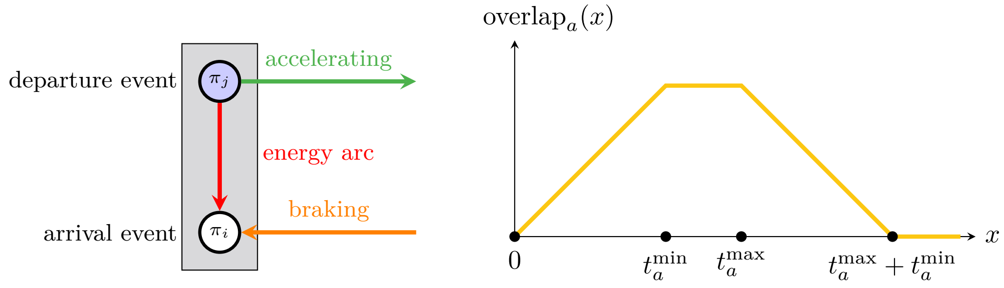
$$\min\{t_{ji}^{\min}, x_{ji}, t_{ji}^{\min} + t_{ji}^{\max} - x_{ji}\} \ge 0 \iff [\pi_j, \pi_j + t_j^{\mathrm{ac}}]_T \cap [\pi_i - t_i^{\mathrm{br}}, \pi_i]_T \neq \emptyset.$$

Hence, for the actual overlap of energy arc $a \in A_{\mathrm{energy}}$ we obtain:

$$o_a = \mathrm{overlap}_a(x) = \max\big\{\min\{x_a,\, t_a^{\min},\, t_a^{\max} + t_a^{\min} - x_a\},\ 0\big\}.$$ ◀

The maximum possible overlap at $a \in A_{\mathrm{energy}}$ is $o_a = t_a^{\min}$, which is achieved if and only if $t_a^{\min} \le x_a \le t_a^{\max}$. In this case, we say that there is *full overlap* on a.

Of course, the fact that energy can only be reused once must be taken into account in the model. Previous work [14] assumed a fixed matching between braking and accelerating trains. In contrast, we integrate these decisions directly into the model. Therefore, the



(a) Energy arc.

(b) Function overlap$_a$ mapping tension x on a to brake-traction overlap.

Figure 2 Energy arc in EAN and brake-traction overlap as a function of the periodic tension.

problem PESP-Energy consists in finding a feasible periodic timetable π together with a matching $M \subset A_{\text{energy}}$ in $\mathcal{E}$ such that the sum of the brake-traction overlaps on the energy arcs in the matching is maximized (cf. (7)):

$$
\begin{aligned}
\text{(PESP-E)}\ \max \quad & \sum_{ji \in A_{\text{energy}}} o_{ji} && && (7)\\
\text{s.t.} \quad & (2)\text{–}(4) && && \\
& o_{ji} \leq x_{ji} && \forall ji \in A_{\text{energy}} && (8)\\
& o_{ji} \leq t_{ji}^{\min} && \forall ji \in A_{\text{energy}} && (9)\\
& o_{ji} \leq t_{ji}^{\max} + t_{ji}^{\min} - x_{ji} + (1 - \alpha_{ji})\Gamma && \forall ji \in A_{\text{energy}} && (10)\\
& o_{ji} \leq \alpha_{ji} \cdot \Gamma && \forall ji \in A_{\text{energy}} && (11)\\
& \sum_{a \in A_{\text{energy}} \cap \delta^-(i)} \alpha_a \leq 1 && \forall i \in E_{\text{arr}} && (12)\\
& \sum_{a \in A_{\text{energy}} \cap \delta^+(j)} \alpha_a \leq 1 && \forall j \in E_{\text{dep}} && (13)\\
& o_{ji} \geq 0,\ \alpha_{ji} \in \{0,1\} && \forall ji \in A_{\text{energy}} && (14)\\
& x_{ij} \geq 0,\ p_{ij} \in \mathbb{Z} && \forall ij \in A && (15)\\
& \pi_i \in \mathbb{Z} && \forall i \in E && (16)
\end{aligned}
$$

As we want to find a feasible timetable, the model also contains the constraints (2)–(4) from the standard PESP. The variable o_{ji} determines the brake-traction overlap and is bounded from above by the constraints (8)–(10) according to Lemma 2. The constant Γ is chosen large enough so that for $\alpha_{ji} \in \{0,1\}$ one of the constraints (10) and (11) does not impose a relevant bound on o_{ji}. It can be set to $\Gamma := \max\big\{\max\{t_{ji}^{\min}, T - (t_{ji}^{\max} + t_{ji}^{\min})\} \bigm| ji \in A_{\text{energy}}\big\}$. Constraints (12) and (13) ensure that the energy arcs chosen at each station form a matching. They set α_{ji} to 0 whenever the arc $ji \in A_{\text{energy}}$ is not chosen to be in the matching. Constraint (11) ensures that the overlap is not counted whenever $\alpha_{ji} = 0$.

We now compare the way to model the brake-traction overlap in (PESP-E) with the formulation of Wang et al. [14] for the timetable adjustment problem. For each energy arc, Wang et al. introduce two binary variables to decide whether there is a brake-traction overlap or not and thereby distinguish cases in which the periodic offset is 0 or 1. The next theorem formally states that the parts maximizing the brake-traction overlap are equivalent in both models. For a proof of this equivalence we refer to the appendix.

▶ **Theorem 3.** *The constraints (8)–(11) are equivalent to the constraints (18)–(26) in the appendix, taken from the model of Wang et al. [14], in the sense that for each energy arc $a \in A_{\text{energy}}$ and periodic timetable π with tension x_a the overlaps in the two models are equal.*

Next, we give an upper bound for the objective value of this maximization problem. To this end, we define weights for the energy arcs $ji \in A_{\text{energy}}$ as $w(ji) := t_{ji}^{\min}$.

▶ **Proposition 4.** *For an instance of PESP-Energy on the EAN $\mathcal{E} = (E, A)$, let $S = (\pi, M)$ be a feasible solution with objective value $f(S)$, and let w^{opt} be the maximum weight of a (perfect) matching in the graph $G = (E, A_{\text{energy}})$ with weights $w(ji)$ as defined above. Then $f(S) \leq w^{\text{opt}}$.*

Proof. Each overlap is bounded from above by both the corresponding acceleration and the corresponding braking time: $o_{ji} \leq t_j^{\text{ac}}$ and $o_{ji} \leq t_i^{\text{br}}$. ◀

2.3 The Bicriteria Problem

For real timetabling problems it is desirable to find timetables that enable the usage of regenerative energy as well as short travel times for the passengers. Hence, it is necessary to consider a bicriteria problem and study Pareto optimal solutions to find a good trade-off. The bicriteria MIP formulation consists of the objectives (1) and (7) under the constraints (2)–(14). However, solving only PESP-Passenger on large networks exactly is already computationally out of scope. To obtain a better understanding of the problem under the two objectives, we investigate the solution structures on a small network of one transfer station.

▶ **Definition 5** (One-Station Network)**.** *An EAN $\mathcal{E}_n = (E, A)$ is called a* one-station network *with n lines if it is based on one station $|\mathcal{V}| = 1$ and n (directed) lines stopping at this station inducing the following events:*

$$E_{\text{arr}} := \{(\ell, \text{arr}) \mid \ell \in [n]\}, \qquad E_{\text{dep}} := \{(\ell, \text{dep}) \mid \ell \in [n]\}.$$

The activities $A = A_{\text{wait}} \,\dot\cup\, A_{\text{trans}} \,\dot\cup\, A_{\text{energy}}$ connect the events as follows:

$$\begin{aligned} A_{\text{wait}} &:= \{((\ell, \text{arr}), (\ell, \text{dep})) \in E_{\text{arr}} \times E_{\text{dep}}\}, \\ A_{\text{trans}} &:= \{((\ell_1, \text{arr}), (\ell_2, \text{dep})) \in E_{\text{arr}} \times E_{\text{dep}} \mid \ell_1 \neq \ell_2\}, \\ A_{\text{energy}} &:= \{((\ell_1, \text{dep}), (\ell_2, \text{arr})) \in E_{\text{dep}} \times E_{\text{arr}}\}. \end{aligned}$$

There are no driving activities in a one-station network. In the following, a one-station network $\mathcal{E}_n^{\text{pass}}$ for PESP-Passenger has the arcs $A_{\text{wait}} \cup A_{\text{trans}}$, while for PESP-Energy the arc set of the one-station network $\mathcal{E}_n^{\text{energy}}$ consists of $A_{\text{wait}} \cup A_{\text{energy}}$.

3 PESP-Passenger on a One-Station Network

The Periodic Event Scheduling Problem is NP-complete for any fixed $T \geq 3$, which can be proved by a reduction from the vertex colouring problem [9]. More recent work shows NP-hardness on a star network with turnaround loops [1]. Here, we show that the problem of finding a timetable minimizing the total transfer time on a *single* station is NP-hard as well.

▶ **Theorem 6.** *The problem PESP-Passenger is NP-hard even on a one-station network.*

Proof. We show NP-hardness by a reduction from the Max-Cut problem. Let I be an arbitrary instance of the Max-Cut problem, consisting of a graph $G = (V, R)$ and a weight function $w\colon R \to \mathbb{R}$. We search a bipartition (S, T) of the vertex set V such that the sum of the weights on the edges between the sets S and T, $\sum_{u\in S, v\in T} w(e_{uv})$, is maximal.

Based on I we define an instance I' of the PESP-Passenger problem on a one-station network: Let $\mathcal{E}_n^{\text{pass}} = (E_{\text{arr}} \mathbin{\dot\cup} E_{\text{dep}}, A_{\text{wait}} \mathbin{\dot\cup} A_{\text{trans}})$ be a one-station network with $n := |V|$ lines, inducing n arrival events and n departure events. Let the period time $T := 2$. The bounds on the waiting activities $ij \in A_{\text{wait}}$ are $l_{ij} = u_{ij} = 0$, and their weights are $w(ij) := 0$. For the transfer activities $ij \in A_{\text{trans}}$, let the bounds $l_{ij} = 1$ and $u_{ij} = 2$ and weights

$$w'(ij) := \begin{cases} w(\{i,j\}) & \text{if } \{i,j\} \in R, \\ 0 & \text{else.} \end{cases}$$

We want to show that any optimal solution to I' can be transformed to an optimal solution of I. Let π be an optimal timetable for I'. We define $S := \{i \in V \mid \pi_{(i,\text{dep})} = 0\}$ and $T := \{i \in V \mid \pi_{(i,\text{dep})} = 1\}$. To see that (S, T) is an optimal solution to the Max-Cut problem, i.e., that $\sum_{u\in S, v\in T} w(\{u, v\})$ is maximum, note that π minimizes the sum of the weights multiplied with the periodic tensions in I'. As every transfer arc has tension 1 or 2, this is the same as maximizing the sum of the weights of arcs with tension 1, which are exactly those between the sets S and T. Hence, (S, T) is a maximum-weight cut. ◀

Now we establish a special case in which we know the structure of an optimal solution.

▶ **Definition 7** (Basel Solution Structure). *A timetable π for a one-station network $\mathcal{E}_n^{\text{pass}}$ has a* Basel solution structure *if all arrival events are scheduled at the same time π^{arr} and all departure events at time π^{dep} such that $(\pi^{\text{dep}} - \pi^{\text{arr}}) \bmod T = l^{\max} := \max\{l_a \mid a \in A\}$.*

▶ **Proposition 8.** *Let $\mathcal{E}_n^{\text{pass}} = (E, A)$ be a one-station network with lower and upper bounds l_a, u_a on the arcs such that $u_a = l_a + T - 1$ for all transfer arcs $a \in A_{\text{trans}}$. Then any timetable π with the Basel solution structure minimizes the total travel time independently of the weights if and only if $l_a = l_{a'}$ for all $a, a' \in A_{\text{trans}} \cup A_{\text{wait}}$.*

Proof. First, let us assume that $l_a = l_{a'}$ for all $a, a' \in A_{\text{trans}} \cup A_{\text{wait}}$. Let π be a timetable with the Basel solution structure. Then the periodic tensions induced by π are $x_{ij} = (\pi_j - \pi_i - l_{ij}) \bmod T + l_{ij} = (l^{\max} - l^{\max}) \bmod T + l^{\max} = l^{\max}$ for all $ij \in A_{\text{trans}} \cup A_{\text{wait}}$. As we cannot do better than attaining the lower bounds on the tensions, π must be optimal.

Let us now assume that there is an arc $a' \in A_{\text{trans}} \cup A_{\text{wait}}$ with $l_{a'} < l^{\max}$. In the following we find a weight vector w for which π is not optimal. Let $a' = i'j'$. Then the following timetable π' achieves a lower objective value than π for the following weight vector w:

$$w(ij) := \begin{cases} 1 & \text{if } ij = i'j', \\ 0 & \text{else,} \end{cases} \qquad \pi'_i := \begin{cases} 0 & \text{if } i = i', \\ l_{i'j'} & \text{if } i = j', \\ \text{arbitrary feasible values} & \text{else.} \end{cases}$$

This is possible as only the waiting activities impose feasibility constraints. The weighted sum of the periodic tensions w.r.t. π is $l^{\max}$, and it is $l_{a'}$ w.r.t. π'. By assumption, $l_{a'} < l^{\max}$, hence, π is not optimal. ◀

4 PESP-Energy on a One-Station Network

4.1 The Timetable for a Given Matching

An EAN for this problem is a bipartite graph with partition classes E_{arr} and E_{dep}. In a one-station network, the set of waiting activities constitutes a perfect matching from E_{arr} to E_{dep}. These activities impose the only feasibility constraints on the timetable π. In contrast, the energy activities solely influence the objective value. Hence, arrival and departure times of different lines are not restricted by any PESP constraint. For any matching $M \subset A_{\text{energy}} = E_{\text{dep}} \times E_{\text{arr}}$, the graph $(E, A_{\text{wait}} \dot\cup M)$ is a union of node-disjoint directed cycles and directed paths. A timetable with maximum brake-traction overlap on the matching arcs can be determined for each connected component of this graph separately. The following proposition describes the structure of an optimal timetable for a directed cycle.

▶ **Proposition 9.** *Let $\mathcal{E}_n^{\text{energy}}$ be a one-station network, let $M \subset A_{\text{energy}}$ be a matching, and let $C \subset A_{\text{wait}} \dot\cup M$ be a directed cycle. We write the cycle as $C = a_1, b_1, a_2, b_2 \dots, a_m, b_m$ with $a_j \in M$, $b_j \in A_{\text{wait}}$, and $t_{a_j}^{\min} \geq t_{a_1}^{\min}$ for all $j \in [m]$. There is an optimal timetable π for PESP-Energy restricted to C such that we have full overlap $o_{a_j} = t_{a_j}^{\min}$ for all $j \in \{2, \dots, m\}$.*

Proof. Let π be a timetable maximizing the brake-traction overlap on the energy arcs of C. Among all such timetables, we consider one with the maximum number of arcs a_j with $j \in \{2, \dots, m\}$ having full overlap. Assume that some arc a_j with $j \in \{2, \dots, m\}$ does not have full overlap. To derive a contradiction, we modify the timetable π on C so that a_j has full overlap, while preserving full overlap on all other arcs and not reducing the total overlap.

To this end, we first define the new tensions x' and then construct a timetable π' inducing them. Let x be the tension induced by π. For $k \in [m]$ we set $x'_{b_k} := x_{b_k}$ and

$$x'_{a_k} := \begin{cases} x_{a_k} & \text{for } k \neq \{1, j\}, \\ (x_{a_1} - \delta) \bmod T & \text{for } k = 1, \\ x_{a_j} + \delta & \text{for } k = j, \end{cases} \qquad \text{where} \quad \delta := \begin{cases} t_{a_j}^{\min} - x_{a_j} & \text{if } x_{a_j} < t_{a_j}^{\min}, \\ t_{a_j}^{\max} - x_{a_j} & \text{if } x_{a_j} > t_{a_j}^{\max}. \end{cases}$$

Note that this covers all cases because for $t_{a_j}^{\min} \leq x_{a_j} \leq t_{a_j}^{\max}$ the activity a_j would have full overlap, contradicting our assumption. We define the periodic timetable π' as follows: We enumerate the nodes of the cycle so that $a_k = (2k-1, 2k)$ for $k \in [m]$ and $b_k = (2k, 2k+1)$ for $k \in [m-1]$. The nodes with even number correspond to arrivals and with odd number to departures. We set $\pi'_1 := 0$, $\pi'_{2k} := (\pi'_{2k-1} + x'_{a_k}) \bmod T$ for $k \in [m]$, and $\pi'_{2k+1} := (\pi'_{2k} + x'_{b_k}) \bmod T$ for $k \in [m-1]$. This adheres to the prescribed tensions on all arcs a_k, $k \in [m]$, and b_k, $k \in [m-1]$. The tension on b_m is congruent to $\pi'_1 - \pi'_{2m} = -\pi'_{2m} = -\sum_{k=1}^{m} x'_{a_k} - \sum_{k=1}^{m-1} x'_{b_k} \equiv -\sum_{k=1}^{m} x_{a_k} - \sum_{k=1}^{m-1} x_{b_k} \equiv x_{b_m} \pmod{T}$. The last congruence holds since the periodic tension x sums up to a multiple of T due to the cycle periodicity.

For $a \in M$ let $o'_a := \text{overlap}_a(x'_a)$. The only matching arcs whose tensions have changed are a_1 and a_j. We have $x'_{a_j} \in \{t_{a_j}^{\min}, t_{a_j}^{\max}\}$, and thus a_j has now full overlap, i.e., $o'_{a_j} = t_{a_j}^{\min}$. It holds that

$$\begin{aligned} o'_{a_j} - o_{a_j} = t_{a_j}^{\min} - o_{a_j} &= \begin{cases} t_{a_j}^{\min} - 0 & \text{if } x_{a_j} > t_{a_j}^{\min} + t_{a_j}^{\max}, \\ t_{a_j}^{\min} - x_{a_j} & \text{if } x_{a_j} < t_{a_j}^{\min}, \\ t_{a_j}^{\min} - (t_{a_j}^{\max} + t_{a_j}^{\min} - x_{a_j}) & \text{if } t_{a_j}^{\max} < x_{a_j} \leq t_{a_j}^{\max} + t_{a_j}^{\min}, \end{cases} \\ &= \begin{cases} t_{a_j}^{\min} & \text{if } x_{a_j} > t_{a_j}^{\min} + t_{a_j}^{\max}, \\ |\delta| & \text{else}, \end{cases} \\ &\geq \min\{|\delta|, t_{a_j}^{\min}\}. \end{aligned}$$

Similarly, $o_{a_1} - o'_{a_1} \leq \min\{|\delta|, t^{\min}_{a_1}\} \leq \min\{|\delta|, t^{\min}_{a_j}\}$, hence the decrease of the overlap on a_1 is at most the increase of the overlap on a_j. Since no other overlaps have changed, the sum of all overlaps cannot have decreased, i.e., we have found a solution whose objective is not worse but which has more arcs a_j with $j \in \{2, \ldots, m\}$ with full overlap. ◀

We can also regard a connected component being a directed path as a cycle whose missing edge has overlap 0. Hence, by Proposition 9, there is an optimal timetable for this component such that all energy edges in the path have full overlap. While the proposition describes the structure of an optimal timetable for each connected component resulting from a fixed matching M, we are interested in a global optimum, comprising the matching. Hence, we need to investigate the structure of an optimal matching.

4.2 The Matching of Energy Arcs

The following result bounds the number of arcs in the matching of a globally optimal solution.

▶ **Theorem 10.** *In every optimal solution $S = (\pi, M)$ to PESP-Energy on a one-station network $\mathcal{E}^{\text{energy}}_n$ the matching M contains at least $n - 1$ arcs.*

Proof. Let $S = (\pi, M)$ be an optimal solution to PESP-Energy with $|M| < n-1$, so at least two connected components of $(E, A_{\text{wait}} \dot\cup M)$ are directed paths P_1, P_2. For $k \in \{1, 2\}$ let $i_k \in E_{\text{arr}}$ be the start and $j_k \in E_{\text{dep}}$ be the end node of P_k.

Let $c \coloneqq \pi_{j_1} + t^{\min}_{j_1 i_2} - \pi_{i_2}$, and let us define the following timetable

$$\pi'_v \coloneqq \begin{cases} \pi_v & \text{if } v \in E \setminus V(P_2), \\ (\pi_v + c) \bmod T & \text{if } v \in V(P_2). \end{cases}$$

Now, let $S' = (\pi', M')$ with $M' = M \cup \{j_1 i_2\}$. For the tensions on M', we obtain:

$$x'_{ji} = \begin{cases} (\pi'_i - \pi'_j) \bmod T = (\pi_i - \pi_j) \bmod T = x_{ji} & \text{for } ji \in M, \\ (\pi'_{i_2} - \pi'_{j_1}) \bmod T = t^{\min}_{j_1 i_2} & \text{for } j = j_1, i = i_2. \end{cases}$$

Hence, we obtain the brake-traction overlaps

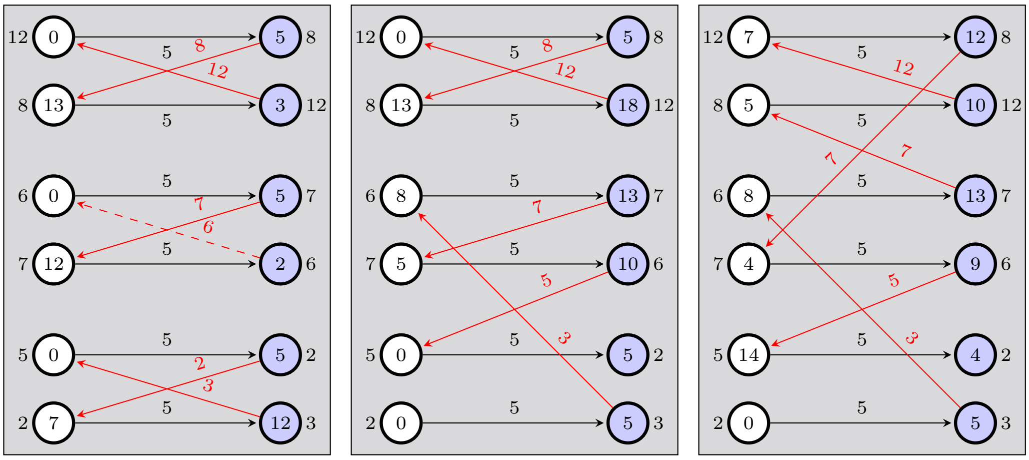
$$o'_{ji} = \begin{cases} o_{ji} & \text{if } ji \in M, \\ t^{\min}_{j_1 i_2} & \text{if } ji = j_1 i_2. \end{cases}$$

Consequently, S' yields a better objective value than S, so S cannot be optimal. ◀

▶ **Corollary 11.** *There is a unique perfect matching M^p which is obtained by extending the matching M of an optimal solution to PESP-Energy on a one-station network.*

This yields another way of looking at an optimal solution to PESP-Energy on a one-station network. A perfect matching $M^p \subset A_{\text{energy}}$ corresponds one-to-one to a permutation $\varphi\colon [n] \to [n]$ of the trains (lines) in a one-station network. This is given by $\varphi(\ell) = k$ if and only if $((\ell, \text{dep}), (k, \text{arr})) \in M^p$. The directed cycles in $M^p \cup A_{\text{wait}}$ then correspond to the cycles of the permutation.

Recall that we can find an upper bound for the objective value of a PESP-Energy instance by calculating the maximum-weight perfect matching on the energy arcs $a \in A_{\text{energy}}$ w.r.t. the weights $w\colon A_{\text{energy}} \to \mathbb{R}$ with $w(a) = t^{\min}_a$ (cf. Proposition 4). This matching can be found easily by a greedy algorithm for the weights in our problem. Sorting both t^{br}_i, $i \in E_{\text{arr}}$, and t^{ac}_j, $j \in E_{\text{dep}}$, according to their sizes, we obtain the permutations ρ and σ with $t^{\text{br}}_{\rho(1)} \leq \cdots \leq t^{\text{br}}_{\rho(n)}$ and $t^{\text{ac}}_{\sigma(1)} \leq \cdots \leq t^{\text{ac}}_{\sigma(n)}$. Then $M_{\text{greedy}} \coloneqq \big\{((\sigma(i), \text{dep}), (\rho(i), \text{arr})) \in E_{\text{dep}} \times E_{\text{arr}} \;\big|\; i \in [n]\big\}$ is a perfect matching with maximum weight in the graph (E, A_{energy}).



(a) Greedy matching, weight: 38, total overlap: 32.

(b) Optimal solution, total overlap: 35.

(c) Hamiltonian path, total overlap: 34.

Figure 3 Example of non-optimal greedy and Hamiltonian path matchings for $T = 15$. The timetable π is written in the nodes. To the left of the arrival nodes (white) the braking times are given, and to the right of the departure nodes (blue) there are the acceleration times. The numbers on the waiting (black) and energy (red) arcs correspond to the periodic tensions. For the full red arcs, they also correspond to the achieved overlap.

4.3 A Hamiltonian Path Algorithm/Heuristic

A lower bound on the optimal objective value of PESP-Energy can be obtained by a maximum-weight Hamiltonian path on $\mathcal{E}_n^{\text{energy}}$ with respect to the weights $w\colon A \to \mathbb{R}$ defined by $w(ji) := t_{ji}^{\min}$ for $ji \in A_{\text{energy}}$ and $w(ij) := \Gamma$ for $ij \in A_{\text{wait}}$, where Γ is a big number. The choice of weights ensures that the path consists of n waiting arcs and $n-1$ energy arcs. Adding the arc from the end node of the path to its start node creates a cycle. Due to Proposition 9, we know that in that cycle, we can obtain full overlap on the best $n-1$ energy arcs, so we obtain at least overlap equal to the weight of the path's energy activities. Hence, this yields a lower bound on the optimal overlap achievable. In Figure 3 we can see that neither weight of the greedy matching is always obtained as overlap nor does a maximum-weight Hamiltonian path necessarily yield an optimal solution. In Figure 3a the greedy matching together with the waiting activity $M \cup A_{\text{wait}}$ decomposes into three cycles. Due to the cycle periodicity, however, we cannot obtain full overlap in the second cycle. There is no overlap on the dashed arc. While the greedy matching has weight 38, only an overlap of 32 can be obtained from the matching. In Figure 3b an optimal solution is depicted. We can see a decomposition of one cycle and one path, which cannot be closed due to cycle periodicity. The achieved overlap is 35. In Figure 3c, we can see a maximum-weight Hamiltonian path with full overlap on all energy arcs. In total an overlap of 34 is achieved. Due to the cycle periodicity it is not possible to obtain overlap on the missing energy arc. We show now that this lower bound can be computed in polynomial time.

▶ **Theorem 12.** *A maximum-weight Hamiltonian path on a one-station network $\mathcal{E}_n^{\text{energy}}$ with weights w can be found in polynomial time.*

In order to prove this theorem, we show that PESP-Energy on a one-station network can be transformed to the problem of sequencing a machine with variable state, for which a polynomial-time algorithm is known, see [4]. To simplify the notation, in this section we write

$t^{\mathrm{ac}}_\ell := t^{\mathrm{ac}}_{(\ell,\mathrm{dep})}$ and $t^{\mathrm{br}}_\ell := t^{\mathrm{br}}_{(\ell,\mathrm{arr})}$ for $\ell \in [n]$. We consider the complete directed graph $G = ([n], \mathcal{A})$ on the set of all lines and weights $w\colon \mathcal{A} \to \mathbb{R}$ defined by $w_G(k\ell) := \min\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\}$. We need the following three lemmas.

▶ **Lemma 13.** *If P_G is a maximum-weight Hamiltonian path in G w.r.t. w_G, then $P := \{((k,\mathrm{dep}),(\ell,\mathrm{arr}) \mid k\ell \in P_G\} \cup \{((\ell,\mathrm{arr}),(\ell,\mathrm{dep})) \mid \ell \in [n]\}$ is a maximum-weight Hamiltonian path in $\mathcal{E}^{\mathrm{energy}}_n$ w.r.t. w.*

Proof. First, P is the arc set of a Hamiltonian path. It has weight $w(P) = w(P_G) + n\Gamma$. Let P^{opt} be a maximum-weight Hamiltonian path in $\mathcal{E}^{\mathrm{energy}}_n$. By the large choice of Γ, this must contain n waiting arcs and hence contains exactly $n-1$ energy arcs $((k,\mathrm{dep}),(\ell,\mathrm{arr}))$. Then $(P^{\mathrm{opt}})_G := \{(k,\ell) \mid ((k,\mathrm{dep}),(\ell,\mathrm{arr})) \in P^{\mathrm{opt}}\}$ is a Hamiltonian path in G of weight $w_G((P^{\mathrm{opt}})_G) = w(P^{\mathrm{opt}}) - n\Gamma$. Therefore, $w(P) = w(P_G) + n\Gamma \geq w((P^{\mathrm{opt}})_G) + n\Gamma = w(P^{\mathrm{opt}})$ and P is a maximum-weight Hamiltonian path in $\mathcal{E}^{\mathrm{energy}}_n$. ◀

▶ **Lemma 14.** *Any maximum-weight Hamiltonian cycle in G w.r.t. w_G contains a maximum-weight Hamiltonian path. Conversely, any maximum-weight Hamiltonian path can be closed to a maximum-weight Hamiltonian cycle.*

Proof. Let $k := \arg\min\{t^{\mathrm{ac}}_{(\ell,\mathrm{dep})}, t^{\mathrm{br}}_\ell \mid \ell \in [n]\}$. W.l.o.g. let us assume $t^{\mathrm{br}}_{(k,\mathrm{arr})} \leq t^{\mathrm{ac}}_k$. We know that all incoming arcs of k have weight $w^{\min}_G := t^{\mathrm{br}}_{(k,\mathrm{arr})}$. Let $\mathcal{C}^{\mathrm{opt}}$ be a maximum-weight Hamiltonian cycle. Since this must visit k, it must contain an arc a of weight $w^{\min}_G$. Then $\mathcal{P} := \mathcal{T}^{\mathrm{opt}} \setminus \{a\}$ is a Hamiltonian path with weight $w(\mathcal{P}) = w_G(\mathcal{C}^{\mathrm{opt}}) - w^{\min}_G$.

Let now $\mathcal{P}^{\mathrm{opt}}$ be a maximum-weight Hamiltonian path, and let v be the first and u be the last vertex in $\mathcal{P}^{\mathrm{opt}}$. Then $\mathcal{C} := \mathcal{P}^{\mathrm{opt}} \cup \{uv\}$ is a Hamiltonian tour with weight $w_G(\mathcal{C}) = w_G(\mathcal{P}^{\mathrm{opt}}) + w_G(uv) \geq w_G(\mathcal{P}^{\mathrm{opt}}) + w^{\min}_G$.

Together, both $\mathcal{P}$ and $\mathcal{C}$ must be optimal since $w_G(\mathcal{P}) = w_G(\mathcal{C}^{\mathrm{opt}}) - w^{\min}_G \geq w_G(\mathcal{C}) - w^{\min}_G \geq w_G(\mathcal{P}^{\mathrm{opt}})$ and $w_G(\mathcal{C}) \geq w_G(\mathcal{P}^{\mathrm{opt}}) + w^{\min}_G \geq w_G(\mathcal{P}) + w^{\min}_G = w(\mathcal{C}^{\mathrm{opt}})$. ◀

▶ **Lemma 15.** *Let $C_1, C_2 \subset \mathcal{A}$ be two Hamiltonian cycles in G. Consider a second weight function $w'\colon \mathcal{A} \to \mathbb{R}$ defined by $w'(k\ell) := |t^{\mathrm{ac}}_k - t^{\mathrm{br}}_\ell|$. If $w(C_1) \leq w(C_2)$, then $w'(C_1) \geq w'(C_2)$. Hence, a maximum-weight Hamiltonian cycle w.r.t. w is a minimum-weight Hamiltonian cycle w.r.t. w'.*

Proof. For the weight w' of a Hamiltonian cycle C we get

$$\begin{aligned} w'(C) &= \sum_{k\ell \in C} |t^{\mathrm{ac}}_k - t^{\mathrm{br}}_\ell| = \sum_{k\ell \in C} \left(\max\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\} - \min\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\}\right) \\ &= \sum_{k\ell \in C} \left(\max\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\} + \min\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\} - 2 \cdot \min\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\}\right) \\ &= \sum_{k\ell \in C} \left(t^{\mathrm{ac}}_k + t^{\mathrm{br}}_\ell - 2 \cdot \min\{t^{\mathrm{ac}}_k, t^{\mathrm{br}}_\ell\}\right) = \sum_{k \in [n]} (t^{\mathrm{ac}}_k + t^{\mathrm{br}}_k) - 2 \cdot w_G(C), \end{aligned}$$

where the first summand in the last expression is constant. Hence, if $w(C_1) \leq w(C_2)$, then $w'(C_1) \geq w'(C_2)$. ◀

Now, we can prove Theorem 12.

Proof of Theorem 12. The problem of sequencing a one state-variable machine from [4] is defined as follows. We consider N jobs $J_1, J_2, \ldots, J_N$ which are to be done on one machine in some order. For each job J_i we know the required starting state of the machine represented by the real number A_i and the machine's state after the completion of job J_i represented by the real number B_i. When job J_l follows job J_k, we need to change the machine's state from B_k to A_l. The cost c_{kl} of this change is defined as

$$c_{kl} := \begin{cases} \int_{B_k}^{A_l} f(x)dx & \text{if } A_l \geq B_k, \\ \int_{A_l}^{B_k} g(x)dx & \text{if } A_l < B_k. \end{cases}$$

Here, f and g are integrable functions such that $f(x) + g(x) \geq 0$ for all $x \in \mathbb{R}$. The problem is to find a sequence of jobs such that the sum of the costs for changing the state of the machine between consecutive jobs is minimized. The polynomial-time algorithm developed in [4] requires the prescription of an initial state B_{N+1} and a final state A_{N+1} of the machine so that it becomes the problem of finding a tour $J_{N+1}J_{i_1} \dots J_{i_N} J_{N+1}$ with the artificial job J_{N+1} minimizing the total state transition cost.

Now, consider finding a maximum-weight Hamiltonian path in $\mathcal{E}_n^{\text{energy}}$, w.r.t. the weights w. By Lemma 13, this is equivalent to finding a maximum-weight Hamiltonian path in G w.r.t. w_G. By means of Lemma 14, this can be reduced to finding a Hamiltonian cycle in G which corresponds to finding a minimum-weight Hamiltonian cycle in G w.r.t. the weights w' defined in Lemma 15. We reduce this problem to solving the sequencing problem of finding a closed tour on a set of jobs on the following instance I^{seq}.

For every directed line $\ell \in [n]$ we define a job ℓ with $A_\ell := t_\ell^{\text{br}}$ and $B_\ell := t_\ell^{\text{ac}}$. The functions for the state transition costs are defined as $f(x) = g(x) := 1$ for all $x \in \mathbb{R}$. This yields the costs $c_{kl} = A_l - B_k$ if $A_l \geq B_k$ and $c_{kl} = B_k - A_l$ if $A_l < B_k$. In other words, $c_{kl} = |A_l - B_k| = w'(k\ell)$. Therefore, the cost of any cyclic tour of the jobs equals the weight of the corresponding Hamiltonian tour in G w.r.t. w'. ◀

The maximum-weight Hamiltonian path yields a feasible solution to PESP-Energy on a one-station network. We can guarantee that the objective value of this solution is not further away from the optimal objective value than the smallest of the largest acceleration and the largest braking time. This follows from the following theorem, which bounds the difference between the lower bound and the upper bound from the greedy matching of Section 4.2.

▶ **Theorem 16.** *Let H be a Hamiltonian path in $\mathcal{E}_n^{\text{energy}}$ of maximum weight. Then it holds*

$$w(M_{\text{greedy}}) - w(H \cap A_{\text{energy}}) \leq \min\big\{\max\{t_\ell^{\text{br}} \mid \ell \in [n]\},\ \max\{t_\ell^{\text{ac}} \mid \ell \in [n]\}\big\}.$$

Proof. We iteratively convert the greedy matching M_{greedy}^0 into a matching inducing a Hamiltonian cycle and bound the total reduction of weight in this process. Finally, we delete one edge to obtain a Hamiltonian path.

Let us assume that $t_1^{\text{ac}} \leq \dots \leq t_n^{\text{ac}}$ holds, and let φ^0 denote the permutation obtained by M_{greedy}^0 such that $t_{\varphi^0(1)}^{\text{br}} \leq \dots \leq t_{\varphi^0(n)}^{\text{br}}$. Then $M_{\text{greedy}}^0 = \{((\ell, \text{dep}), (\varphi^0(\ell), \text{arr})) \mid \ell \in [n]\}$. The permutation φ^i corresponds to the perfect matching M^i obtained in iteration i.

In each iteration, we obtain the matching M^i as follows from the matching M^{i-1}. Let $C \subseteq M^{i-1} \cup A_{\text{wait}}$ be the cycle containing $(1, \text{dep})$. If C is a Hamiltonian cycle, we are done. Otherwise, there is a smallest ℓ such that $(\ell, \text{dep}) \in C$ but $(\ell + 1, \text{dep}) \notin C$. We define the new permutation φ^i as follows:

$$\varphi^i(x) := \begin{cases} \varphi^{i-1}(\ell + 1) & \text{if } x = \ell, \\ \varphi^{i-1}(\ell) & \text{if } x = \ell + 1, \\ \varphi^{i-1}(x) & \text{else.} \end{cases}$$

For the new matching M^i we have, $M^i = M^{i-1} \cup \{((\ell, \mathrm{dep}), (\varphi^{i-1}(\ell+1), \mathrm{arr})), ((\ell+1, \mathrm{dep}), (\varphi^{i-1}(\ell), \mathrm{arr}))\} \setminus \{((\ell, \mathrm{dep}), (\varphi^{i-1}(\ell), \mathrm{arr})), ((\ell+1, \mathrm{dep}), (\varphi^{i-1}(\ell+1), \mathrm{arr}))\}$. The cycle's length increases by this operation and for the weight of the new matching M^i, we get

$$\begin{aligned} w(M^i) = w(M^{i-1}) &+ \min\{t^{\mathrm{ac}}_\ell, t^{\mathrm{br}}_{\varphi^{i-1}(\ell+1)}\} + \min\{t^{\mathrm{ac}}_{\ell+1}, t^{\mathrm{br}}_{\varphi^{i-1}(\ell)}\} \\ &- \min\{t^{\mathrm{ac}}_\ell, t^{\mathrm{br}}_{\varphi^{i-1}(\ell)}\} - \min\{t^{\mathrm{ac}}_{\ell+1}, t^{\mathrm{br}}_{\varphi^{i-1}(\ell+1)}\} \\ = w(M^{i-1}) &- |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^{i-1}(\ell)}, t^{\mathrm{br}}_{\varphi^{i-1}(\ell+1)}]|. \end{aligned}$$

Further, we know that $[t^{\mathrm{br}}_{\varphi^{i-1}(\ell)}, t^{\mathrm{br}}_{\varphi^{i-1}(\ell+1)}] \subset [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]$. Hence, the length of the intersection can be bounded by

$$|[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^{i-1}(\ell)}, t^{\mathrm{br}}_{\varphi^{i-1}(\ell+1)}]| \le |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]|$$

and, therefore $w(M^i) \ge w(M^{i-1}) - |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]|$.

Let k be the number of iterations until we obtain a matching M^k such that $M^k \cup A_{\mathrm{wait}}$ corresponds to a Hamiltonian cycle. In the following ℓ_i denotes the smallest $\ell \in [n-1]$ such that $(\ell, \mathrm{dep}) \in C$ and $(\ell+1, \mathrm{dep}) \notin C$ in iteration i. Further, it holds $k \le n-1$ as there are n different trains. Hence, we can overestimate the sum as follows:

$$\sum_{i=1}^{k} |[t^{\mathrm{ac}}_{\ell_i}, t^{\mathrm{ac}}_{\ell_i+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(\ell_i+1)}]| \le \sum_{\ell=1}^{n-1} |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]|$$

Instead of just summing up the length of the intervals for the corresponding train ℓ in each iteration, we sum over the lengths of all possible choices for ℓ. We get the following bound:

$$\begin{aligned} w(M^k) &\ge w(M^0_{\mathrm{greedy}}) - \sum_{i=1}^{k} |[t^{\mathrm{ac}}_{\ell_i}, t^{\mathrm{ac}}_{\ell_i+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(\ell_i+1)}| \\ &\ge w(M^0_{\mathrm{greedy}}) - \sum_{\ell=1}^{n-1} |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]|. \end{aligned}$$

As the intervals $[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}]$ intersect only in one point (of length 0), we can bound the sum of the intersections as follows:

$$\begin{aligned} &\sum_{\ell=1}^{n-1} |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]| \le \left| \bigcup_{\ell=1}^{n-1} [t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \right| = t^{\mathrm{ac}}_n - t^{\mathrm{ac}}_1, \\ &\sum_{\ell=1}^{n-1} |[t^{\mathrm{ac}}_\ell, t^{\mathrm{ac}}_{\ell+1}] \cap [t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]| \le |[t^{\mathrm{br}}_{\varphi^0(1)}, t^{\mathrm{br}}_{\varphi^0(n)}]| = t^{\mathrm{br}}_{\varphi^0(n)} - t^{\mathrm{br}}_{\varphi^0(1)}. \end{aligned}$$

Thus, $w(M^k) \ge w(M^0_{\mathrm{greedy}}) - \min\{t^{\mathrm{ac}}_n - t^{\mathrm{ac}}_1, t^{\mathrm{br}}_{\varphi^0(n)} - t^{\mathrm{br}}_{\varphi^0(1)}\}$. In order to receive a Hamiltonian path $H \subseteq M^k \cup A_{\mathrm{wait}}$, we delete one edge from the matching M^k. As we want to maximize the path's weight, we choose the edge with the lowest weight. Due to the weight structure, this weight is $\min\{t^{\mathrm{ac}}_1, t^{\mathrm{br}}_{\varphi^0(1)}\}$. For the difference of the weights of the greedy matching M_{greedy} and the weight of the energy arcs in H, we get:

$$\begin{aligned} w(M_{\mathrm{greedy}}) - w(H \cap A_{\mathrm{energy}}) &\le \min\{t^{\mathrm{ac}}_n - t^{\mathrm{ac}}_1, t^{\mathrm{br}}_{\varphi^0(n)} - t^{\mathrm{br}}_{\varphi^0(1)}\} + \min\{t^{\mathrm{ac}}_1, t^{\mathrm{br}}_{\varphi^0(1)}\} \\ &\le \min\{t^{\mathrm{ac}}_n, t^{\mathrm{br}}_{\varphi^0(n)}\}. \end{aligned}$$

The weight of the path H is a lower bound for the weight of an optimal Hamiltonian path. ◀

4.4 Two Special Cases Solvable in Polynomial Time

There are some special cases in which we can solve PESP-Energy on a one-station network in polynomial time. In the first case, all braking times and all acceleration times are equal.

▶ **Proposition 17.** *Let I be an instance of PESP-Energy on a one-station network with n lines such that all acceleration times are equal and all braking times are equal, i.e., $t^{\mathrm{ac}} = t^{\mathrm{ac}}_j$ and $t^{\mathrm{br}} = t^{\mathrm{br}}_i$ for all $i, j \in [n]$. Then, there is an optimal solution to I consisting of one cycle of all lines in arbitrary order. This can be found in polynomial time.*

Proof. Assume that there is no optimal solution consisting of a single cycle, and consider an optimal solution (M, π) with the minimum number of cycles. Let C_1, C_2 be two different cycles in $M \cup A_{\mathrm{wait}}$, and let $a_k = (j_k, i_k) \in A_{\mathrm{energy}} \cap C_k$ for $k = 1, 2$. Consider the alternative solution with $M' := (M \setminus \{a_1, a_2\}) \cup \{j_1 i_2, j_2 i_1\}$. Then M' induces a big cycle on the node set $V(C_1) \cup V(C_2)$. Let $c := \pi_{j_1} + x_{a_1} - \pi_{i_2}$, and set

$$\pi'_v := \begin{cases} \pi_v & \text{if } v \in E \setminus V(C_2), \\ (\pi_v + c) \bmod T & \text{if } v \in V(C_2). \end{cases}$$

Then the arc $j_1 i_2$ has new tension $x'_{j_1 i_2} = (\pi'_{i_2} - \pi'_{j_1}) \bmod T = (\pi_{i_2} + c - \pi_{j_1}) \bmod T = x_{a_1}$, i.e., it also has the same overlap because all energy arcs a have the same function $\mathrm{overlap}_a$ mapping tensions to overlaps. Moreover, the arc $j_2 i_1$ has tension $x'_{j_2 i_1} = (\pi'_{i_1} - \pi'_{j_2}) \bmod T = (\pi_{i_1} - \pi_{j_2} - c) \bmod T = (\pi_{i_1} - \pi_{j_1} + \pi_{i_2} - \pi_{j_2} - x_{a_1}) \bmod T = x_{a_2}$, i.e., the overlap is also equal. Therefore, together the overlap on the two new arcs is the same as on the two old arcs. So we have found an optimal solution with less cycles, which constitutes a contradiction. ◀

In the second case, we consider a period time that is so large that no energy cycle can exist. This corresponds to an aperiodic timetabling problem.

▶ **Proposition 18.** *Let I be an instance of PESP-Energy on a one-station network with n lines. Let $u^{\max} := \max\{u_a \mid a \in A_{\mathrm{wait}}\} + \max\{t^{\min}_{a'} + t^{\max}_{a'} \mid a' \in A_{\mathrm{energy}}\}$ such that $T > n \cdot u^{\max}$. Then, any matching M^H inducing a Hamiltonian path of maximum weight w.r.t. w is part of an optimal solution $S = (\pi, M^H)$.*

Proof. We show that for every optimal solution $S^{\mathrm{opt}} = (\pi^{\mathrm{opt}}, M^{\mathrm{opt}})$ the set $M^{\mathrm{opt}} \cup A_{\mathrm{wait}}$ contains a Hamiltonian path. By Corollary 11, M^{opt} can be extended to an optimal perfect matching M^p. Let C be an arbitrary directed cycle in $M^p \cup A_{\mathrm{wait}}$. By Proposition 9 we can assume that π^{opt} induces full overlap on all but at most one arc of C. Let a_0 denote this energy arc such that $t^{\min}_{a_0} = \min\{t^{\min}_a \mid a \in C\}$. It holds:

$$\begin{aligned} & t^{\max}_{a_0} + t^{\min}_{a_0} + \sum_{a \in C \cap A_{\mathrm{wait}}} u_a + \sum_{a' \in C \cap A_{\mathrm{energy}} \setminus \{a_0\}} t^{\max}_{a'} \\ & < \sum_{a \in C \cap A_{\mathrm{wait}}} u_a + \sum_{a' \in C \cap A_{\mathrm{energy}}} t^{\max}_{a'} + t^{\min}_{a'} < n \cdot u^{\max} < T. \end{aligned}$$

Therefore, we have $\sum_{a \in C \cap A_{\mathrm{wait}}} u_a + \sum_{a' \in C \cap A_{\mathrm{energy}} \setminus \{a_0\}} t^{\max}_{a'} < T - (t^{\max}_{a_0} + t^{\min}_{a_0})$. Since there is full overlap on all $a' \in C \cap A_{\mathrm{energy}} \setminus \{a_0\}$, we know that the periodic tensions induced by π^{opt} satisfy $x^{\mathrm{opt}}_{a'} \leq t^{\max}_{a'}$ for all these arcs. Hence, for the tension $x^{\mathrm{opt}}_{a_0}$ on a_0 we have $x^{\mathrm{opt}}_{a_0} > t^{\max}_{a_0} + t^{\min}_{a_0}$ as by the cycle periodicity constraints all periodic tensions in C need to sum up to a multiple of T. Therefore, there is no overlap on a_0. By the same argument, every other cycle in $M^p \cup A_{\mathrm{wait}}$ has an arc without overlap. We can remove all these arcs from M^p without reducing the objective value. However, by Theorem 10, any optimal matching

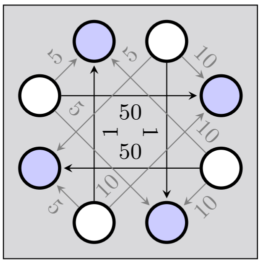


Figure 4 Number of passengers on transfer and waiting activities in instance I_1 from Section 5.

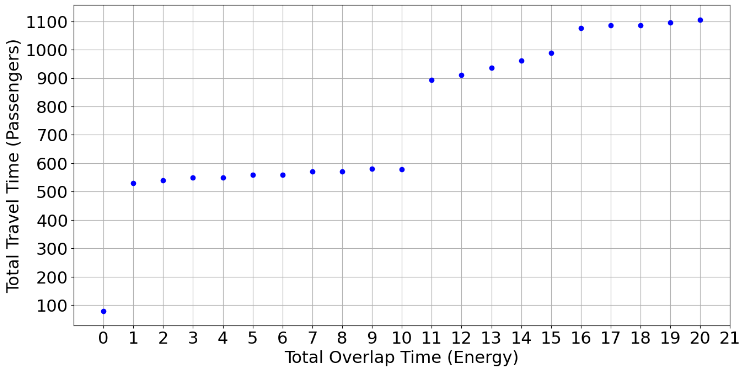


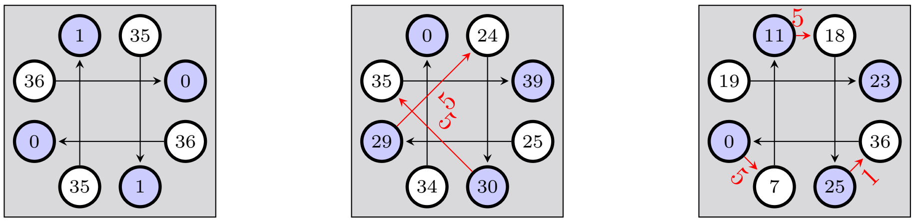
Figure 5 Pareto frontier of instance I_1.

has at least $n-1$ arcs. Therefore, C must be the only cycle in $M^p \cup A_{\text{wait}}$, i.e., it is a Hamiltonian cycle. Moreover, the objective value of S^{opt} is equal to the total weight of $(C \setminus \{a_0\}) \cap A_{\text{energy}}$. An arbitrary maximum-weight Hamiltonian path contains all waiting arcs and then maximizes the weight of the chosen energy arcs. Therefore, it yields the optimal objective value. ◀

5 Examples of Bicriteria Timetables – Numerical Results

In this section, we present some computational results of bicriteria timetabling problems at a single transfer station. We use the MIP formulation from Section 2.3 and solve it with the CPLEX solver on a 13th Gen Intel(R) Core(TM) i5-1335U with 1.30 GHz, 16,0 GB RAM, and a 64-bit processor. We use an ε-constraint method bounding the total brake-traction overlap from below in order to obtain a set of Pareto-optimal solutions. The objective then seeks for the minimal weighted sum of travel times. In the one-station network this equals the weighted sum of the periodic tensions on the waiting and the transfer activities.

The instance I_1 under consideration is based on a one-station network with 2 lines into both directions and a period time of $T = 40$. The acceleration and braking times are all set to $t^{\text{ac}} = 5$ and $t^{\text{br}} = 7$. On the waiting activities, we have the bounds $l = 4$, $u = 8$, and the transfers have a lower bound of $l = 5$ and are non-restricted with $u = 44$. There are no transfers into opposite directions of one line. We assume a symmetric passenger distribution on the arcs, see Figure 4. Figure 5 shows the optimal weighted sum of the periodic tensions (total travel time of the passengers) at this station depending on the required overlap time for the braking and acceleration phases. We observe that the travel times increase with



(a) Overlap ≥ 0
Travel Time: 80.

(b) Overlap ≥ 10
Travel Time: 579.

(c) Overlap ≥ 11
Travel Time: 894.

Figure 6 Exemplary solution structures of I_1. The numbers in the nodes represent the scheduled times of the events. Black arcs represent waiting activities and red arcs represent energy arcs with the numbers indicating the overlap times.

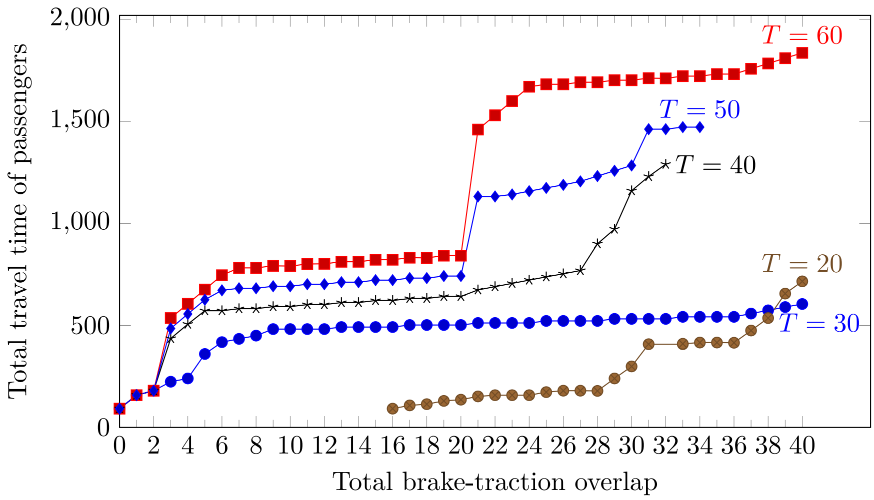


Figure 7 Objective values for instance I_1 with different period times.

increasing required overlap time. Further, there is one huge gap, where the increase from a required overlap time from 10 to 11 results in a huge increase of the total travel time from 579 to 894. In Figure 6 we can see the corresponding timetables for the scenario of no enforced overlap and for a required overlap time of at least 10 and 11. Without an enforced overlap, the timetable has almost a Basel solution structure separately for both the horizontal and the vertical line. If the required overlap is 10, still two trains arrive and depart at almost the same times. This structure disappears for an overlap of at least 11.

Figure 7 shows the Pareto frontiers of instances with the same parameters as I_1 but with varying period time T. We can observe that in general the total travel time of the passengers increases with an increasing period time, which is reasonable as there are transfers with the same number of passengers into both directions of each pair of lines. Further, we can observe that it depends on the period time whether it is possible to attain a maximum overlap of 40 time units. While this is possible for the cases of $T \in \{20, 30, 60\}$, for $T = 50$ we obtain at most 34 time units overlap and for $T = 40$ we obtain a maximum overlap of 32 time units. Due to the cycle periodicity constraints which depend on the period length it is not always possible to obtain full overlap on all chosen energy arcs.

6 Outlook

We have introduced the new periodic timetabling problem PESP-Energy and its bicriteria version. Apart from giving a MIP formulation we characterize the structure of optimal solutions for both single objective problems on a one-station network. On this small network, a polynomial-time algorithm with an additive performance guarantee is obtained for the problem with energy objective. Further, some bicriteria instances on a one-station network were solved numerically and analysed. We plan to continue our work investigating the complexity of the single objective PESP-Energy on a one-station network and developing algorithms for the bicriteria problem on larger networks.

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A Comparison with the Model of Wang et al.

In the timetable adjustment problem considered by Wang et al. [14], there is an explicitly given set $A_{\text{sab}} \subseteq E_{\text{dep}} \times E_{\text{arr}}$ of activities (j, i) specifying that the acceleration after the departure j (taking time t_j) and the braking before i (taking time t_i) should be synchronized. Their model reads

$$\max\ O = \sum_{ji \in A_{\text{sab}}} L_{ji} \tag{17}$$

$$\text{s.t.}\ \ L^*_{ji} = \min\{\pi_i - \pi_j + \beta_{ji} T,\ \ t_j + t_i - \pi_i + \pi_j - \beta_{ji} T,\ \ t_j,\ \ t_i\} \qquad \forall ji \in A_{\text{sab}} \tag{18}$$

$$M \cdot (\alpha_{ji} + \beta_{ji} - 1) \le L_{ji} - L^*_{ji} \le -M \cdot (\alpha_{ji} + \beta_{ji} - 1) \qquad \forall ji \in A_{\text{sab}} \tag{19}$$

$$-M \cdot (\alpha_{ji} + \beta_{ji}) \le L_{ji} \le M \cdot (\alpha_{ji} + \beta_{ji}) \qquad \forall ji \in A_{\text{sab}} \tag{20}$$

$$\alpha_{ji} \ge \min\Big\{\frac{\pi_i - \pi_j}{M}, \frac{\pi_j - \pi_i + t_j + t_i}{M}\Big\} \qquad \forall ji \in A_{\text{sab}} \tag{21}$$

$$\alpha_{ji} \le 1 + \min\Big\{\frac{\pi_i - \pi_j}{M}, \frac{\pi_j - \pi_i + t_j + t_i}{M}\Big\} \qquad \forall ji \in A_{\text{sab}} \tag{22}$$

$$\beta_{ji} \ge \min\Big\{\frac{\pi_i + T - \pi_j}{M}, \frac{\pi_j - \pi_i - T + t_j + t_i}{M}\Big\} \qquad \forall ji \in A_{\text{sab}} \tag{23}$$

$$\beta_{ji} \le 1 + \min\Big\{\frac{\pi_i + T - \pi_j}{M}, \frac{\pi_j - \pi_i - T + t_j + t_i}{M}\Big\} \qquad \forall ji \in A_{\text{sab}} \tag{24}$$

$$\alpha_{ji}, \beta_{ji} \in \{0, 1\} \qquad \forall ji \in A_{\text{sab}} \tag{25}$$

$$L_{ji}, L^*_{ji} \in \mathbb{Z} \qquad \forall ji \in A_{\text{sab}} \tag{26}$$

The brake-traction overlap between the events j and i is represented by the variable L_{ji}. The auxiliary variable L^*_{ji} indicates the value of the minimum in the expression for the overlap in Lemma 2, which equals the overlapping time if the overlap is non-empty. This is enforced by Constraint (18). The binary variables α_{ji} and β_{ji} model whether there is some overlap or not: the constraints (21)–(25) ensure that

$$\alpha_{ji} = \begin{cases} 1 & \text{if } \pi_i - \pi_j > 0 \text{ and } \pi_j + t_j > \pi_i - t_i, \\ 0 & \text{if } \pi_i - \pi_j < 0 \text{ or } \pi_j + t_j < \pi_i - t_i, \\ 0 \text{ or } 1 & \text{else} \end{cases} \tag{27}$$

and

$$\beta_{ji} = \begin{cases} 1 & \text{if } \pi_j + t_j > \pi_i + T - t_i, \\ 0 & \text{if } \pi_j + t_j < \pi_i + T - t_i, \\ 0 \text{ or } 1 & \text{else} \end{cases} \tag{28}$$

for all $ji \in A_{\text{sab}}$. Finally, constraints (19) and (20) ensure that the actual overlap L_{ji} is set to L^*_{ji} in the case that $\alpha_{ji} + \beta_{ji} = 1$ and to 0 if $\alpha_{ji} + \beta_{ji} = 0$. Further, the case that $\alpha_{ji} + \beta_{ji} = 2$ is prevented by constraint (19). So it holds

$$\alpha_{ji} + \beta_{ji} \leq 1 \qquad \forall ji \in A_{\text{sab}} \tag{29}$$

Note that the constraint involving a minimum are not linear. To linearize them, we would need to introduce two constraints for each constraint. Further, whenever the minimum bounds from below, we would have to introduce a new binary variable.

Proof of Theorem 3. We begin by showing that the variable L_{ji} in this model measures the same overlap as o_{ji} from PESP-Energy. Hence, we show the following:

1. If there exists an overlap then $L_{ji} = L^*_{ji} = o_{ji}$.
2. If there is no overlap, then $L_{ji} = 0 = o_{ji}$.

The equalities $o_{ji} = L^*_{ji}$ if there is an overlap and $o_{ji} = 0$ else follow from Lemma 2. So we need to investigate the value of L_{ji} for the cases in which there is an overlap (Case 1.1 and 1.2) and in which there is no overlap (Cases 2.1, 2.2 and 2.3).

Case 1.1 $[\pi_j < \pi_i$ and $\pi_j + t_j > \pi_i - t_i]$ In this case, $\alpha_{ji} = 1$ by (27) and $\beta_{ji} = 0$ by (28). (23) allows β_{ji} to take the value 0. Hence, $\alpha_{ji} + \beta_{ji} = 1$ and we have $L_{ji} = L^*_{ji}$ by (19).

Case 1.2 $[\pi_j > \pi_i$ and $\pi_j + t_j > \pi_i + T - t_i]$ In this case, $\beta_{ji} = 1$ by (28) and $\alpha_{ji} = 0$ by (27). Also by (21) allows α_{ji} to take the value 0. Hence, $\alpha_{ji} + \beta_{ji} = 1$ and we have $L_{ji} = L^*_{ji}$ by (19).

Case 2.1 $[\pi_j < \pi_i$ and $\pi_j + t_j \leq \pi_i - t_i]$ In this case, $\alpha_{ji} = 0$ by (27). Further, $\beta_{ji} = 0$ by (28) as $\pi_j + t_j \leq \pi_i - t_i$ implies that $\pi_j + t_j < \pi_i + T - t_i$. Hence, $\alpha_{ji} + \beta_{ji} = 0$ and we have $L_{ji} = 0$ by (19).

Case 2.2 $[\pi_j > \pi_i$ and $\pi_j + t_j \leq \pi_i + T - t_i]$ We have that $\alpha_{ji} = 0$ by (27). Further, if $\pi_j + t_j < \pi_i + T - t_i$ then $\beta_{ji} = 0$ by (28). Hence, $\alpha_{ji} + \beta_{ji} = 0$ and we have $L_{ji} = 0$ by (19). On the other hand, if $\pi_j + t_j = \pi_i + T - t_i$ then $\beta_{ji} = 0$ or $\beta_{ji} = 1$ by (29). If $\beta_{ji} = 0$ then it holds, as just discussed, that $L_{ji} = 0$. If $\beta_{ji} = 1$, then it holds that $L_{ji} = L^*_{ji} = \min\{\pi_i - \pi_j + \beta_{ji}T,\ t_j + t_i - \pi_i + \pi_j - \beta_{ji}T,\ t_j,\ t_i\} = \min\{\pi_i - \pi_j + \beta_{ji}T,\ 0,\ t_j,\ t_i\} = 0$ as $t_j, t_i > 0$ and $\pi_i + T - \pi_j > 0$.

Case 2.3 $[\pi_j = \pi_i]$ By (27) we get that α_{ji} could be 0 or 1 in this case, also the value for β_{ji} is unclear. Therefore, we rest with the two options $\alpha_{ji} + \beta_{ji} = 0$ and $\alpha_{ji} + \beta_{ji} = 1$. If $\alpha_{ji} + \beta_{ji} = 0$, we know that $L_{ji} = 0$ by constraint (20). If $\alpha_{ji} + \beta_{ji} = 1$, then $L_{ji} = L^*_{ji} = \min\{\pi_i - \pi_j + \beta_{ji}T,\ t_j + t_i - \pi_i + \pi_j - \beta_{ji}T,\ t_j,\ t_i\} = \min\{0,\ t_j + t_i - 0,\ t_j,\ t_i\} = 0$ as $t_j, t_i > 0$ and therefore $t_j + t_i > 0$. $\alpha_{ji} = 1$ by constraint (21) $\alpha_{ji} \leq 1$ by constraint (22) $\beta_{ji} = 0$ by constraint (19). ◀