Generalized Fibonacci Cubes Based on Swap and Mismatch Distance

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Abstract

The hypercube of dimension n is the graph with 2^n vertices associated to all binary words of length n and edges connecting pairs of vertices with Hamming distance equal to 1. Here, an edit distance based on swaps and mismatches is considered and referred to as *tilde-distance*. Accordingly, the *tilde-hypercube* is defined, with edges linking words having tilde-distance equal to 1. The focus is on the subgraphs of the tilde-hypercube obtained by removing all vertices having a given word as factor. If the word is 11, then the subgraph is called *tilde-Fibonacci cube*; in the case of a generic word, it is called *generalized tilde-Fibonacci cube*. The paper surveys recent results on the definition and characterization of those words that define generalized tilde-Fibonacci cubes that are *isometric* subgraphs of the tilde-hypercube. Finally, a special attention is given to the study of the tilde-Fibonacci cubes.

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1 Introduction

The n-dimensional *hypercube*, Q_n, is the well-known graph whose vertices are in correspondence with the 2^n words of length n over the binary alphabet $\{0, 1\}$ and two vertices are connected by an edge if the corresponding words differ in one position, that is, if their

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Hamming distance is 1. Hence, the distance (number of edges in a minimal path) between two vertices in the graph is the Hamming distance of the corresponding words. The notion of hypercube has been extensively investigated because it is used to design interconnection networks (cf. [13, 17]) and also finds applications in theoretical chemistry (cf. [27] and [20, 24] for surveys). However, the critical limitation of the hypercube lies in its exponential growth in size, specifically in the number of vertices. Exploring some of its isometric subgraphs can improve efficiency. A subgraph of the n-dimensional hypercube is *isometric* to Q_n if the distance between any pair of vertices in such subgraphs is the same as the distance in the complete hypercube. With this aim, in 1993, Hsu introduced the *Fibonacci cubes* [21], obtained from Q_n by selecting only those vertices that do not contain 11 as a factor. They have a Fibonacci number of vertices making them useful in applications to reduce the complexity and to limit resources. They have many remarkable properties, also related to Fibonacci numbers (cf. [15]).

In 2012, Generalized Fibonacci cubes have been defined by means of a binary word f; the graph $Q_n(f)$ is a subgraphs of Q_n whose vertices do not contain f as a factor, i.e. the vertices are f-free binary words [22]. Then, the property of $Q_n(f)$ being an isometric subgraph of Q_n is related to some combinatorial properties of the avoided word f that in such cases is called *isometric*. More formally, a binary word f is isometric (or Ham-isometric) when, for any $n \geq 1$, $Q_n(f)$ is an isometric subgraph of Q_n, and *non-isometric*, otherwise [25]. The definition can also be done without mentioning graphs and only in terms of the Hamming distance. A word f is Ham-isometric, if for any pair of f-free words u and v of the same length, u can be transformed into v by a minimal sequence of symbol replacements that each time produce an f-free word, as well. Binary Ham-isometric words have been characterized in [23, 25, 35, 38, 39] and research on the topic remains very active [11, 36, 37, 12, 6, 7, 16, 8, 9, 4, 10].

Over the years, many variations of the hypercube have been introduced to enhance certain features. For example, folded hypercubes (cf. [18]) and enhanced hypercubes (cf. [32]) have been defined by adding some edges to the original structure, providing several advantages in terms of topological properties. Motivated by the same reasons, we introduced a type of generalized hypercube that employs a different distance, enabling the definition of isometric subgraphs. The inspiration and motivation came from many applications of computational biology, where many processes involve complex transformations and it is natural to consider not only replacement operations but also *swap* operations that exchange two adjacent different symbols in a word. The edit distance based on swaps and replacements is worth considering [1, 19, 29] instead of the Hamming distance. Actually, this distance was defined in the 70s by Wagner and Fischer [34, 33] who proved that it can be efficiently computed.

In [3] this distance is referred to as *tilde-distance*, since the $\sim$ symbol somehow evokes the swap operation. In [2], the tilde-distance is taken as the base to define the *tilde-hypercube*, $\tilde{Q}_n$; it has again all the n-binary strings as vertices, but two vertices are adjacent if the tilde-distance is equal to 1. This implies that $\tilde{Q}_n$ has more edges than Q_n; in particular, since a swap corresponds to two replacements of consecutive characters, some vertices at distance 2 in Q_n, become adjacent in $\tilde{Q}_n$.

This paper surveys most of the recent results on tilde-isometric words and generalized tilde-Fibonacci cubes. In particular, we collect the main definitions to give a recursive construction of tilde-hypercubes and enumerate their edges and vertices. Then, the subgraphs $\tilde{Q}_n(f)$ of the tilde-hypercubes are considered. They are obtained by selecting the vertices corresponding to f-free words, for a given word f. It is also reported the characterization of words f such that $\tilde{Q}_n(f)$ is an isometric subgraph of $\tilde{Q}_n$ as proved in [5]. We also exhibit an infinite family of tilde-isometric words that are not Hamming-isometric and vice versa. Finally, the last part of the paper focuses on the word $f = 11$, that is both a Hamming- and a tilde-isometric word; the subgraph $\tilde{Q}_n(11)$ is referred to as the *tilde-Fibonacci cube*. We describe its recursive construction and present new simple results on diameter and radius that prove that the tilde-Fibonacci cubes are self-centered graphs.

2 Basic on Strings and Fibonacci Cubes

In this paper we focus only on the binary alphabet $\Sigma = \{0, 1\}$. A word (or string) w over Σ of length $|w| = n$, is $w = a_1 a_2 \cdots a_n$, where $a_1, a_2, \ldots, a_n$ are symbols in Σ. The set of all words over Σ is denoted Σ^*. Finally, ϵ denotes the *empty word* and $\Sigma^+ = \Sigma^* - \{\epsilon\}$. For any word $w = a_1 a_2 \cdots a_n$, the *reverse* of w is the word $w^{rev} = a_n a_{n-1} \cdots a_1$. If $x \in \Sigma$, $\overline{x}$ denotes the opposite of x, i.e $\overline{x} = 1$ if $x = 0$ and vice versa. Then we define the *complement* of w the word $\overline{w} = \overline{a}_1 \overline{a}_2 \cdots \overline{a}_n$.

Let $w[i]$ denote the symbol of w in position i, i.e. $w[i] = a_i$. Then, $w[i..j] = a_i \ldots a_j$, for $1 \leq i \leq j \leq n$, denotes a *factor* u of w of length $j - i + 1$ placed from the i-th to the jth position of w. We say that $I = [i..j]$ is the *interval* where the factor u occurs in w. The *prefix* (resp. *suffix*) of w of length l, with $1 \leq l \leq n - 1$ is $\text{pre}_l(w) = w[1..l]$ (resp. $\text{suf}_l(w) = w[n - l + 1..n]$). When $\text{pre}_l(w) = \text{suf}_l(w) = u$ then u is here referred to as an *overlap* of w of length l; in other frameworks, it is also called the border, or bifix of w (cf. [30]). A word w *avoids a word* f if w does not contain f as a factor: we also say that w is f*-free*.

An *edit operation* is a function $O : \Sigma^* \to \Sigma^*$ that transforms a word into another one. Let OP be a *set of edit operations*. The *edit distance* of words $u, v \in \Sigma^*$, with respect to the set OP, is the minimum number of edit operations in OP needed to transform u into v.

In this paper, we consider the edit distance that uses only *swap* and *replacement* operations. Note that these operations preserve the length of the word.

▶ **Definition 1.** *Let $w = a_1 a_2 \ldots a_n$ be a word over Σ.*
The replacement operation (or replacement, for short) on w at position i, with $i = 1, \ldots, n$, is defined by

$$R_i(a_1 a_2 \ldots a_{i-1} \boldsymbol{a_i} a_{i+1} \ldots a_n) = a_1 a_2 \ldots a_{i-1} \boldsymbol{\overline{a}_i} a_{i+1} \ldots a_n.$$

The swap operation (or swap, for short) on w at position i, with $i = 1, \ldots, n-1$ and $a_i \neq a_{i+1}$, is defined by

$$S_i(a_1 a_2 \ldots a_{i-1} \boldsymbol{a_i a_{i+1}} a_{i+2} \ldots a_n) = a_1 a_2 \ldots a_{i-1} \boldsymbol{a_{i+1} a_i} a_{i+2} \ldots a_n.$$

Note that one swap corresponds to two replacements of consecutive symbols.

The *Hamming distance* $\text{dist}_H(u, v)$ of equal-length words $u, v \in \Sigma^*$ is defined as the minimum number of replacements needed to obtain v from u.

Let $G = (V(G), E(G))$ be a graph, $V(G)$ be the set of its nodes and $E(G)$ be the set of its edges. The distance of $u, v \in V(G)$, $\text{dist}_G(u, v)$, is the length of the shortest path that connects u and v in G. A subgraph $S = (V(S), E(S))$ of a (connected) graph G is an *isometric subgraph* if for any $u, v \in V(S)$, $\text{dist}_S(u, v) = \text{dist}_G(u, v)$.

The n*-hypercube*, or binary n-cube, Q_n, is a graph with 2^n vertices, labeled with the words of length n and edges connecting two vertices u and v in Q_n when their labels differ exactly in 1 position, i.e. when $\text{dist}_H(u, v) = 1$. Therefore, $\text{dist}_{Q_n}(u, v) = \text{dist}_H(u, v)$.

Denote by f_n the n-th Fibonacci number, defined by $f_1 = 1, f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$, for $i \geq 3$. The *Fibonacci cube* (cf. [22]) F_n of order n is the subgraph of Q_n whose vertices are binary words of length n avoiding the factor 11. It is well known that F_n is an isometric subgraph of Q_n (cf. [24]).

One of the main properties of Q_n and F_n is their *recursive structure* that have been extensively studied (cf. [21, 28, 26, 24]).

The following results are well-known, but are hereby stated for future reference.

▶ **Proposition 2.** *Let Q_n be the hypercube of order n. Then*

- $|V(Q_n)| = 2^n$,
- $|E(Q_n)| = n2^{n-1}$.

▶ **Proposition 3.** *Let F_n be the Fibonacci cube. Then:*

- $|V(F_n)| = f_{n+2}$,
- $|E(F_n)| = \frac{2(n+1)f_n + nf_{n+1}}{5}$.

In other terms, if the number of vertices $N = 2^n$ of the hypercube is taken as main parameter, the number of edges of a hypercube with N vertices is $(N \log N)/2$.

The sequence $|E(F_n)|$ is Sequence A001629 in [31]. Hence, the number of edges of a Fibonacci cube with N vertices, $N = f_{n+2}$, is $O(N \log N)$, asymptotically equal to the number of edges of a hypercube with the same number of vertices.

In order to generalize the notion of Fibonacci cube, one can consider the subgraphs of the n-hypercube whose nodes are f-free, for some word $f \in \Sigma^*$, denoted by $Q_n(f)$, called *generalized Fibonacci cubes*. Of course, if $n < |f|$, then $Q_n(f) = Q_n$; so it makes sense to consider $n \geq |f|$. Unfortunately, not all the words f make $Q_n(f)$ an isometric subgraph of Q_n. The words having this property have been widely studied (cf. [23, 25, 35, 38, 39]). These words are in fact called isometric words, because they reflect on this isometry property on the corresponding graphs.

Under the combinatorial point of view, isometric words are defined as follows. A word $f \in \Sigma^*$ is *Ham-isometric* (or simply *isometric*) if and only if $Q_n(f)$ is an isometric subgraph of Q_n. A word that is not Ham-isometric is said to be *Ham-non-isometric*. In terms of transformations, isometric words can be characterized by the following:

▶ **Proposition 4.** *A word f is isometric iff for any pair of f-free words u and v, there exists a sequence of $\mathrm{dist}_H(u, v)$ replacements that transforms u into v where all the intermediate words are also f-free.*

A word w has a 2-*error overlap* if there exists $l \leq |w|$ such that $\mathrm{pre}_l(w)$ and $\mathrm{suf}_l(w)$ have Hamming distance 2. Then, the following characterization of Ham-non-isometric words is proved (cf. [35]).

▶ **Proposition 5** ([35])**.** *A word f is Ham-non-isometric if and only if f has a 2-error overlap.*

3 Tilde-distance and Tilde-hypercube

In this section, the edit distance based on swap and replacement operations is considered. In [3] it is called tilde-distance and denoted by $dist_\sim$. According to this definition one can define the n-tilde-hypercube. We start with the definition of tilde-distance.

▶ **Definition 6.** *Let $u, v \in \Sigma^*$ be words of equal length. The tilde-distance $\mathrm{dist}_\sim(u, v)$ between u and v is the minimum number of replacements and swaps needed to transform u into v.*

Note that for all $u, v \in \Sigma^*$, $\mathrm{dist}_\sim(u, v) \leq \mathrm{dist}_H(u, v)$, since a swap is a shortcut for two adjacent replacements.

▶ **Example 7.** The words $u = 1011$ and $v = 0110$ have tilde-distance $\text{dist}_\sim(u,v) = 2$. In fact, v can be obtained from u with a swap S_1 of the first and the second bits, and a replacement R_4 in the fourth position. Note that, in order to compute the Hamming distance, three replacements are needed in positions $1, 2$ and 4, therefore $\text{dist}_H(u,v) = 3$.

In order to describe the sequence of the operations that are used to compute the tilde-distance of two words, we need the following definition of a tilde-transformation:

▶ **Definition 8.** *Let $u, v \in \Sigma^*$ be words of equal length and $dist_\sim(u,v) = d$. A tilde-transformation τ from u to v is a sequence of $d+1$ words $(w_0, w_1, \ldots, w_d)$ such that $w_0 = u$, $w_d = v$, and for any $k = 0, 1, \ldots, d-1$, $\text{dist}_\sim(w_k, w_{k+1}) = 1$. Further, τ is f-free if for any $i = 0, 1, \ldots, d$, word w_i is f-free.*

A tilde-transformation $(w_0, w_1, \ldots, w_d)$ from u to v with $dist_\sim(u,v) = d$ is associated to a sequence of d operations $(O_{i_1}, O_{i_2}, \ldots, O_{i_d})$ such that, for any $k = 1, \ldots, d$, $O_{i_k} \in \{R_{i_k}, S_{i_k}\}$ and $w_k = O_{i_k}(w_{k-1})$; it can be graphically represented as follows:

$$u = w_0 \xrightarrow{O_{i_1}} w_1 \xrightarrow{O_{i_2}} \cdots \xrightarrow{O_{i_h}} w_h = v.$$

With a little abuse of notation, in the sequel we will refer to a tilde-transformation both as a sequence of words and as a sequence of operations. Furthermore, as a consequence of the definition, in a tilde-transformation, the positions i_1, i_2, …, i_d are all distinct.

In the following we give some examples that show some special features of tilde-transformations, which never occur when transformations using only replacements are considered. This highlights, on the one hand, the richness of new situations that arise from introducing the swap operation, and on the other hand, it anticipates the need for new and more sophisticated techniques to handle these increasingly complex scenarios. First of all, we point out that when only replacements are used, the different transformations from u to v use the same set of operations, possibly applied in a different order. This is not the case for the different tilde-transformations between two words. The following two examples show two singular cases of different tilde-transformations which use different sets of operations.

▶ **Example 9.** Let $u = 100$, $v = 001$. In this case $\sigma_1 = (S_1, S_2)$ and $\sigma_2 = (R_1, R_2)$ are two tilde-transformations from u to v on different sets of operations. In particular, observe that σ_1 flips twice the bit in the second position, whereas in σ_2 the second bit is not involved.

$$\sigma_1 : 100 \xrightarrow{S_1} 010 \xrightarrow{S_2} 001 \qquad \sigma_2 : 100 \xrightarrow{R_1} 000 \xrightarrow{R_3} 001$$

▶ **Example 10.** Consider $u' = 010$ and $v' = 101$ and the tilde-transformations $\rho_1 = (S_1, R_3)$ and $\rho_2 = (S_2, R_1)$ from u' to v'. Here, different sets of operations are used and, differently from Example 9 in both transformations each symbol is changed just once:

$$\rho_1 : 010 \xrightarrow{S_1} 100 \xrightarrow{R_3} 101 \qquad \rho_2 : 010 \xrightarrow{S_2} 001 \xrightarrow{R_1} 101$$

The variety of situations described above translates into a higher degree of difficulty of the tilde-transformations (compared to the Hamming transformations) when handling some property like, for instance, isometricity.

▶ Remark 11. Let $u, v \in \Sigma^m$ and τ be a tilde-transformation from u to v. Referring to Example 9, if τ contains two swaps, S_i and S_{i+1}, at consecutive positions i and $i+1$ of u, then such swaps can be substituted by two replacements, namely R_i and R_{i+2}, still obtaining a tilde-transformation from u to v of length equal to their distance. Hence, in particular, if $u = 00u'$ and $v = 10v'$, among all sequences of swaps and replacements of minimal length that transform u into v there is one starting with the replacement R_1.

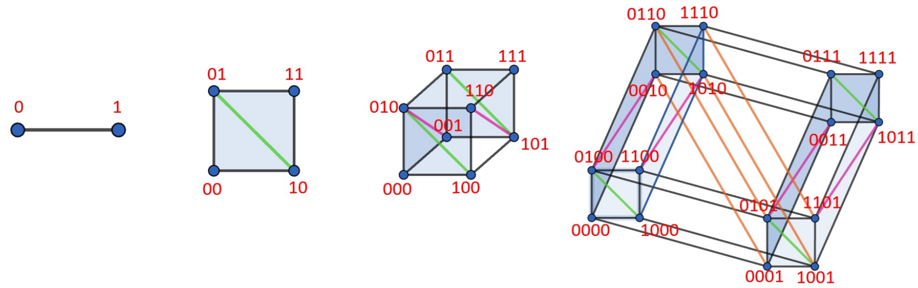

Figure 1 Tilde-hypercubes with n=1,2,3,4 (the colored edges are those added to the traditional hypercube).

▶ Remark 12. Let $u, v \in \Sigma^m$ and τ be a tilde-transformation from u to v. Referring to Example 10. If τ contains a swap S_{i+1} and a replacement R_i then they can be substituted by S_i and R_{i+2}. As a consequence, if $u = 01u'$ and $v = 10v'$ among all sequences of swaps and replacements of minimal length that transform u into v there is one starting with the replacement R_1.

Based on the definition of tilde-distance, a natural extension of the concept of n-hypercube is given in [2] as follows:

▶ **Definition 13.** *The n-tilde-hypercube $\tilde{Q}_n$, or tilde-hypercube of order n, is a graph with 2^n vertices, labeled with binary words of length n. Two vertices in $\tilde{Q}_n$, are adjacent whenever the tilde-distance of their labels is 1.*

Figure 1 shows the tilde-hypercubes of order $1, 2, 3, 4$.

▶ Remark 14. Q_n is a proper subgraph of $\tilde{Q}_n$. In fact, for $u, v \in \Sigma^*$, $\text{dist}_H(u, v) = 1$ implies $\text{dist}_\sim(u, v) = 1$. Note that Q_n is not an isometric subgraph of $\tilde{Q}_n$. Indeed, for any $n \geq 2$, there exists a pair of words (u_n, v_n) of length n such that $\text{dist}_\sim(u_n, v_n) = 1$ and $\text{dist}_H(u_n, v_n) \neq 1$. For instance, for any $0 \leq k, h \leq n-2$, $h + k = n - 2$, consider the words $u_n = 0^h 010^k$ and $v_n = 0^h 100^k$; then $\text{dist}_\sim(u_n, v_v) = 1$ and $\text{dist}_H(u_n, v_n) = 2$, therefore (u_n, v_n) is an edge in $\tilde{Q}_n$ but not in Q_n.

The following lemma is the main tool for exhibiting a recursive definition of the tilde-hypercube, in analogy with the classical hypercube.

▶ **Lemma 15.** *For any $u, v \in \Sigma^{n-1}$, $\text{dist}_\sim(u0, v0) = \text{dist}_\sim(u, v) = \text{dist}_\sim(u1, v1)$ and $\text{dist}_\sim(u0, u1) = 1$. Moreover, for any $u' \in \Sigma^{n-2}$, $\text{dist}_\sim(u'01, u'10) = 1$.*

▶ **Proposition 16** ([2]). *The n-tilde-hypercubes $\tilde{Q}_n$, with $n \geq 1$, can be recursively defined.*

Proof. If $n = 1$, $\tilde{Q}_1$ has just two vertices 0 and 1 connected by an edge.

Suppose all the tilde-hypercubes of dimension smaller than n have been defined. The hypercube $\tilde{Q}_n$ is recursively constructed as follows. Start with a copy of $\tilde{Q}_{n-1}$ where every vertex u is replaced by $u0$, and denote this copy by $\tilde{Q}_{n-1}0$ and a second copy where every vertex u is replaced by $u1$, and denoted by $\tilde{Q}_{n-1}1$.

By Lemma 15, if $(u, v) \in E(\tilde{Q}_{n-1})$, then $(u0, v0), (u1, v1) \in E(\tilde{Q}_{n-1})$, this means that $\tilde{Q}_{n-1}0$ and $\tilde{Q}_{n-1}1$ are subgraphs of $\tilde{Q}_n$. Moreover, for any $u \in \Sigma^{n-1}$, there is an edge $(u0, u1)$ in $\tilde{Q}_n$ with $u0 \in V(\tilde{Q}_{n-1}0)$ and $u1 \in V(\tilde{Q}_{n-1}1)$. Finally, for any $v \in \Sigma^{n-2}$ $(v10, v01) \in E(\tilde{Q}_n)$ with $v10 \in V(\tilde{Q}_{n-1}0)$ and $v01 \in V(\tilde{Q}_{n-1}1)$. In Fig. 1, these latter edges added in the fourth step of recursion, are depicted with orange edges. For any other pair of words $u, v \in \{0, 1\}^n$, $\text{dist}_\sim(u, v) > 1$, then (u, v) is not an edge of $\tilde{Q}_n$. ◀

▶ **Corollary 17.** *Let $\tilde{Q}_n$ be the tilde-hypercube of order n. Then, $|E(\tilde{Q}_1)| = 1$ and, for any $n \geq 2$ $|E(\tilde{Q}_n)| = 2|E(\tilde{Q}_{n-1})| + 2^{n-1} + 2^{n-2}$.*

By solving the above recurrence, we find the exact formula $|E(\tilde{Q}_n)| = (3n-1) \cdot 2^{n-2}$ (Sequence A053220 in [31]). Let $\tilde{EQ}(N)$ be the number of edges of the tilde-hypercube with N vertices, $N = 2^n$. Then,

$$\tilde{EQ}(N) = \frac{3N(\log N - 1)}{4}. \tag{1}$$

4 Tilde-hypercube Avoiding a Word and Tilde-isometric Words

In analogy with the n-hypercube avoiding a word based on the Hamming distance, referred to as generalized Fibonacci cube in [22], in this section, we consider the n-tilde-hypercube avoiding a word, based on the tilde-distance, here named generalized tilde-Fibonacci cubes.

In [2] the following definition is given:

▶ **Definition 18.** *The n-tilde-hypercube avoiding a word f, or the tilde-hypercube of order n avoiding a word f, denoted $\tilde{Q}_n(f)$, is the subgraph of $\tilde{Q}_n$ obtained by removing those vertices which contain f as a factor.*

We are interested in those words f such that $\tilde{Q}_n(f)$ is an isometric subgraph of $\tilde{Q}_n$, i.e. the distance of two vertices of $\tilde{Q}_n(f)$ is equal to the tilde-distance of the corresponding labels.

▶ **Definition 19.** *A word $f \in \Sigma^*$ is tilde-isometric if and only if for all $n \geq |f|$, $\tilde{Q}_n(f)$ is an isometric subgraph of $\tilde{Q}_n$.*

The following proposition characterizes isometric words re-stating the definition of isometric word under a combinatorial point of view.

▶ **Proposition 20.** *Let $f \in \Sigma^*$ be a word of length n with $n \geq 1$. The word f is tilde-isometric if and only if for any pair of f-free words u and v of equal length $m \geq n$, there exists a tilde-transformation from u to v that is f-free. It is tilde-non-isometric if it is not tilde-isometric.*

In order to show that a word is tilde-non-isometric it is sufficient to exhibit a pair (u, v) of f-free words such that all the tilde-transformations from u to v are not f-free. Such pair of words is called *a pair of tilde-witnesses*. More challenging is to prove that a word is tilde-isometric.

▶ **Example 21.** The word $f = 1010$ is tilde-non-isometric. In fact, let $u = 11000$ and $v = 10110$; then (u, v) is a pair of tilde-witnesses for f. In fact u and v are f-free; moreover there are only two possible tilde-transformations from u to v, namely $11000 \xrightarrow{S_2} 10100 \xrightarrow{R_4} 10110$ and $11000 \xrightarrow{R_4} 11010 \xrightarrow{S_2} 10110$, and in both cases 1010 appears as factor after the first step. On the other side, observe that f is Ham-isometric by Proposition 5.

The following straightforward property of tilde-isometric binary words is very helpful to simplify proofs and computations.

▶ Remark 22. A word f is tilde-isometric iff $\overline{f}$ is tilde-isometric iff f^{rev} is tilde-isometric.

In view of Remark 22 , we will focus on words starting with 1.

5 Characterization of Tilde-isometric Words

The characterization of Ham-isometric words given in [38] and here reported as Proposition 5, uses the notion of 2-error overlap. In this section we introduce the corresponding definition that refers to the tilde-distance. Tilde-error overlaps will have a main role in the characterization of tilde-isometric words but the presence of swap operations will force us to handle them with care.

▶ **Definition 23.** *Let $f \in \Sigma^n$. Then, f has a q-tilde-error overlap of length ℓ and shift $r = n - \ell$, with $1 \le \ell \le n-1$ and $0 \le q \le \ell$, if* $\mathrm{dist}_\sim(\mathrm{pre}_\ell(f), \mathrm{suf}_\ell(f)) = q$.

▶ **Example 24.** The word $f = 1101110101101$ has a 2-tilde-error overlap of length 6 and shift 7. Indeed, $\mathrm{pre}_6(f) = 110111$, $\mathrm{suf}_6(f) = 101101$ and $\mathrm{dist}_\sim(110111, 101101) = 2$.

For our proofs, given a word f with a q-tilde-error overlap of length ℓ, we will study the tilde-transformations τ from $\mathrm{pre}_\ell(f)$ to $\mathrm{suf}_\ell(f)$ with q operations. For this reason, it is useful to refer to the alignment of the two strings $\mathrm{pre}_\ell(f)$ to $\mathrm{suf}_\ell(f)$. Furthermore, for our purpose, it is relevant to consider also the bits adjacent to a tilde-error overlap of a word. For this reason, we introduce the following notation.

Let f be a word in $\{0,1\}^*$ and \$ be a symbol different from $0, 1$, here used as delimiter of a word, that by definition "matches" any symbol of the word. Consider f with its delimiters $\$f\$$. A q-tilde-error overlap of length ℓ is denoted by $\binom{\$xb}{ay\$}$ where xb, ay are a prefix and a suffix, respectively, of f, $a, b \in \Sigma$, $x, y \in \Sigma^*$ with $|x| = |y| = \ell$, and $\mathrm{dist}_\sim(\$xb, ay\$) = \mathrm{dist}_\sim(x, y) = q$. This notation makes evident the fact that in f the prefix x is followed by b and the suffix y is preceded by a. Moreover, a q-tilde-error overlap $\binom{\$xb}{ay\$}$ is sometimes factorized into blocks to highlight the significant parts. For example, the 2-tilde-error overlap in Example 24 is denoted by $\binom{\$1}{01}\binom{1011}{0110}\binom{10}{1\$}$ because $\mathrm{dist}_\sim(110111, 101101) = 2 = \mathrm{dist}_\sim(1011, 0110)$.

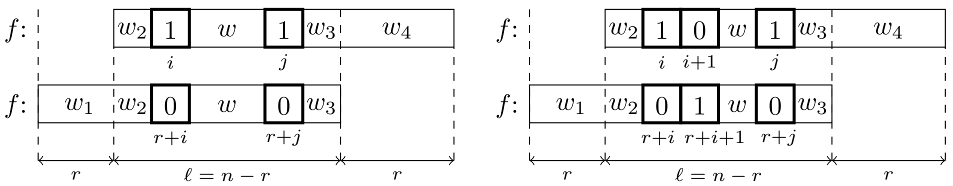

Figure 2 A word f and its 2-tilde-error overlap of shift r and length $\ell = n - r$, with tilde-transformation $(O_i, O_j) = (R_i, R_j)$ (left), and $(O_i, O_j) = (S_i, R_j)$ (right).

In the sequel, we will be interested in the specific case of 1-tilde-error overlap where the single error in the alignment is a swap and in all the cases of 2-tilde-error overlaps. Consider a 2-tilde-error overlap of f of shift r, length $\ell = n - r$, and let (O_i, O_j), $1 \le i < j \le \ell$, be a tilde-transformation from $\mathrm{pre}_\ell(f)$ to $\mathrm{suf}_\ell(f)$. Observe that the positions in $\mathrm{pre}_\ell(f)$ modified by O_i are either i or both i and $i+1$, following that O_i is a replacement or a swap. Hence, the number of the positions modified by O_i and O_j may be 2, 3 or 4. Fig. 2 shows a word f with its 2-tilde-error overlap of shift r and length $\ell = n - r$. With our notation, the 2-tilde-error overlap is $\binom{\$w_2 1 w 1 w_3 b}{a w_2 0 w 0 w_3 \$}$ in the figure on the left and $\binom{\$w_2 10 w 1 w_3 b}{a w_2 01 w 0 w_3 \$}$ in the figure on the right, where b is the first letter of w_4 and a the last letter of w_1. A tilde-transformation from $\mathrm{pre}_\ell(f)$ to $\mathrm{suf}_\ell(f)$ is given by $(O_i, O_j) = (R_i, R_j)$ in the first case and by $(O_i, O_j) = (S_i, R_j)$ in the second case. We say that a 2-tilde-error overlap has *non-adjacent errors* when there is at least one character interleaving the positions modified by O_i and those modified by O_j.

The 2-tilde-error overlap $\binom{\$x}{ax}\binom{100}{001}\binom{yb}{y\$}$ is also considered as having non-adjacent errors because it admits the tilde-transformation $(O_i, O_j) = (R_i, R_{i+2})$, despite it has also the other tilde-transformation $(O_i, O_j) = (S_i, S_{i+1})$.

In all the other cases, we say that the 2-tilde-error overlap has *adjacent errors*.

Let us state the characterization of tilde-isometric words in terms of special configurations in their overlap proved in [5].

▶ **Theorem 25.** *A word $f \in \Sigma^n$ is tilde-non-isometric if and only if one of the following cases occurs (up to complement, reverse and inversion of rows):*

(C0) *f has a 1-tilde-error overlap $\binom{\$x}{ax}\binom{01}{10}\binom{yb}{y\$}$ with $x, y \in \Sigma^*$, $a, b \in \Sigma$;*

(C1) *f has a 2-tilde-error overlap with non-adjacent errors, different from $\binom{\$x}{ax}\binom{000}{101}\binom{yb}{y\$}$ with $x, y \in \Sigma^+$, $a, b \in \Sigma$;*

(C2) *f has a 2-tilde-error overlap $\binom{\$x}{ax}\binom{0101}{1010}\binom{yb}{y\$}$ or $\binom{\$x}{ax}\binom{0110}{1001}\binom{yb}{y\$}$ with $x, y \in \Sigma^*$, $a, b \in \Sigma$;*

(C3) *f has a 2-tilde-error overlap $\binom{\$x}{ax}\binom{010}{101}\binom{yb}{y\$}$ with $x, y \in \Sigma^*$, $a, b \in \Sigma$;*

(C4) *f has a 2-tilde-error overlap $\binom{\$x}{ax}\binom{011}{100}\binom{0}{\$}$ with $x \in \Sigma^*$, $a \in \Sigma$;*

(C5) *f has a 2-tilde-error overlap $\binom{\$}{0}\binom{00}{11}\binom{1}{\$}$.*

Thanks to Theorem 25, we can classify any word as isometric or non-isometric. In the following, several examples are given. The following are two examples of words with a 2-tilde-error overlap with adjacent errors; the first one is tilde-isometric, the second one is tilde-non-isometric.

▶ **Example 26.** The word $f = 010110000$ is tilde-isometric; indeed its unique 2-tilde-error overlap has shift 5 and length 4, $\binom{\$0}{10}\binom{101}{000}\binom{1}{\$}$. Note this is the case of non-adjacent errors but of the type prohibited by the condition in (C1).

▶ **Example 27.** The word $f = 1011000$ is tilde-non-isometric; indeed it has the 2-tilde-error overlap, of shift 4 and length 3, $\binom{\$}{1}\binom{101}{000}\binom{1}{\$}$, that satisfies (C1). Note that the pair $(u, v) = (10110011000, 10101001000)$ is a pair of tilde-witnesses for f.

The following example shows an infinite family of words that are tilde-isometric and Ham-non-isometric.

▶ **Example 28.** All the words $f = 1^n0^m$ (and their complement $f = 0^n1^m$) for $n, m > 2$ are Ham-non isometric, by Proposition 5, and tilde-isometric. In fact, for $n, m > 2$, $f = 1^n0^m$ has only two 2-tilde-error overlaps and none of them fall into a case in the statement of Theorem 25. The first one is the tilde-error overlap with shift 2, $\binom{\$1^{n-2}}{11^{n-2}}\binom{11}{00}\binom{0^{n-2}0}{0^{n-2}\$}$, the other one has shift $n + m - 2$, and it is $\binom{\$}{0}\binom{11}{00}\binom{1}{\$}$.

Instead, if $n = m = 2$, $f = 1100$ is both tilde-non-isometric and Ham-non-isometric. In fact, f has only one 2-tilde-error overlap $\binom{\$}{1}\binom{11}{00}\binom{0}{\$}$, corresponding to case $(C5)$ of Theorem 25. Moreover, one can verify that $(u, v) = (110100, 101010)$ is a pair of tilde-witnesses for f.

The next corollary shows that there exist Ham-isometric words that are not tilde-isometric. This fact, together with Example 5, proves that the families of Ham-isometric and tilde-isometric words are incomparable.

▶ **Corollary 29.** *The word $f = 1010$ is Ham isometric and tilde non-isometric.*

Proof. The word 1010 has no 2-error overlap, therefore is isometric. Instead, it has a 2-tilde-error overlap $\binom{1}{\$}\binom{010}{101}\binom{\$}{0}$ corresponding to case $(C3)$ of Theorem 25. ◀

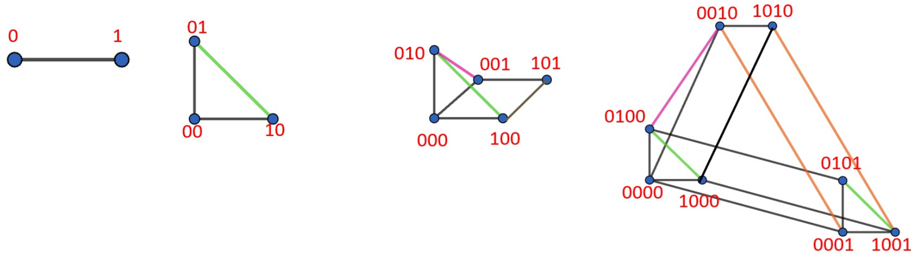

Figure 3 Tilde-Fibonacci cubes with n=1,2,3,4 (the colored edges are those added to the traditional hypercube).

6 Tilde-Fibonacci Cubes

Theorem 25 allows also to give light to isometric subgraphs of $\tilde{Q}_n$ avoiding a word. In fact they are those avoiding a tilde-isometric word.

In analogy with the definition of Fibonacci cubes introduced by Hsu [21], we give the following:

▶ **Definition 30.** *Let $n \geq 1$ and $\tilde{Q}_n$ be the tilde-hypercube of dimension n. The n-tilde-Fibonacci cube, or tilde-Fibonacci cube of dimension n is $\tilde{F}_n = \tilde{Q}_n(11)$.*

Figure 3 shows the tilde-Fibonacci cube of order $1, 2, 3, 4$.

From Theorem 25 and Proposition 19, the following corollary derives.

▶ **Corollary 31.** *The tilde-Fibonacci cube $\tilde{F}_n = \tilde{Q}_n(11)$ is an isometric subgraph of $\tilde{Q}_n$.*

Further, about other hypercubes avoiding words of length 2, we have the following:

▶ Remark 32. For each $n \geq 1$, $\tilde{Q}_n(10)$ and $Q_n(10)$ coincide. In fact, $V(\tilde{Q}_n(10)) = \{0^h 1^k |\ h, k \geq 0,\ \ h + k = n\} = V(Q_n(10))$ and $E(\tilde{Q}_n(10)) = \{(0^i 1^j, 0^{i-1} 1^{j+1}) |\ 1 \leq i \leq n,\ 0 \leq j \leq n-1,\ i + j = n\} = E(Q_n(10))$.

By complement, also $\tilde{Q}_n(01)$ and $Q_n(01)$ coincide (see Remark 22).

Note also that $\tilde{Q}_n(01)$ is obtained from $\tilde{Q}_n(10)$ by complementing all the bits in the vertices labels, i.e. they are isomorphic.

The tilde Fibonacci cube admits also a recursive construction that allows one to enumerate its edges.

By Proposition 2, $|V(\tilde{F}_n)| = |V(F_n)| = f_{n+2}$. Among these vertices, f_{n+1} end with a 0 and f_n end with a 1. Figure 3 shows the tilde-Fibonacci cube of order 4.

▶ Remark 33. Let $u \in V(F_{n-1})$, $x \in \Sigma$. If u ends with 1, then $ux \in V(\tilde{F}_n)$ iff $x = 0$. If u ends with 0 then $ux \in V(\tilde{F}_n)$, for any $x \in \{0, 1\}$.

▶ **Proposition 34.** *The n-tilde-Fibonacci cubes $\tilde{F}_n$, with $n \geq 1$, can be defined recursively.*

Proof. If $n = 1$, $\tilde{F}_1$ has two vertices 0 and 1 connected by an edge. If $n = 2$, $\tilde{F}_2$ has three vertices 00, 01 and 10 and $E(\tilde{F}_2) = \{(00, 10), (00, 01), (01, 10)\}$.

Let $n \geq 3$ and suppose that $\tilde{F}_i$ are defined for all $i < n$. Then, $\tilde{F}_n$ can be constructed from a copy of $\tilde{F}_{n-1}$ where each vertex u is replaced by $u0$, denoted by $\tilde{F}_{n-1}0$, and a copy of $\tilde{F}_{n-2}$, where each vertex v is replaced by $v01$, and denoted by $\tilde{F}_{n-2}01$. If there is an edge (u, v) in $\tilde{F}_{n-1}$, then there is en edge $(u0, v0)$ $\tilde{F}_n$, i.e. $\tilde{F}_{n-1}0$ is a subgraph of $\tilde{F}_n$. For similar reasons $\tilde{F}_{n-2}01$ is a subgraph of $\tilde{F}_n$. Further, for any $u0 \in V(\tilde{F}_{n-1})$, then there is an edge in

$\tilde{F}_n$ connecting $v00$ in $\tilde{F}_{n-1}0$ and $u01$ in $\tilde{F}_{n-2}01$ and for any $u1 \in V(\tilde{F}_{n-1})$ there is an edge linking $u10$ in $\tilde{F}_{n-1}0$ and $u01$ in $\tilde{F}_{n-2}01$ (see the orange edges in Fig. 3). By Remark 33 and Lemma 15 no further edges exist in $\tilde{F}_n$. ◀

▶ **Corollary 35.** *Let $\tilde{F}_n$ be the n-tilde-Fibonacci cube. Then, $|E(\tilde{F}_1)| = 1, |E(\tilde{F}_2)| = 3$ and, for any $n \geq 2$*

$$|E(\tilde{F}_n)| = |E(\tilde{F}_{n-1})| + |E(\tilde{F}_{n-2})| + f_{n+1}.$$

Hence, we can give the following exact formula:

$$|E(\tilde{F}_n)| = \frac{(n+1)f_{n+3} + (n-2)f_{n+1}}{5},$$

(Sequence A023610 in [31] for $|E(\tilde{F}_{n+1}|)$. Since the number of vertices of $\tilde{F}_n$ is f_{n+2}, from the previous formula it follows that the tilde-Fibonacci cube has $O(N \log N)$ edges, where N is the number of vertices, as for the tilde-hypercube (see Equation (1)). Let us conclude the section with some structural properties of tilde-Fibonacci cubes such as the diameter and the radius.

The *eccentricity* of a vertex v of a connected graph G is defined as

$$e(v) = \max_{u \in V(G)} d_G(u, v),$$

where $d_G(u, v)$ is the length of a shortest path from u to v in G. The *diameter* and the *radius* of G are respectively defined by

$$d(G) = \max_{v \in V(G)} e(v) \quad \text{and} \quad r(G) = \min_{v \in V(G)} e(v).$$

In [21] it is proven that $d(F_n) = n$ and that the maximal distance involves the words $(10)^{n/2}$ and $(01)^{n/2}$ for even n, and $(01)^{\lfloor n/2 \rfloor}0$ and $(10)^{\lfloor n/2 \rfloor}1$ for odd n. Note that, since the swap operation adds new edges, it shortens the distances between vertices. Moreover, the distances can even be halved because a swap replaces two replacement operations. More precisely, we have the following proposition.

▶ **Proposition 36.** *Let $\tilde{F}_n$ be the n-tilde-Fibonacci cube. Then, for any $n \geq 1$, $d(\tilde{F}_n) = r(\tilde{F}_n) = \lceil n/2 \rceil$.*

Proof. First, we prove that for any $n \geq 1$ and $u, v \in V(\tilde{F}_n)$, $dist_\sim(u, v) \leq \lceil n/2 \rceil$ by induction on n.

The case where $n = 1$ is trivial. If $n = 2$, then for any $u, v \neq 11$, we have $\text{dist}_\sim(u, v) = 1$. Let now u, v be 11-free words of same length n, with $n > 2$. Then, $u = xyu'$ and $v = ztv'$, with $x, y, z, t \in \{0, 1\}$ and $u', v' \in \{0, 1\}^*$. Recall that $\text{dist}_\sim(u, v)$ is the minimal number of swaps and replacements to transform u into v. Since xy can be transformed into zt with at most a single swap or replacement, a possible way to transform u in v is to transform xy into zt and then u' into v'. Therefore, $\text{dist}_\sim(u, v) \leq 1 + \text{dist}_\sim(u', v') \leq 1 + \lceil (n-2)/2 \rceil \leq \lceil n/2 \rceil$.

Moreover, for each 11-free word u of length n, with $n \geq 1$, there exists a 11-free word v of length n such that $dist_\sim(u, v) = \lceil n/2 \rceil$. In fact, for each word u of length n, the word v is obtained by replacing in u, from left to right, the blocks 00 with 10, the blocks 10 with 00, the blocks 01 with 10 and finally, for odd n, the last bit with its complement. Trivially, if u is 11-free then v is 11-free, as well. We prove that $\text{dist}_\sim(u, v) = \lceil n/2 \rceil$ by induction on n. The case $n = 1$ is trivial. If $n = 2$ then $\text{dist}_\sim(01, 10) = \text{dist}_\sim(10, 00) = \text{dist}_\sim(00, 10) = 1$. If

$n > 2$ then either $u = 00u'$ ($v = 10v'$) or $u = 01u'$ ($v = 10v'$). In the first case, by Remark 11, one has $dist_\sim(u, v) = 1 + \lceil (n-2)/2 \rceil = \lceil n/2 \rceil$. In the second case, by Remark 12, one has $dist_\sim(u, v) = 1 + \lceil (n-2)/2 \rceil = \lceil n/2 \rceil$. In any case, $dist_\sim(u, v) = \lceil n/2 \rceil$. Finally, since $\tilde{F}_n$ is an isometric subgraph of $\tilde{Q}_n$ then $d_{\tilde{F}_n}(u, v) = dist_\sim(u, v)$ and $r(\tilde{F}_n) = d(\tilde{F}_n) = \lceil n/2 \rceil$. ◀

The previous proposition proves that $\tilde{F}_n$ is a *self-centered* graph, that is, a graph in which radius and diameter coincide (cf. [14] for a survey).

7 Conclusions

The paper surveys some recent results on isometric words with respect to the edit distance that allows swap and replacement operations, here referred to as tilde-distance. Moreover, the tilde-hypercube and the tilde-Fibonacci cube are presented as a generalization of the corresponding classical notions, with the tilde-distance in place of the Hamming distance.

Compared with the setting of Hamming distance, all the problems appear to be more complicated since, when using swaps, the order of performing the operations does matter. Nevertheless, isometric words and generalized Fibonacci cubes based on this tilde-distance open up new scenarios and present interesting new situations that surely deserve further investigation as it can serve as base for string and graph algorithmic developments.

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