# Computing User Equilibria for Schedule-Based Transit Networks with Hard Vehicle Capacities

**Tobias Harks**
University of Passau, Germany

**Sven Jäger**
RPTU Kaiserslautern-Landau, Germany

**Michael Markl**
University of Passau, Germany

**Philine Schiewe**
Aalto University, Finland

## Abstract

Modelling passenger assignments in public transport networks is a fundamental task for city planners, especially when deliberating network infrastructure decisions. A key aspect of a realistic model for passenger assignments is to integrate selfish routing behaviour of passengers on the one hand, and the limited vehicle capacities on the other hand. We formulate a side-constrained user equilibrium model in a schedule-based time-expanded transit network, where passengers are modelled via a continuum of non-atomic agents that want to travel with a fixed start time from a user-specific origin to a destination. An agent's route may comprise several rides along given lines, each using vehicles with hard loading capacities. We give a characterization of (side-constrained) user equilibria via a quasi-variational inequality and prove their existence by generalizing a well-known existence result of Bernstein and Smith (Transp. Sci., 1994). We further derive a polynomial time algorithm for single-commodity instances and an exact finite time algorithm for the multi-commodity case. Based on our quasi-variational characterization, we finally devise a fast heuristic computing user equilibria, which is tested on real-world instances based on data gained from the Hamburg S-Bahn system and the Swiss long-distance train network. It turns out that w.r.t. the total travel time, the computed user-equilibria are quite efficient compared to a system optimum, which neglects equilibrium constraints and only minimizes total travel time.

**2012 ACM Subject Classification** Theory of computation → Network games; Applied computing → Transportation; Theory of computation → Exact and approximate computation of equilibria

**Keywords and phrases** traffic assignment, side-constrained equilibrium, public transportation

**Digital Object Identifier** 10.4230/OASIcs.ATMOS.2024.17

**Related Version** *Full Version*: `https://arxiv.org/abs/2406.17153` [19]

**Supplementary Material**
*Software (Source Code)*: `https://github.com/ArbeitsgruppeTobiasHarks/sbta/tree/atmos` [20]
archived at `swh:1:dir:a3e48de8912d81166b2517c3c826e11bcda2c356`

**Funding** This research was funded by the German Federal Ministry of Education and Research (BMBF). FK 05M22UKB, 05M22WPA.

**Acknowledgements** We thank the anonymous reviewers for their constructive feedback.

## 1 Introduction

In the domain of public transport, models describing the assignment of passengers over a transit network are crucial for infrastructure planners to understand congestion phenomena and possible investments into the infrastructure. The existing approaches can roughly be


24th Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2024).
Editors: Paul C. Bouman and Spyros C. Kontogiannis; Article No. 17; pp. 17:1–17:17
OpenAccess Series in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

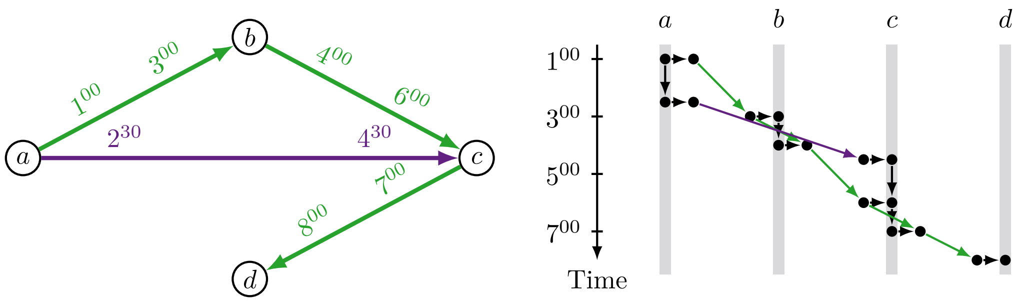


**Figure 1** Two scheduled vehicle trips in the physical network (left) and their representation in the time-expanded transit network (right).

categorized into *frequency-based* and *schedule-based* models, see [13, 12] for a survey. The former model class operates operate with line frequencies and implicitly defines resulting travel times and capacities of lines and vehicles, cf. [38, 11, 40, 3, 6, 5, 24]. With variations in the demand profile during peak hours, the frequency-based approach only leads to approximate vehicle loads, with the error increasing as variability grows. In contrast, schedule-based approaches are more fine-grained and capable of explicitly modelling irregular timetables of lines. They are usually based on a *time-expanded transit network* derived from the physical transit network and augmented by (artificial) edges such as waiting, boarding, alighting, dwelling, and driving edges to connect different stations. This construct, also known as diachronic graph [29] or space-time network [4], is illustrated in Figure 1.

An assignment of passengers to paths in this network encompasses their entire travel strategies, including line changes, waiting times, etc. It corresponds to a path-based multi-commodity network flow satisfying all demand and supply. One key obstacle in the analysis of such a schedule-based model is the integration of *strategic behaviour* of passengers, opting for shortest routes, and the *limited vehicle capacity*, which bounds the number of passengers able to use a vehicle at any point in time. If a vehicle is already at capacity, further passengers might not be able to enter this vehicle at the next boarding stop, which can make their (shortest) route infeasible. On the other hand, the passengers already in the vehicle are not affected by the passengers wishing to board.

A key issue of such a capacitated model is to choose the right equilibrium concept. Consider for instance the simple example in Figure 1, and suppose that the vehicles operating the violet and the green line have a capacity of 1 unit each. A demand volume of 2 units start their trip at node $a$ at time $1^{00}$, and all particles want to travel as fast as possible to the destination node $c$. The violet line arrives at $4^{30}$, while the green line arrives at $6^{00}$. Then there exists no capacity-feasible *Wardrop equilibrium* [39, 8], i.e., a flow only using quickest paths.

Most works in the literature deal with this non-existence by either assuming soft vehicle capacities (cf. [9, 30, 28]) or by considering more general travel strategies and a probabilistic loading mechanism (cf. [26, 23, 27, 17]). An alternative approach that inherently supports capacities are so-called *side-constrained user equilibria*. These are feasible flows such that for any used path there is no *available alternative path* with lower cost (cp. [10]). The precise definition of what is an available lower-cost alternative has spurred a whole series of works. Larsson and Patriksson [32, 25] consider only paths with available residual capacity as available alternative paths. This excludes the change from a used path $p$ to another path $q$ that shares some saturated edges with $p$. For a path $p$, Smith [37] considers all paths $q$ to which a part of the flow on $p$ can change over so that the resulting flow is feasible.

This concept has the drawback that it allows for coordinated deviations of bundles of users, which is unrealistic and, as shown by Smith, leads to non-existence of such side-constrained equilibria for monotonic, continuous, but non-separable cost functions. In response, Bernstein and Smith [2] propose an alternative equilibrium concept (see Definition 3 of *BS-equilibrium*) characterized by the property that "no arbitrarily small bundle of drivers on a common path can strictly decrease its cost by switching to another path" [7].

In this paper, we consider side-constrained user equilibria for schedule-based time-expanded networks in the sense of Bernstein and Smith [2]. Here, whether a path is an available alternative depends only on the available capacity of the vehicle when the passenger boards it, but not on whether capacity is exceeded on a later edge of the vehicle trip. Hence, a path can be an available alternative for some user even if arbitrarily small deviations to that path make the resulting flow infeasible (for some other users). Similar to [28], the priority of passengers in the vehicle can be modelled by expressing the capacity limitations using discontinuous costs on the boarding edges in the time-expanded transit network. The resulting cost map is not separable, and it turns out that it does not satisfy the regularity conditions imposed by Bernstein and Smith [2] to prove existence of equilibria. Our approach works for elastic demands where a user only travels if the travel cost does not exceed the user's willingness to travel. This elastic demand model is quite standard in the transportation science literature, see [41] and references therein.

## 1.1 Our Contribution

We define a user equilibrium for schedule-based time-expanded networks using the notion of deviations. For a given flow, an *admissible $\varepsilon$-deviation* corresponds to shifting an $\varepsilon$-amount of flow from a path $p$ to another path $q$ without exceeding the capacity of any boarding edge along $q$. A feasible flow is a *side-constrained user equilibrium* if there are no improving admissible $\varepsilon$-deviations for arbitrarily small $\varepsilon$. We summarize our contribution as follows.

1. We prove that for schedule-based time-expanded networks, side-constrained user equilibria can be characterized by a quasi-variational inequality defined over the set of admissible deviations (Theorem 2). Moreover, by moving the side-constraints into discontinuous cost functions, we can express side-constrained user equilibria as BS-equilibria (Theorem 4).
2. We study the central question of the existence of BS-equilibria. We first give an example showing that our cost map does not fall into the category of *regular* cost maps, defined by Bernstein and Smith [2], for which they showed the existence of equilibria. Instead, we introduce a more general condition for cost maps, which we term *weakly regular*. As our main theoretical result, we prove in Theorem 7 that BS-equilibria do exist for weakly regular cost maps. We further show (Theorem 8) that the cost maps in schedule-based time-expanded networks are weakly regular, hence Theorem 7 applies. However, the generalization of Bernstein and Smith's result might also be of interest for other traffic models.
3. We then turn to the computation of BS-equilibria. For single-commodity time-expanded networks, we present an algorithm that computes a BS-equilibrium in quadratic time relative to the number of edges of the input graph (Theorem 12). For multi-commodity networks, we give an exact finite-time algorithm. As this algorithm is too slow for practical computations, we further develop a heuristic based on our quasi-variational inequality formulation. It starts with an arbitrary feasible flow and updates this flow along elementary admissible deviations in the sense of Theorem 2.
4. Finally, we test our heuristic on realistic instances drawn from the Hamburg S-Bahn network and the Swiss long-distance train network. It turns out that user-equilibria can be computed with our heuristic and that they are quite efficient compared to a system

optimum, which neglects equilibrium constraints and minimizes total travel time. More specifically, for the computed instances the total travel times in the user equilibria are less than 8% higher compared to the system optimum.

### 1.2 Related Work

Schedule-based transit assignment has been studied extensively in the past. Since the works by [29, 4], most authors use a time-expanded graph as their modelling basis. For example, Carraresi et al. [4] consider a model with hard capacity constraints where passengers accept routes that are at most a factor of $1+\varepsilon$ worse than an optimal path without any congestion. This is approximated in several papers [9, 28, 30] by incorporating the vehicle capacities as continuous penalties representing the discomfort experienced by using an overcrowded edge.

Marcotte and Nguyen [26] address hard capacities by defining an agent's *strategy* as preference orderings of outgoing edges at each node, similar to so-called hyperpaths, and by assuming a random loading mechanism for congested edges, where the probability of entering an edge is proportional to its capacity and decreases with the amount of flow desiring to traverse it. Every passenger wants to minimize the *expected* travel cost resulting from their strategy. A whole series of works [28, 17, 16, 15, 33, 18] build upon this model and expand it by incorporating boarding priorities of passengers, departure time choice with early/late arrival penalties, different costs for seated and standing passengers, risk aversion, or stochastic link travel times modelling variation due to weather or incidents.

An alternative to time-expanded graphs is the use of dynamic flows. For example, [31] defines dynamic flows that traverse the public-transport edges in discrete chunks and employ the method of successive averages (MSA) to find approximate equilibria. Side-constrained equilibria for dynamic flows have been studied in [14] where also a dynamic variant of BS-equilibria is analysed without stating any existence results for them.

## 2 Side-Constrained Equilibria for Schedule-Based Transit Networks

We first describe a schedule-based time-expanded network (cf. [29, 4]) and then formally define the side-constrained user equilibrium concept.

### 2.1 The Schedule-Based Time-Expanded Network

Consider a set of geographical stations $S$ (e.g., metro stations or bus stops) and a set of vehicle trips $Z$ (e.g., trips of metro trains or buses), specified by their sequence of served stations and adhering to a fixed, reliable timetable. This timetable specifies the arrival and departure times at all stations of the trip, where the arrival time at a station is always strictly later than the departure time of the previous station. Each vehicle trip $z \in Z$ also has an associated capacity $\nu_z$ which represents the maximum number of users the corresponding vehicle may hold at any time. We use the term *vehicle* synonymously with *vehicle trip*.

To represent the passengers' routes through the network, we construct a time-expanded, directed, acyclic graph $G = (V, E)$ with a time $\theta(v) \in \mathbb{R}$ assigned to each node $v \in V$.

There are three categories of nodes: an *on-platform node* for each station $s \in S$ and each time $\theta$ at which at least one vehicle departs or arrives in $s$, a *departure node* for each vehicle $z \in Z$ and each time $\theta$ at which $z$ departs from a station $s$, and an *arrival node* for each vehicle $z \in Z$ and each time $\theta$ at which $z$ arrives at a station $s$.

There are five categories of edges connecting these nodes: A *waiting edge* connects two on-platform nodes $v$, $w$ of the same station $s$ with consecutive times $\theta(v) < \theta(w)$. A *boarding edge* connects an on-platform node with a departure node of a vehicle $z$ of common time $\theta$

and station $s$. A *driving edge* connects a departure node with the next stop's arrival node of the shared vehicle $z$. An *alighting edge* connects an arrival node of a vehicle $z$ with the on-platform node of common time $\theta$ and station $s$. Finally, a *dwelling edge* connects an arrival node of a vehicle $z$ with the corresponding departure node at the same station $s$.

For ease of notation, let $E_B$ and $E_D$ denote the set of all boarding and driving edges, respectively. We denote the time delay of an edge $e = vw$ by $\tau_e := \theta(w) - \theta(v)$, the delay of a $v$-$w$-path $p = (e_1, \dots, e_k)$ by $\tau_p := \sum_{e \in p} \tau_e = \theta(w) - \theta(v)$. For a driving edge $e \in E_D$ belonging to a vehicle $z \in Z$, we write $\nu_e := \nu_z$. Waiting and driving edges are always time-consuming, dwelling edges may be time-consuming, and boarding and alighting edges are instantaneous. For $e \in E_B$, we denote the succeeding driving edge by $e^+$.

Figure 1 shows a possible generated graph for two vehicles, a green one and a violet one, and four stations $a$, $b$, $c$, and $d$. The nodes on the grey rectangles represent the on-platform nodes, the other nodes are the departure and arrival nodes.

The non-atomic users of the network are partitioned into several groups: First, they are grouped into finitely many origin-destination and departure-time triplets $(s_j, t_j, \theta_j) \in \mathcal{T} \subseteq S \times S \times \mathbb{R}$. To model the elastic demand, there is a non-increasing function $Q_j \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for each triplet that given some travel time $\tau$ assigns the volume of particles of triplet $j$ that are willing to travel at cost $\tau$. We assume $Q_j(\tau) = 0$ for all $\tau \geq T$ for some $T \in \mathbb{R}$. Let $\mathcal{P}_j^\circ$ denote the set of paths from the on-platform node at time $\theta_j$ and station $s_j$ to any on-platform node of station $t_j$.

As the travel times of the paths are fixed, we can subdivide all triplets into a finite number of *commodities* $I$ of common willingness to travel and introduce an outside option for each commodity: Let $\{\tau_{j,1}, \dots, \tau_{j,k_j}\} = \{\tau_p \mid p \in \mathcal{P}_j^\circ\}$ be the set of travel times of all paths $p \in \mathcal{P}_j^\circ$ ordered by $\tau_{j,1} < \dots < \tau_{j,k_j}$. For each $j' \in \{1, \dots, k_j + 1\}$, we introduce a commodity $i_{j,j'}$ consisting of all particles of triplet $j$ whose willingness to travel is contained in the interval $[\tau_{j,j'-1}, \tau_{j,j'})$ with $\tau_{j,0} := 0$ and $\tau_{j,k_j+1} := T$. Thus, commodity $i := i_{j,j'}$ has a demand volume of

$$Q_i := Q_j(\tau_{j,j'-1}) - Q_j(\tau_{j,j'}),$$

and we assign it an outside option $p_i^{\text{out}}$ with some constant cost $\tau_{p_i^{\text{out}}}$ chosen from $(\tau_{j,j'-1}, \tau_{j,j'})$. Finally, we write $\mathcal{P}_i := \mathcal{P}_j^\circ \cup \{p_i^{\text{out}}\}$, and denote the set of all commodity-path pairs by $\mathcal{P} := \{(i,p) \mid p \in \mathcal{P}_i\}$.

## 2.2 Side-Constrained User Equilibrium

A *(path-based) flow* $f$ is a vector $(f_{i,p})_{(i,p) \in \mathcal{P}}$ with $f_{i,p} \in \mathbb{R}_{\geq 0}$. We call the flow $f$

- *demand-feasible*, if $\sum_{p \in \mathcal{P}_i} f_{i,p} = Q_i$ holds for all $i \in I$,
- *capacity-feasible*, if $f_e := \sum_{i,p : e \in p} f_{i,p} \leq \nu_e$ holds for all driving edges $e \in E_D$,
- *feasible*, if $f$ is both demand- and capacity-feasible.

Let $\mathcal{F}_Q$, $\mathcal{F}^\nu$, $\mathcal{F}_Q^\nu$ denote the sets of all demand-feasible, capacity-feasible, and feasible flows, respectively. For a given demand-feasible flow $f$ and two paths $p, q \in \mathcal{P}_i$ with $f_{i,p} \geq \varepsilon$, we define the *$\varepsilon$-deviation* from $p$ to $q$

$$f_{i,p \to q}(\varepsilon) := f + \varepsilon \cdot (1_{i,q} - 1_{i,p})$$

as the resulting flow when shifting an $\varepsilon$-amount of flow of commodity $i$ from $p$ to $q$. We call $f_{i,p \to q}(\varepsilon)$ an *admissible* deviation, if $f_{i,p \to q}(\varepsilon)_{e^+} \leq \nu_{e^+}$ holds for all boarding edges $e$ of $q$. We say, $q$ is an *available alternative* to $p$ for $i$ with respect to $f$, if there is some positive $\varepsilon$ such that $f_{i,p \to q}(\varepsilon)$ is an admissible deviation. In other words, $q$ is an available alternative, if after switching some arbitrarily small amount of flow from $p$ to $q$, the path $q$ does not involve boarding overcrowded vehicles. Equivalently, all boarding edges $e \in q$ fulfil $f_{e^+} < \nu_{e^+}$, if $e^+ \notin p$, and $f_{e^+} \leq \nu_{e^+}$, if $e^+ \in p$. We denote the set of available alternatives to $p$ for $i$ with respect to $f$ by $A_{i,p}(f)$.

▶ **Definition 1.** *A feasible flow $f$ is a* (side-constrained) user equilibrium *if for all $i \in I$ and $p \in \mathcal{P}_i$ the following implication holds:*

$$f_{i,p} > 0 \implies \forall q \in A_{i,p}(f) : \tau_p \leq \tau_q.$$

This means, a feasible flow is a side-constrained user equilibrium if and only if a path is only used if all its faster alternative routes are unavailable due to the boarding capacity constraints. For the rest of this work, we use the shorthand *user equilibrium* for this concept.

## 3 Characterization and Existence

We characterize user equilibria as defined above in two different ways: as solutions to a quasi-variational inequality and as BS-equilibria by defining appropriate, discontinuous cost functions. By generalizing the existence result of Bernstein and Smith [2] for BS-equilibria, we show that user equilibria exist. Some proofs are omitted in this conference paper due to space constraints. They can be found in the full version [19] of the paper.

### 3.1 Quasi-Variational Inequalities

Traditional types of user equilibria without hard capacity constraints can be equivalently formulated as a solution to a variational inequality of the form

$$\text{Find } f^* \in D \text{ such that:} \qquad \langle c(f^*), f - f^* \rangle \geq 0 \quad \text{ for all } f \in D, \qquad (\text{VI}(c, D))$$

where $D$ is a closed, convex set, and $c$ is a continuous cost function.

With the introduction of hard capacity constraints together with boarding priorities, an admissible $\varepsilon$-deviation might lead to capacity violations. Therefore, such deviations may leave the feasible set $D = \mathcal{F}_Q^\nu$ and, thus, are not representable in this variational inequality, leading us to the concept of quasi-variational inequalities. We define the set-valued function

$$M : \mathcal{F}_Q^\nu \rightrightarrows \mathbb{R}_{\geq 0}^{\mathcal{P}}, \quad f \mapsto \{f_{i,p\to q}(\varepsilon) \mid f_{i,p\to q}(\varepsilon) \text{ is an admissible } \varepsilon\text{-deviation}, \varepsilon > 0\}$$

that returns for any given flow $f$ the set of all possible flows obtained by any admissible $\varepsilon$-deviation with respect to $f$. We now consider the following quasi-variational inequality:

$$\text{Find } f^* \in \mathcal{F}_Q^\nu \text{ such that:} \qquad \langle (\tau_p)_{i,p}, f - f^* \rangle \geq 0 \quad \text{ for all } f \in M(f^*). \qquad (\text{QVI})$$

A feasible flow $f$ is a solution to this quasi-variational inequality if and only if there is no commodity $i$ and a pair of paths $p, q \in \mathcal{P}_i$ with $\tau_q < \tau_p$ such that $f_{i,p\to q}(\varepsilon)$ is an admissible $\varepsilon$-deviation for arbitrary $\varepsilon > 0$. Hence, we can characterize user equilibria as follows:

▶ **Theorem 2.** *A feasible flow $f^*$ is a user equilibrium if and only if it is a solution to the quasi-variational inequality* (QVI)*.*

While the existence of solutions to customary variational inequalities in the form of (VI($c, D$)) can be shown using Brouwer's fixed point theorem, the existence of solutions to quasi-variational inequalities is not clear upfront. To establish an existence result, we therefore introduce an alternative characterization of our problem in the next section.

## 3.2 Bernstein-Smith Equilibrium

In this section, we will reformulate the user equilibrium as a Bernstein-Smith equilibrium for suitably chosen edge cost functions $c_e \colon \mathcal{F}_Q \to \mathbb{R}_{\geq 0}$. This way we dispense with the explicit side-constraints and instead incorporate them as discontinuities into the cost functions, so that any equilibrium must correspond to a feasible flow.

▶ **Definition 3** ([2, Definition 2]). *We are given a directed graph $G = (V, E)$ and a set of commodities $I$, each equipped with a demand $Q_i \geq 0$, a finite, non-empty set of paths $\mathcal{P}_i$, and a cost function $c_{i,p} \colon \mathbb{R}^{\mathcal{P}}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for every $p \in \mathcal{P}_i$. A demand-feasible flow $f \in \mathcal{F}_Q$ is a* Bernstein-Smith equilibrium (BS-equilibrium) *if, for all $i \in I$ and $p \in \mathcal{P}_i$, $f_{i,p} > 0$ implies $c_{i,p}(f) \leq \min_{q \in \mathcal{P}_i} \liminf_{\varepsilon \downarrow 0} c_{i,q}(f_{i,p \to q}(\varepsilon))$.*

We first define cost functions for the edges of our schedule-based transit network as follows: The cost of a non-boarding edge $e$ is given by the time it takes to traverse the edge, i.e., $c_e(f) \coloneqq \tau_e \geq 0$. Passing a boarding edge takes no time, however, it is only possible to board until the capacity of the vehicle is reached. We realize this by raising the cost of the boarding edge when the capacity is exceeded to a sufficiently large constant $M$, which is guaranteed to be higher than the cost of any available path, e.g., $M \coloneqq \max_{i \in I, p \in \mathcal{P}_i} \tau_p + 1$. This means, for a boarding edge $e \in E_B$, the experienced cost is

$$c_e(f) \coloneqq \begin{cases} 0, & \text{if } f_{e^+} \leq \nu_{e^+}, \\ M, & \text{if } f_{e^+} > \nu_{e^+}. \end{cases} \tag{1}$$

For a $v$-$w$-path $p$ in our time-expanded graph, we assign the cost

$$c_{i,p}(f) \coloneqq \sum_{e \in p} c_e(f) = \theta(w) - \theta(v) + \sum_{e \in p \cap E_B} c_e(f). \tag{2}$$

For the outside options $p_i^{\text{out}}$, we assume that they are virtual paths consisting of a single edge in $E$, which is exclusively used by the path $p_i^{\text{out}}$, and we set the cost of this edge to $c_{i,p_i^{\text{out}}}(f) \coloneqq \tau_{p_i^{\text{out}}}$. The BS-equilibria with respect to these cost functions are exactly the user equilibria, as the following theorem shows.

▶ **Theorem 4.** *A demand-feasible flow $f$ is a user equilibrium if and only if it is a BS-equilibrium with respect to the cost functions $c_{i,p}$ defined above.*

**Proof.** For any $f \in \mathcal{F}_Q^{\nu}$ and $p, q \in \mathcal{P}_i$ with $f_{i,p} > 0$, it holds that

$$\limsup_{\varepsilon \downarrow 0} c_{i,q}(f_{i,p \to q}(\varepsilon)) \begin{cases} = \tau_q, & \text{if } \forall e \in E_B \cap q : e^+ \in p \text{ or } f_{e^+} < \nu_{e^+}, \\ \geq M, & \text{otherwise.} \end{cases}$$

Note that the condition in the case distinction is equivalent to $q \in A_{i,p}(f)$.

If $f$ is a BS-equilibrium, then $c_{i,p}(f) \leq \tau_{p_i^{\text{out}}} < M$ holds by the BS-equilibrium condition and, thus, we must have $c_{i,p}(f) = \tau_p$. The same holds true, if we instead assume $f$ to be a user equilibrium. Therefore, the following equivalence holds:

$$\forall q \in A_{i,p}(f) : \tau_p \leq \tau_q \iff \forall q \in \mathcal{P}_i : c_{i,p}(f) \leq \limsup_{\varepsilon \downarrow 0} c_{i,q}(f_{i,p \to q}(\varepsilon)).$$

Hence, $f$ is a BS-equilibrium if and only if it is a user equilibrium. ◀

### 3.3 Existence

Bernstein and Smith [2, Theorem 2] proved the existence of BS-equilibria in the case that each path cost function has the form $c_{i,p} = \sum_{e \in p} c_e$, where $c_e \colon \mathcal{F}_Q \to \mathbb{R}_{\geq 0}$, $e \in E$, are lower-semicontinuous, bounded functions that satisfy the following regularity condition.

▶ **Definition 5** ([2]). *A cost structure $c \colon \mathcal{F}_Q \to \mathbb{R}^E_{\geq 0}$ is* regular *if it satisfies*

$$\liminf_{\varepsilon \downarrow 0} c_{i,q}(f_{i,p \to q}(\varepsilon)) = \sum_{e \in p \cap q} c_e(f) + \sum_{e \in q \setminus p} \bar{c}_e(f)$$

*for all $f \in \mathcal{F}_Q$, $i \in I$, and paths $p, q \in \mathcal{P}_i$ with $f_{i,p} > 0$, where $\bar{c}_e$ is the upper hull of $c_e$ defined as*

$$\bar{c}_e(f) \coloneqq \lim_{\varepsilon \downarrow 0} \sup\{c_e(x) \mid x \in \mathcal{F}_Q, \|x - f\| < \varepsilon\}.$$

Unfortunately, the cost functions in our case as defined in (1) do not fulfil this condition, as the network in Figure 1 illustrates: Assume there is a single commodity with source $a$ and sink $d$ with demand 2, and assume that both vehicles have a capacity of one. Let $p$ be the $a$-$d$-path using only the green vehicle, and let $q$ be the $a$-$d$-path using both vehicles. Let $f$ be the flow sending one unit along $p$ and the remaining unit along the commodity's outside option $p_i^{\text{out}}$. Then, for the boarding edge $e$ of the green vehicle at station $c$, it holds that $\bar{c}_e(f) = M$ (as $\mathcal{F}_Q$ contains $f_{i,p_i^{\text{out}} \to p}(\varepsilon)$ for $\varepsilon \leq 1$). This implies

$$\liminf_{\varepsilon \downarrow 0} c_q(f_{i,p \to q}(\varepsilon)) = \tau_q < M \leq \sum_{e \in p \cap q} c_e(f) + \sum_{e \in q \setminus p} \bar{c}_e(f).$$

On the left-hand side it is noticed that the flow on the last driving edge is unchanged and boarding remains possible, whereas the right-hand side is oblivious to the flow reduction along $p$.

Therefore, we introduce a weaker condition that is satisfied in our time-expanded networks but still guarantees the existence of equilibria.

▶ **Definition 6.** *A cost structure $c \colon \mathcal{F}_Q \to \mathbb{R}^E_{\geq 0}$ is called* weakly regular *if the following implication holds for all demand-feasible flows $f \in \mathcal{F}_Q$, $i \in I$, and $p \in \mathcal{P}_i$ with $f_{i,p} > 0$:*

$$c_{i,p}(f) \leq \min_{q \in \mathcal{P}_i} \sum_{e \in p \cap q} c_e(f) + \sum_{e \in q \setminus p} \bar{c}_e(f) \implies c_{i,p}(f) \leq \min_{q \in \mathcal{P}_i} \liminf_{\varepsilon \downarrow 0} c_{i,q}(f_{i,p \to q}(\varepsilon)).$$

It is easy to see that any regular cost structure is also weakly regular.

▶ **Theorem 7.** *If $c \colon \mathcal{F}_Q \to \mathbb{R}^E_{\geq 0}$ is a lower-semicontinuous, bounded, and weakly regular cost structure, a BS-equilibrium exists.*

The proof uses a similar approach to that in [2, Theorem 2], but we show that it applies to the broader class of weakly regular cost structures.

**Proof.** Let $M$ be an upper bound for all functions $c_e$. There exists, for each $e \in E$, a sequence of continuous functions $c_e^{(n)} \colon \mathcal{F}_Q \to [0, M]$ such that $c_e^{(n)}(f) \uparrow c_e(f)$ holds for all $f \in \mathcal{F}_Q$. For each $n \in \mathbb{N}$, there is a Wardrop equilibrium $f^{(n)} \in \mathcal{F}_Q$ w.r.t. the path cost function $(c_{i,p}^{(n)})_{(i,p) \in \mathcal{P}}$ defined by $c_{i,p}^{(n)}(f) \coloneqq \sum_{e \in p} c_e^{(n)}(f)$, i.e., $f_{i,p}^{(n)} > 0$ implies $c_{i,p}^{(n)}(f^{(n)}) \leq c_{i,q}^{(n)}(f^{(n)})$ for all $i \in I$ and paths $p, q \in \mathcal{P}_i$ [36]. Equivalently, we have

$$f_{i,p}^{(n)} > 0 \implies \sum_{e \in p \setminus q} c_e^{(n)}(f^{(n)}) \leq \sum_{e \in q \setminus p} c_e^{(n)}(f^{(n)}). \tag{3}$$

The sequence $(f^{(n)}, c^{(n)}(f^{(n)}))$ is contained in the compact set $\mathcal{F}_Q \times [0, M]^E$, and therefore has a convergent sub-sequence with some limit $(f, x)$; we pass to this sub-sequence.

By the upper-semicontinuity of the upper hull and the monotonicity of the sequence of cost functions, we have for all $e \in E$

$$\bar{c}_e(f) \geq \limsup_{n\to\infty} \bar{c}_e(f^{(n)}) \geq \limsup_{n\to\infty} c_e(f^{(n)}) \geq \lim_{n\to\infty} c_e^{(n)}(f^{(n)}) = x_e. \tag{4}$$

Let $\alpha > 0$. First, since $(c_e^{(n)})_n$ converges pointwise to $c_e$, there exist $n_0 \in \mathbb{N}$ such that $c_e^{(n_0)}(f) \geq c_e(f) - \alpha/2$. Second, since $c_e^{(n_0)}$ is continuous, there is $\delta > 0$ such that for all $g \in \mathcal{F}_Q$ with $\|f - g\| < \delta$ we have $c_e^{(n_0)}(g) \geq c_e^{(n_0)}(f) - \alpha/2$. As $(c_e^{(n)})_n$ is a pointwise increasing sequence, we then have for all $n \geq n_0$ that $c_e^{(n)}(g) \geq c_e(f) - \alpha$. Third, since $(f^{(n)})_n$ converges to $f$, there is $n_1$ such that $\left\|f - f^{(n)}\right\| < \delta$ holds for all $n \geq n_1$. In conclusion, $c_e^{(n)}(f^{(n)}) \geq c_e(f) - \alpha$ holds for $n \geq \max\{n_0, n_1\}$. Since $\alpha > 0$ was arbitrary, we deduce $x_e \geq c_e(f)$.

Let $i \in I$ and $p, q \in \mathcal{P}_i$ with $f_{i,p} > 0$. There exists $n_0 \in \mathbb{N}$ with $f_{i,p}^{(n)} > 0$ for $n \geq n_0$. Taking the limit of (3) yields $\sum_{e\in p\setminus q} x_e \leq \sum_{e\in q\setminus p} x_e$, and applying (4) and $x_e \geq c_e(f)$ we get

$$\sum_{e\in p\setminus q} c_e(f) \leq \sum_{e\in q\setminus p} \bar{c}_e(f).$$

Adding $c_e(f)$ for each $e \in p \cap q$ to both sides , this shows

$$c_p(f) \leq \sum_{e\in p\cap q} c_e(f) + \sum_{e\in q\setminus p} \bar{c}_e(f).$$

Thus, we can apply weak regularity. ◀

While it is clear that the cost functions in (1) are lower-semicontinuous and bounded, some effort is required to show that they fulfil weak regularity. The idea is that given a path $p$ and a path $q$ minimizing $\liminf_{\varepsilon\downarrow 0} c_{i,q}(f_{i,p\to q}(\varepsilon))$ we consider the last common node $v$ of $p$ and $q$, and define $q'$ as the path that consists of $p$ up until $v$ concatenated with the suffix of $q$ starting from $v$. For boarding edges $e$ on the second part of $q'$, we can then show $\liminf_{\varepsilon\downarrow 0} c_e(f_{i,p\to q'}(\varepsilon)) = \bar{c}_e(f)$ while boarding edges $e$ on the first part fulfil $\liminf_{\varepsilon\downarrow 0} c_e(f_{i,p\to q'}(\varepsilon)) = c_e(f)$. The observation $\liminf_{\varepsilon\downarrow 0} c_{i,q'}(f_{i,p\to q'}(\varepsilon)) \leq \liminf_{\varepsilon\downarrow 0} c_{i,q}(f_{i,p\to q}(\varepsilon))$ then concludes the argument.

▶ **Theorem 8.** *For schedule-based transit networks, a user equilibrium always exists.*

▶ Remark 9. Even for single-commodity networks, the price of stability, i.e., the ratio of the total travel times in a best user equilibrium and in a system optimum, is unbounded. However, Section 5 shows that this ratio is well-behaved in experiments on real-world networks.

## 4 Computation of Equilibria

We continue by discussing the computation of user equilibria. After describing an $\mathcal{O}(|E|^2)$ algorithm for single-commodity networks, we consider the multi-commodity case, for which we outline a finite algorithm. To compute multi-commodity equilibria in practice, we propose a heuristic based on insights gained by the characterization with the quasi-variational inequality.

## 4.1 An Efficient Algorithm for Single-Commodity Networks

We begin with the description of an efficient algorithm for single-commodity networks. To reduce noise, we omit the index $i$ where applicable, i.e., we write $\mathcal{P}$ instead of $\mathcal{P}_i$, etc.

▶ **Definition 10.** *Let $p, q \in \mathcal{P}$. We say that a driving edge $e \in p \cap q$ is a* conflicting edge of $p$ and $q$ *if its corresponding boarding edge $e_B$ lies either on $p$ or on $q$ (but not on both).*

*Assume $p$ and $q$ have a conflicting edge, and let $e \in E$ be the first conflicting edge. We say $p$* has priority over $q$ *if the boarding edge $e_B$ preceding $e$ lies on $q$ (and not on $p$). Let $\prec \subseteq \mathcal{P} \times \mathcal{P}$ denote this relation.*

A $\prec$-minimal path ending in a given reachable node $w$ can be computed by a simple backward-search on the sub-graph of reachable nodes prioritizing non-boarding edges over other edges. This in fact returns a $\prec$-minimal path as for any conflicting edge $e$ with a path $q \in \mathcal{P}$, the corresponding boarding edge $e_B$ must lie on $q$. As the graph is acyclic, this backward-search terminates in $\mathcal{O}(|E|)$ time.

▶ **Lemma 11.** *For the end node $w$ of any path in $\mathcal{P}$, a $\prec$-minimal path ending in $w$ can be computed in $\mathcal{O}(|E|)$ time.*

In order to compute single-commodity equilibrium flows, we can now successively send flow along $\prec$-minimal and $\tau$-optimal paths. In every iteration, the flow on this path is increased until an edge becomes fully saturated. Then, we reduce the capacity on the edges of $p$ by the added flow, remove zero-capacity edges, and repeat this procedure until the demand is met.

▶ **Theorem 12.** *For single-commodity networks, a user equilibrium can be computed in $\mathcal{O}(|E|^2)$ time. The resulting user equilibrium uses at most $|E|$ paths.*

For general (aperiodic) schedules our algorithm is strongly polynomial in the input. For compactly describable periodic schedules it is only pseudo-polynomial as it depends on the size of the time-expanded network. The actual blow-up of the network depends on the ratio of the time horizon and the period length.

## 4.2 The General Multi-Commodity Case

The approach of the previous section fails for the general multi-commodity case as the set of paths $\bigcup_i \mathcal{P}_i$ may not necessarily have a $\prec$-minimal element if there are commodities that do not share the same destination station. Hence, we now describe a finite-time algorithm and a heuristic for computing multi-commodity equilibria in practice.

### 4.2.1 A finite-time algorithm

In the following, we describe a finite-time algorithm for computing exact multi-commodity user equilibria. As we know that an equilibrium $f$ exists, the idea is now to guess the subset $E_O$ of driving edges that are at capacity, i.e., $E_O = \{e \in E_D \mid f_e = \nu_e\}$, as well as a positive, lower bound $\varepsilon$ on the available capacity on all other edges, i.e., $\min_{e \in E_D \setminus E_O} \nu_e - f_e \geq \varepsilon$. Then, we check the following set defined by linear constraints for feasibility:

$$\mathcal{F}(E_O, \varepsilon) := \left\{ f \in \mathcal{F}_Q \;\middle|\; \begin{array}{rl} f_e = \nu_e, & \text{for } e \in E_O, \\ f_e \leq \nu_e - \varepsilon, & \text{for } e \in E_D \setminus E_O, \\ f_{i,p} = 0, & \text{for } i \in I,\, p \in \mathcal{P}_i(E_O) \end{array} \right\}, \tag{5}$$

where $\mathcal{P}_i(E_O)$ is the set of paths $p \in \mathcal{P}_i$ for which there exists a $q \in \mathcal{P}_i$ with $\tau_q < \tau_p$ such that $e^+ \notin E_O$ holds for all edges $e \in E_B \cap q$ with $e^+ \notin p$.

▶ **Lemma 13.** *The set of user equilibria coincides with* $\bigcup_{\varepsilon>0}\bigcup_{E_O\subseteq E_D}\mathcal{F}(E_O,\varepsilon)$.

As the set $\mathcal{F}(E_O,\varepsilon)$ only grows when reducing $\varepsilon$, a finite-time algorithm can iteratively decrease $\varepsilon$ and then check the feasibility of $\mathcal{F}(E_O,\varepsilon)$ for every subset $E_O$ of $E_D$. As a user equilibrium must exist, an equilibrium will be found for small enough $\varepsilon$.

▶ **Corollary 14.** *The procedure described above computes a user equilibrium in finite time.*

### 4.2.2 Heuristic for computing multi-commodity equilibria

In the following, we describe a heuristic for computing multi-commodity equilibria. Start with some initial feasible flow $f \in \mathcal{F}_Q^\nu$, e.g., by sending all flow along their outside option. Then, iteratively, find a direction $d \in \mathbb{R}^{\mathcal{P}}$ and change the flow along this direction while preserving feasibility, until an equilibrium is found. More specifically, we replace $f$ with $f' = f + \alpha \cdot d$ where $\alpha$ is maximal such that $f'$ is feasible.

▶ **Definition 15.** *Let $f$ be a feasible flow. A* direction $d \in \mathbb{R}^{\mathcal{P}}$ *is called*
- balanced *if* $\sum_{p\in\mathcal{P}_i} d_{i,p} = 0$ *for* $i \in I$, *and*
- feasible for $f$ *if the flow* $f + \alpha \cdot d$ *is feasible for small enough* $\alpha > 0$.

Clearly, the choice of the direction is essential for this heuristic to approach an equilibrium. The characterization in Theorem 2 indicates using a direction $d$ such that $f + \alpha \cdot d$ is a deviation violating the quasi-variational inequality, i.e., $d := 1_{i,q} - 1_{i,p}$ for some path $p \in \mathcal{P}_i$ with $f_{i,p} > 0$ for which $q$ is a better available alternative, i.e., $q \in A_{i,p}(f)$ and $\tau_q < \tau_p$. However, not all such directions are feasible; even worse, sometimes no feasible direction is of this form. Therefore, our approach is to start with such a direction $d$ and, if necessary, transform it to make it feasible.

If this heuristic terminates, it provides an equilibrium, but termination is not always guaranteed, as we will see later. We first describe how we achieve feasibility of the direction. For this, a key observation is stated in the following proposition:

▶ **Proposition 16.** *Let $f$ be a feasible flow and $d$ a balanced direction that fulfils $f_{i,p} > 0$ whenever $d_{i,p} < 0$. Then, $d$ is a feasible direction for $f$ if and only if there exists no boarding edge $e$ such that $f_{e^+} = \nu_{e^+}$, $d_{e^+} > 0$ and ($f_e > 0$ or $d_e > 0$) hold.*

In the case that $d = 1_{i,q} - 1_{i,p}$ is an infeasible direction, we apply the following transformation: As long as $d$ is infeasible, there exists a boarding edge such that $f_{e^+} = \nu_{e^+}$, $d_{e^+} > 0$, and $f_e > 0 \lor d_e > 0$, and we repeat the following procedure: Let $(i, p')$ be such that $p'$ is a path containing $e$ with positive flow $f_{i,p'} > 0$ or whose entry in the direction vector is positive, i.e., $d_{i,p'} > 0$. We decrease $d_{i,p'}$ by $\delta := d_e$, if $f_{i,p'} > 0$, or by $\delta := \min(d_e, d_{i,p'})$, otherwise. Next, we determine a best path $q' \in \mathcal{P}_i$ that does not use full driving edges, i.e., driving edges $\tilde{e}$ with $f_{\tilde{e}} = \nu_{\tilde{e}}$ and $d_{\tilde{e}} \geq 0$. We increase $d_{i,q'}$ by $\min(\{\delta\} \cup \{-d_e \mid e \in q', f_e = \nu_e\})$, and afterwards, decrease $\delta$ by the same amount. We repeat this until $\delta$ equals zero. A detailed description of this transformation of the direction can be found in Algorithm 1.

▶ **Proposition 17.** *Algorithm 1 transforms any direction $d$, that fulfils $f_{i,p} > 0$ whenever $d_{i,p} < 0$, to a feasible direction.*

Using Algorithm 1 in the main loop, we can implement the heuristic mentioned above. However, in some situations, the heuristic might apply changes along directions $d_1, \dots, d_k$ in a cyclic behaviour. We distinguish between *terminating* cycles, for which the heuristic breaks out of the cyclic behaviour after some finite but potentially very large number of

**Algorithm 1** Establishing feasible directions.

**Data:** Time-expanded graph with outside options, feasible flow $f$, balanced direction $d \in \mathbb{Z}^{\mathcal{P}}$ s.t. $\forall (i,p) : d_{i,p} < 0 \implies f_{i,p} > 0$
**Result:** A feasible direction
**while** $\exists e \in E_B$ *with* $f_{e^+} = \nu_{e^+} \wedge d_{e^+} > 0 \wedge (f_e > 0 \vee d_e > 0)$ **do**
  $(i,p) \leftarrow$ any commodity $i$ and path $p$ containing $e$ with $f_{i,p} > 0$ or $d_{i,p} > 0$;
  $\delta \leftarrow \begin{cases} d_{e^+}, & \text{if } f_{i,p} > 0, \\ \min(d_{e^+}, d_{i,p}), & \text{otherwise.} \end{cases}$;
  Decrease $d_{i,p}$ by $\delta$;
  **while** $\delta > 0$ **do**
    $q \leftarrow$ best alternative to $p$ not containing any $e' \in E_D$ with $f_{e'} = \nu_{e'} \wedge d_{e'} \geq 0$;
    $\delta' \leftarrow \min(\{\delta\} \cup \{-d_{e'} \mid e' \in q, f_{e'} = \nu_{e'}\})$;
    Increase $d_{i,q}$ by $\delta'$;
    Decrease $\delta$ by $\delta'$;
**return** $d$

iterations, and *non-terminating* cycles. In practice, most terminating cycles can be detected and prohibited by changing the flow along the common direction $\sum_{i=1}^{k} d_i$. Non-terminating cycles, however, constitute a more serious problem. We can detect these cycles, as their common direction $\sum_{i=1}^{k} d_i$ vanishes. Randomizing the path selection in the main loop of the heuristic might help in breaking out of the cycle.

We employ a technique to reduce the initial complexity of a given instance: There is a class of paths for which the procedure will never remove flow from. Thus, we first fill these paths directly when initializing the heuristic. In our experiments, this has shown to handle between 10% and 25% of the total demand before entering the heuristic.

## 5 Computational Study

To gain insights into the applicability of the proposed heuristic, we conduct a computational study on real world train networks. We analyse the performance of the heuristic and compare the computed equilibrium solutions with system optima.

### 5.1 Experiment Setup

For each network in our dataset, we use a dynamic demand profile. As our dataset only provides aggregate, static demand data, we generate a dynamic profile using a uniform distribution of the demand over time horizon of a typical work day. This means, for every origin-destination (OD) pair, we generate commodities $c_i$ with varying start time.

Since the performance of our heuristic varies with the utilization of network capacities, we also apply it to rescaled demands. A demand scale factor of 1 reflects the real-world demand.

The system optima are computed using a linear-programming based approach with delayed column generation. A system optimum is an optimal solution to the linear program

$$\min_{f \in \mathcal{F}_Q^{\nu}} \sum_{(i,p) \in \mathcal{P}} \tau_p \cdot f_{i,p}.$$

This problem is solved using a column generation approach. It is initialized with some feasible solution, e.g., by sending all particles along their outside option. Then, after every


**Figure 2** Map of the considered S-Bahn Hamburg as operated until 2023 [1].

iteration, paths that reduce the objective function are added to the so-called Master-LP, and if any path was added, the Master-LP is solved again. For this, we used software developed by [21, 22].

Our implementation of the heuristic (available in [20]) uses the Rust programming language. All experiments were conducted on an AMD Ryzen 9 5950X CPU.

## 5.2 Data

A pool of periodic timetables of real-world public transportation networks is provided in the publicly available TimPassLib [35]. In our computational study, we unroll these periodic timetables to match an observation period of a day (18 hours) and feed our heuristic the resulting schedule. Table 1 describes the considered networks and schedules in more detail.

We present results for the Hamburg S-Bahn network (Figure 2) and the Swiss long-distance train network. The nominal demand of 750.000 passengers per day in the Hamburg S-Bahn network was taken from [34]. Due to a lack of real-world data, we set the capacity of all vehicles to 1000 passengers, a number closely aligned to real-world train systems. For example, a 2-unit train system of the DBAG Class 490 can hold up to 1028 passengers. The average number of stops per vehicle is 19.14 in the Hamburg network and 8.25 in the Swiss network. We equip all commodities in our experiments with an outside option of 180 minutes.

**Table 1** Details of the considered networks.

| Name | # stations | # vehicles | nominal demand | # commodities |
|---|---|---|---|---|
| Hamburg S-Bahn | 68 | 1,512 | 750,000 | 219,240 |
| Swiss Long Distance | 140 | 2,772 | 1,347,686 | 2,609,712 |

## 5.3 Results

For real-world demands, the heuristic computes user equilibria in both networks in less than 30 seconds. Starting with a demand factor of 0.5 for the Hamburg S-Bahn network and 0.55 for the Swiss network, capacity conflicts occur; for demand factors below these values, any system-optimal flow is also an equilibrium flow as all agents can be assigned to a path $p$ minimizing $\tau_p$ over all paths $p \in \mathcal{P}_i$. With an increase in demand, the number of iterations of the heuristic increases heavily. For example in the Hamburg instance, when initializing the heuristic by filling optimal paths until all of them have a full driving edge, only 1,015 iterations are necessary for the nominal demand while 10,280,576 iterations were necessary for a demand factor of 2.5. For the Swiss long-distance train network, the computation time increased similarly dramatically already when doubling the demand.

The average travel time in the computed equilibria is up to 3.3% higher, in the Hamburg instance, and up to 7.4% higher, in the Swiss instance, compared to a system optimum. For the nominal demand, these numbers are below 1%. Figure 3 shows the computation time of the heuristic, and the mean travel times in the equilibrium and in the system optimum.

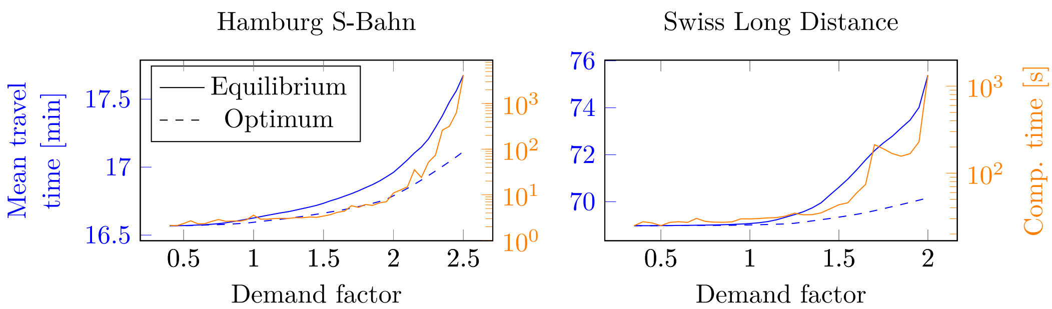


**Figure 3** Comparison of the mean travel time of the user equilibrium and system optimum, and the computation time of the heuristic in the two networks.

## 6 Conclusion

We presented a side-constrained user equilibrium model for a schedule-based transit network incorporating hard vehicle capacities. As our main results, we proved that equilibria exist and can be computed efficiently for single-commodity instances. The existence result generalizes a classical result of Bernstein and Smith [2]; its proof is based on a new condition (weak regularity) implying existence of BS-equilibria for a class of discontinuous and non-separable cost maps. For general multi-commodity instances we devised a heuristic, which was implemented and tested on several realistic networks based on data of the Hamburg S-Bahn and the Swiss railway.

### Limitations of the Model

Our model assumes that passengers are associated with fixed start times for their travel. For flexible departure time choices, however, a side-constrained user equilibrium need not exist (see [28]). Non-existence also applies to a model with transfer penalties.

## Open Problems

First of all, a side-constrained user equilibrium is not unique and, hence, the issue of equilibrium selection or determining which equilibrium is likely to be observed in practice remains unclear and deserves further study. From an algorithmic point of view, the hardness of the equilibrium computation problem for multi-commodity networks is open. The computational complexity for single-commodity networks and periodic timetables (which leads to a compactly representable time-expanded graph) is also open.

## References


1 Arbalete. Liniennetzplan der S-Bahn Hamburg, 2016. Licensed under CC BY-SA 4.0. Accessed 19-Dec-2023. URL: https://commons.wikimedia.org/wiki/File:Karte_der_S-Bahn_Hamburg.svg.

2 David Bernstein and Tony E. Smith. Equilibria for networks with lower semicontinuous costs: With an application to congestion pricing. *Transp. Sci.*, 28(3):221–235, August 1994. doi:10.1287/trsc.28.3.221.

3 Belgacem Bouzaïene-Ayari, Michel Gendreau, and Sang Nguyen. Passenger assignment in congested transit networks: A historical perspective. In Patrice Marcotte and Sang Nguyen, editors, *Equilibrium and Advanced Transportation Modelling*, pages 47–71. Springer, Boston, MA, 1998. doi:10.1007/978-1-4615-5757-9_3.

4 Paolo Carraresi, Federico Malucelli, and Stefano Pallottino. Regional mass transit assignment with resource constraints. *Transp. Res. B: Methodol.*, 30(2):81–98, April 1996. doi:10.1016/0191-2615(95)00027-5.

5 M. Cepeda, R. Cominetti, and M. Florian. A frequency-based assignment model for congested transit networks with strict capacity constraints: characterization and computation of equilibria. *Transp. Res. B: Methodol.*, 40(6):437–459, July 2006. doi:10.1016/j.trb.2005.05.006.

6 Roberto Cominetti and José R. Correa. Common-lines and passenger assignment in congested transit networks. *Transp. Sci.*, 35(3):250–267, 2001. doi:10.1287/trsc.35.3.250.10154.

7 José R. Correa, Andreas S. Schulz, and Nicolás E. Stier-Moses. Selfish routing in capacitated networks. *Math. Oper. Res.*, 29(4):961–976, November 2004. doi:10.1287/moor.1040.0098.

8 José R. Correa and Nicolás E. Stier-Moses. Wardrop equilibria. In James J. Cochran, Louis A. Cox, Jr, Pinar Keskinocak, Jeffrey P. Kharoufeh, and J. Cole Smith, editors, *Wiley Encyclopedia of Operations Research and Management Science*. John Wiley & Sons, Ltd, 2011. doi:10.1002/9780470400531.eorms0962.

9 Umberto Crisalli. Dynamic transit assignment algorithms for urban congested networks. *WIT Trans. Built Environ.*, 44, 1999. URL: https://www.witpress.com/elibrary/wit-transactions-on-the-built-environment/44/5985.

10 Stella C. Dafermos and Frederick T. Sparrow. The traffic assignment problem for a general network. *J. Res. Natl. Inst. Stand.*, 73B(2):91–118, April 1969. doi:10.6028/jres.073B.010.

11 Joaquín de Cea and Enrique Fernández. Transit assignment for congested public transport systems: An equilibrium model. *Transp. Sci.*, 27(2):133–147, 1993. doi:10.1287/trsc.27.2.133.

12 Qian Fu, Ronghui Liu, and Stephane Hess. A review on transit assignment modelling approaches to congested networks: a new perspective. *Procedia Soc. Behav. Sci.*, 54:1145–1155, October 2012. doi:10.1016/j.sbspro.2012.09.829.

13 Guido Gentile, Michael Florian, Younes Hamdouch, Oded Cats, and Agostino Nuzzolo. The theory of transit assignment: Basic modelling frameworks. In Guido Gentile and Klaus Noekel, editors, *Modelling Public Transport Passenger Flows in the Era of Intelligent Transport Systems*, STTT, pages 287–386. Springer, Cham, 2016. doi:10.1007/978-3-319-25082-3_6.

14 Lukas Graf and Tobias Harks. Side-constrained dynamic traffic equilibria. In Kevin Leyton-Brown, Jason D. Hartline, and Larry Samuelson, editors, *Proceedings of the 24th ACM Conference on Economics and Computation*, page 814. ACM, 2023. doi:10.1145/3580507.3597801.

15 Younes Hamdouch, H.W. Ho, Agachai Sumalee, and Guodong Wang. Schedule-based transit assignment model with vehicle capacity and seat availability. *Transp. Res. B: Methodol.*, 45(10):1805–1830, December 2011. doi:10.1016/j.trb.2011.07.010.

16 Younes Hamdouch and Siriphong Lawphongpanich. Schedule-based transit assignment model with travel strategies and capacity constraints. *Transp. Res. B: Methodol.*, 42(7-8):663–684, August 2008. doi:10.1016/j.trb.2007.11.005.

17 Younes Hamdouch, Patrice Marcotte, and Sang Nguyen. Capacitated transit assignment with loading priorities. *Math. Program.*, 101(1):205–230, 2004. doi:10.1007/s10107-004-0542-7.

18 Younes Hamdouch, W. Y. Szeto, and Y. Jiang. A new schedule-based transit assignment model with travel strategies and supply uncertainties. *Transp. Res. B: Methodol.*, 67:35–67, September 2014. doi:10.1016/j.trb.2014.05.002.

19 Tobias Harks, Sven Jäger, Michael Markl, and Philine Schiewe. Computing user equilibria for schedule-based transit networks with hard vehicle capacities, 2024. doi:10.48550/arXiv.2406.17153.

20 Tobias Harks, Sven Jäger, Michael Markl, and Philine Schiewe. A heuristic for computing user equilibria in schedule-based transit networks with hard capacity constraints, 2024. Software, swhId: swh:1:dir:a3e48de8912d81166b2517c3c826e11bcda2c356 (visited on 2024-08-20). URL: https://github.com/ArbeitsgruppeTobiasHarks/sbta/tree/atmos.

21 Tobias Harks, Ingo Kleinert, Max Klimm, and Rolf H. Möhring. Computing network tolls with support constraints. *Networks*, 65(3):262–285, March 2015. doi:10.1002/net.21604.

22 Olaf Jahn, Rolf H. Möhring, Andreas S. Schulz, and Nicolás E. Stier-Moses. System-optimal routing of traffic flows with user constraints in networks with congestion. *Oper. Res.*, 53(4):600–616, August 2005. doi:10.1287/opre.1040.0197.

23 Fumitaka Kurauchi, Michael G. H. Bell, and Jan-Dirk Schmöcker. Capacity constrained transit assignment with common lines. *J. Math. Model. Algorithms*, 2(4):309–327, December 2003. doi:10.1023/b:jmma.0000020426.22501.c1.

24 Homero Larrain, Hemant K. Suman, and Juan Carlos Muñoz. Route based equilibrium assignment in congested transit networks. *Transp. Res. C: Emerg. Technol.*, 127:103125, June 2021. doi:10.1016/j.trc.2021.103125.

25 Torbjörn Larsson and Michael Patriksson. An augmented lagrangean dual algorithm for link capacity side constrained traffic assignment problems. *Transp. Res. B: Methodol.*, 29(6):433–455, December 1995. doi:10.1016/0191-2615(95)00016-7.

26 Patrice Marcotte and Sang Nguyen. Hyperpath formulations of traffic assignment problems. In Patrice Marcotte and Sang Nguyen, editors, *Equilibrium and Advanced Transportation Modelling*, pages 175–200. Springer, Boston, MA, 1998. doi:10.1007/978-1-4615-5757-9_9.

27 Patrice Marcotte, Sang Nguyen, and Alexandre Schoeb. A strategic flow model of traffic assignment in static capacitated networks. *Oper. Res.*, 52(2):191–212, April 2004. doi:10.1287/opre.1030.0091.

28 Sang Nguyen, Stefano Pallottino, and Federico Malucelli. A modeling framework for passenger assignment on a transport network with timetables. *Transp. Sci.*, 35(3):238–249, August 2001. doi:10.1287/trsc.35.3.238.10152.

29 Agostino Nuzzolo and Francesco Russo. Stochastic assignment models for transit low frequency services: Some theoretical and operative aspects. In Lucio Bianco and Paolo Toth, editors, *Advanced Methods in Transportation Analysis*, Transportation Analysis, pages 321–339. Springer, 1996. doi:10.1007/978-3-642-85256-5_14.

30 Agostino Nuzzolo, Francesco Russo, and Umberto Crisalli. A doubly dynamic schedule-based assignment model for transit networks. *Transp. Sci.*, 35(3):268–285, August 2001. doi:10.1287/trsc.35.3.268.10149.

31 Natale Papola, Francesco Filippi, Guido Gentile, and Lorenzo Meschini. Schedule-based transit assignment: new dynamic equilibrium model with vehicle capacity constraints. In Agostino Nuzzolo and Nigel H. M. Wilson, editors, *Schedule-Based Modeling of Transportation Networks*, volume 46 of *ORCS*, pages 1–26. Springer, 2009. doi:10.1007/978-0-387-84812-9_8.

**32** Michael Patriksson. *The traffic assignment problem: models and methods*. VSP, Utrecht, 1994.

**33** Normen Rochau, Klaus Nökel, and Michael G. H. Bell. Cost functions for strategies in schedule-based transit networks with limited vehicle capacities. *Transp. Res. Rec.*, 2284(1):62–69, January 2012. doi:10.3141/2284-08.

**34** S-Bahn Hamburg GmbH. Zahlen, Daten, Fakten rund um die S-Bahn Hamburg, 2023. URL: https://www.s-bahn-hamburg.de/service/schulklassen/daten-zahlen-fakten.

**35** Philine Schiewe, Marc Goerigk, and Niels Lindner. Introducing TimPassLib – A library for integrated periodic timetabling and passenger routing. *Oper. Res. Forum*, 4, August 2023. doi:10.1007/S43069-023-00244-1.

**36** Mike J. Smith. The existence, uniqueness and stability of traffic equilibria. *Transp. Res. B: Methodol.*, 13(4):295–304, December 1979. doi:10.1016/0191-2615(79)90022-5.

**37** Mike J. Smith. Two alternative definitions of traffic equilibrium. *Transp. Res. B: Methodol.*, 18(1):63–65, February 1984. doi:10.1016/0191-2615(84)90006-7.

**38** Heinz Spiess and Michael Florian. Optimal strategies: A new assignment model for transit networks. *Transp. Res. B: Methodol.*, 23(2):83–102, April 1989. doi:10.1016/0191-2615(89)90034-9.

**39** John Glen Wardrop. Some theoretical aspects of road traffic research. *Proc. Inst. Civ. Eng.*, 1(3):325–378, May 1952.

**40** Jia Hao Wu, Michael Florian, and Patrice Marcotte. Transit equilibrium assignment: A model and solution algorithms. *Transp. Sci.*, 28(3):193–203, August 1994. doi:10.1287/trsc.28.3.193.

**41** Z. X. Wu and William H. K. Lam. Network equilibrium for congested multi-mode networks with elastic demand. *J. Adv. Transp.*, 37(3):295–318, 2003. doi:10.1002/atr.5670370304.