

Design of Distance Tariffs in Public Transport

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Abstract

Setting the ticket prices is a crucial decision in public transport. Its basis, relevant for all related questions, such as dynamic prices or prices for different passenger groups, is the underlying fare strategy. Popular fare strategies are based on zones or on distances. Transitions from one fare strategy to another occur frequently, e.g., if public transport operators are joined to a larger association, or if structural decisions in a region have taken place.

In this paper we report practically relevant issues when a fare structure should be changed to a distance tariff, a problem frequently arising when a ticket system based on mobile devices is introduced. We present mixed-integer linear programs for finding the parameters of a distance tariff, analyze rounding properties, and reflect how the change in revenue for the operator and the number of highly affected passengers can be controlled. Additionally, we evaluate the developed models experimentally.

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1 Introduction

The choice of ticket prices for public transport usage is a crucial decision. Ticket prices are important for covering the costs of the public transport operator. Through price elasticities (see, e.g., [9, 2, 5]), they affect the number of people traveling by public transport. They also contribute to the passenger satisfaction, and they even may affect the routes passengers choose [3, 12]. A variety of fare strategies is implemented worldwide, each with a different focus and purpose. In the following we sketch three basic types which come with many variations in practice.

Flat tariffs. In a flat tariff, every ticket has the same price. This has the advantage that it is very easy to understand, but on the other hand it is often perceived as unfair because passengers with a short journey pay the same price as passengers with a long journey.

Zone tariffs. Zone tariffs are more granular. They group stations to zones and set fares based on the traversed zones. Within each fare zone a flat tariff is applied, but traversing different zones yields different prices. It is common that the price for a journey in a zone tariff depends on the number of traversed zones, usually with a few exceptions. Such a zone



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tariff is also called *counting zone* tariff. In the past years, zone tariffs have been very popular and are implemented in many cities in Europe (e.g., Berlin, London, Paris, Copenhagen) as well as worldwide (e.g., Vancouver, Melbourne, Johannesburg). The design of and the transition to zone tariffs with practical applications has been broadly investigated in the literature, e.g., [8, 4, 13, 27].

Distance tariffs. Distance tariffs are the most differentiated fare strategy. They determine the price of a ticket based on the length of the corresponding journey. This can for example be the actual distance traveled in the network (*network distance tariff*) or the beeline (Euclidean) distance between the start and the end station of the journey (*beeline distance tariff*, also called airline distance tariff). In this paper, we consider affine distance tariffs, which are composed of a price per kilometer and an additional base amount. Distance tariffs are the standard system in most long-distance (railway) transportation systems and get nowadays more popular also in regional bus transportation. There are several reasons for this: First of all, charging a price which is proportional to the beeline distance seems to be fair in an area where rivers and mountains do not impose barriers for traveling. Furthermore, the rising popularity of mobile tickets and smart cards has led to an increased interest in distance tariffs because check-in/check-out systems can rather easily be used to determine the length of a journey. A particular promising setting is that passengers use mobile devices that track the coordinates and compute the distance traveled when the destination is reached. This is easy to handle for people that use public transport only occasionally and therefore in Germany an alternative to the so-called *Deutschlandticket* that offers unlimited access to all urban and regional public transport systems for frequent users.

The transition to a distance tariff has been regarded mainly with respect to demand change. [12] simulates route choice effects when changing from a zone tariff to a distance tariff and evaluates travel times as well as amount of fares paid. In [28], a bilevel approach maximizing the demand under the consideration of route choice based on cost and time components is discussed. A different kind of distance tariff that takes the number of stations into account instead of a kilometer distance is considered in [14, 15, 10].

In this paper, we consider a different topic than previous papers. We develop mathematical models for the design of a distance tariff focusing on practice-oriented questions arising. The models are based on standard data available for ticket sales, and the formulations presented can be implemented directly. Therefore, this paper provides the means to generate relevant tariff setting and make informed decisions.

The design of an affine distance tariff seems to be an easy exercise: one only has to determine the price per kilometer and a fixed base amount. Nevertheless, when we discussed the introduction of a distance tariff with partners from public transport operators, we learned of several issues to be considered. In this paper we share our findings that occurred during such projects by presenting how such practical requirements can be modeled.

The remainder of the paper is structured as follows. We present the general model for the transition to a distance tariff in Section 2. We then discuss the modeling issues that stem from our partners within a (confidential) real-world case study: Distance tariffs should be integral (Section 3). In most real cases there exists an upper limit as a cap on the ticket price. This upper limit is a third variable in the model as shown in Section 4. Public transport operators wish to control the deviation in revenues (Section 5.1) and are also interested in the passengers' perspective, which means that there should be a bound on the number of highly affected passengers as introduced in Section 5.2. An experimental evaluation of the developed models is presented in Section 6. We conclude in Section 7.

2 A model for the transition to a distance tariff

Informally speaking, a fare structure assigns a ticket price to every passengers' journey. Hence, in order to define formally what a fare structure and a distance tariff are, we first need the network in which the passengers travel.

Let us define the *public transport network* PTN as a graph $PTN = (V, E)$ with V being the stations and E being the links between stations on which a regular service exists. A *passengers' journey* is a path in the PTN. In order to reflect all possible passengers journeys through the network, we define $\mathcal{J}$ as the set of all paths in the PTN.

A *fare structure* assigns a ticket price to every path in the PTN, realizing a fare strategy, which may be a distance tariff, a zone tariff or a flat tariff. This paper deals with models to determine distance tariffs.

Given an *org-dest-path* J in the PTN, we consider two common options to assign a distance (or length) $l(J)$ to path J :

Network distance: In this case, $l(J)$ is the length of path J in the PTN.

Beeline distance: In this case, $l(J)$ is the beeline distance between the origin and the destination of path J . The path J itself is not needed.

We assume that that all paths have positive lengths and that all distances are rounded up to full kilometers. The latter means that a distance tariff as described in the following requires passengers to pay the price per kilometer for every kilometer started on their journey. In small regions of operation a smaller unit might be used to round to. Note that, while network and beeline distance are the most common options, any other way to determine the length of a path $l(J) \in \mathbb{N}$ can be employed as well, where $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers starting from 1.

► **Notation 1.** Let $\mathcal{J}$ be the set of all paths in the PTN. A fare structure $\pi : \mathcal{J} \rightarrow \mathbb{R}$ is a distance tariff if the price for every path $J \in \mathcal{J}$ is defined as

$$\pi(J) := p \cdot l(J) + f,$$

where the two parameters p and f describe the price per kilometer and the base amount of the distance tariff, and $l(J)$ is the length of path J .

We denote a distance tariff π with price per kilometer p and base amount f by (p, f) . The corresponding price list of π is given as $(\pi_i^{\text{km}})_{i \in \mathbb{N}}$ with $\pi_i^{\text{km}} := p \cdot i + f$ for all $i \in \mathbb{N}$.

The goal of this paper is to develop models for the transition from an existing tariff to a distance tariff as defined in Notation 1. Since neither the operator would like to have a (big) loss in its income nor the passengers would like to have a (large) increase in their ticket prices, the idea is to design the distance tariff such that the new prices are as close as possible to the current prices. This leads to the concept of *reference prices*, which may be the current ticket prices or other preferable prices that should be realized. The goal then is to minimize the deviation to the reference prices. This objective has been introduced in [7], and is followed in many other publications such as [1, 8, 14, 15, 4].

In order to model the deviations, let a set of *passenger groups* $\mathcal{D}$ be given. For each group $d \in \mathcal{D}$ let $w_d \in \mathbb{N}$ be the number of passengers of group d , J_d the path the passengers of group d wish to travel, $l_d \in \mathbb{N}$ its length, and $r_d \in \mathbb{R}_{\geq 0}$ the reference price.

- The set $\mathcal{D}$ of passenger groups can for example be a set of origin-destination (OD) pairs $\mathcal{D} \subseteq V \times V$, which means to consider at most one passenger group (and hence one path) between each pair of an origin station and a destination station. We also can allow different passenger groups within the same OD pair, representing different paths between the same origin and destination.

11:4 Design of Distance Tariffs in Public Transport

- The distance l_d can be derived as network or beeline distance of path J_d of passenger group d , i.e., $l_d = l(J_d) > 0$. Note that all paths with the same pair of origin and destination station are assigned the same distance l_d if the beeline distance is applied, whereas the network distance may lead to passenger groups with the same pair of origin and destination station but with different distances because their paths differ.
- As the reference price r_d , we apply the current price to be paid by passenger group $d \in \mathcal{D}$ for traveling along path J_d .
- Note that two passenger groups d_1, d_2 with $l_{d_1} = l_{d_2}, r_{d_1} = r_{d_2}$ can be joined to a new passenger group d with $l_d = l_{d_1} = l_{d_2}, r_d = r_{d_1} = r_{d_2}$ and $w_d = w_{d_1} + w_{d_2}$. We hence may assume that passenger groups are pairwise disjoint regarding distance or reference price.

Finally, the new price of the passenger group d is denoted by π_d . If it is determined by a distance tariff (p, f) , we have $\pi_d = p \cdot l_d + f$. The vector $(\pi_d)_{d \in \mathcal{D}}$ contains all the new prices.

The basic *distance tariff design* model for a transition to a distance tariff can now be stated. It looks for a distance tariff π with price per kilometer p and base amount f , requiring $p, f \geq 0$, and minimizing the sum of absolute deviations between the new prices $(\pi_d)_{d \in \mathcal{D}}$ and the reference prices $(r_d)_{d \in \mathcal{D}}$:

$$\begin{aligned}
 \min_{p, f, \pi_d} \quad & \sum_{d \in \mathcal{D}} w_d |r_d - \pi_d| \\
 \text{s.t.} \quad & \pi_d = p \cdot l_d + f \quad \text{for all } d \in \mathcal{D} \\
 & p, f \geq 0.
 \end{aligned} \tag{1}$$

The variables π_d with $d \in \mathcal{D}$ can be replaced such that the program consists of only two variables, p and f , and the absolute value can be replaced by a bottleneck variable z_d for each $d \in \mathcal{D}$ such that we equivalently obtain its linear version:

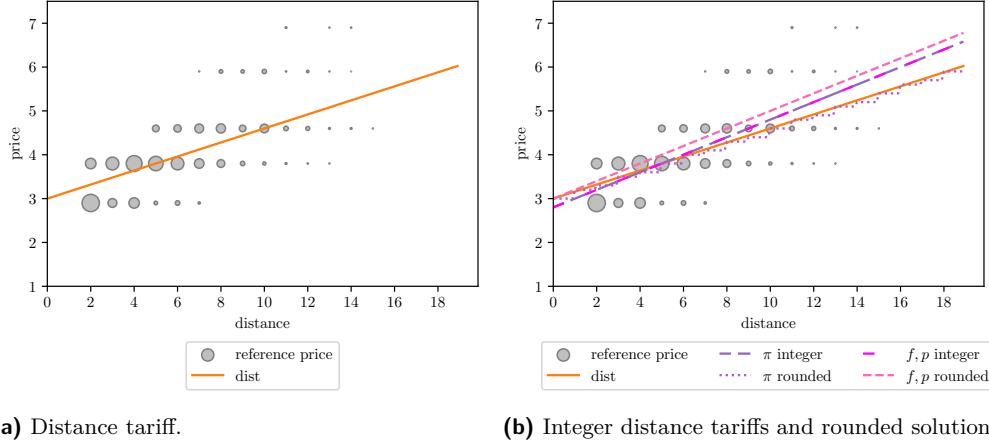
$$\begin{aligned}
 \min_{p, f, z_d} \quad & \sum_{d \in \mathcal{D}} w_d z_d \\
 \text{s.t.} \quad & r_d - p \cdot l_d - f \leq z_d \quad \text{for all } d \in \mathcal{D} \\
 & p \cdot l_d + f - r_d \leq z_d \quad \text{for all } d \in \mathcal{D} \\
 & p, f \geq 0.
 \end{aligned} \tag{2}$$

► **Example 2.** To illustrate the different tariff types considered in this paper, we consider an example derived from the `sioux_falls` data set, see Figure 4, of the software library LinTim [18, 19]. To generate passenger groups, we compute the network distance of each origin destination pair and a reference price based on the zone tariff depicted in Figure 4a. As noted above, we combine passenger groups which share the same distance and reference price. Note that this example is part of the experimental evaluation in Section 6 (where it is denoted as data set A with `ratio_zone = 1`).

In the graphical representations, e.g. in Figure 1a, each passenger group is marked as a point (l_d, r_d) showing its distance (rounded to full kilometers) and reference price. The size of the marker depends on the number of passengers w_d of the group. The fare structure π is plotted as a function of the distance l , e.g., in Figure 1a, the orange line $l \mapsto p \cdot l + f$ represents the distance tariff. The price list $\pi_i^{\text{km}} := p \cdot i + f$ can be read off as the function values at $i \in \mathbb{N}$.

We illustrate an optimized distance tariff for Example 2 in Figure 1a.

Recall that $l_d > 0$ for all $d \in \mathcal{D}$. We can add additional bounds to model (1):



■ **Figure 1** Example data from Example 2 with optimal solutions. Each passenger group is marked as a point (l_d, r_d) showing its distance (rounded to full kilometers) and reference price. The size of the marker depends on the number of passengers w_d of the group. The optimized fare structures are plotted as functions of the distance.

► **Lemma 3.** Let $r_{\max} := \max_{d \in \mathcal{D}} r_d$. Every optimal solution (p, f) to model (1) satisfies $p \leq \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$ and $f \leq r_{\max}$. In particular, these are valid inequalities for model (1).

Proof. Let (p, f) be an optimal distance tariff. First assume that $f > r_{\max}$. This means that $\pi_d = p \cdot l_d + f > p \cdot l_d + r_{\max} \geq p \cdot l_d + r_d \geq r_d$ for all $d \in \mathcal{D}$. Decreasing f to $r_{\max}$ hence decreases the objective function value, which is a contradiction to (p, f) being an optimal distance tariff.

Second assume that $p > \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$. This yields

$$\pi_d = p \cdot l_d + f > \max_{d' \in \mathcal{D}} \frac{r_{d'}}{l_{d'}} \cdot l_d + f \geq \frac{r_d}{l_d} \cdot l_d + f = r_d + f \geq r_d$$

for all $d \in \mathcal{D}$. Decreasing p to $\max_{d \in \mathcal{D}} \frac{r_d}{l_d}$ hence decreases the objective function value, again a contradiction to (p, f) being an optimal distance tariff. ◀

Note that instead of the sum of absolute deviations $\sum_{d \in \mathcal{D}} w_d |r_d - \pi_d|$ as in (1) also the maximum absolute deviation $\max_{d \in \mathcal{D}} |r_d - \pi_d|$ or the sum of squared deviations $\sum_{d \in \mathcal{D}} w_d (r_d - \pi_d)^2$ may be considered as objective functions [21, 8, 1, 14, 15, 4]. Minimizing the maximum absolute deviation is strongly dependent on outliers, and hence does not lead to good results in practice. Also, while minimizing the sum of squared deviations is easy to solve by a simple regression analysis, it is not what practitioners want; it is still more dependent on outliers than minimizing the sum of absolute deviations. Another reason that makes minimizing the sum of absolute deviations attractive for practical settings is that optimization problem (1) satisfies the following two properties (which are not satisfied if the sum of absolute deviations is replaced by the maximum absolute deviation or by the the sum of squared deviations):

(P1) There is always an optimal solution with a passenger group $d \in \mathcal{D}$ for which the reference price r_d and the new price π_d coincide, i.e., $r_d = \pi_d$. If $p \neq 0$ and $f \neq 0$, then there are even two passenger groups with this property [26]. This means that there always exist two passenger groups $d \in \mathcal{D}$ whose ticket prices stay the same when the old fare structure is transformed to a distance tariff.

(P2) As we show in Lemma 4, with a few exceptions, the new ticket price is larger than the reference price for at most half of the passengers and it is smaller for at most half of the passengers, i.e., it may be considered as balanced. Consequently, at most half of the passengers experience increasing ticket prices when the fare structure is transformed to a distance tariff.

Moreover, some detailed analysis [26] shows that the optimization problem (1) can be solved in linear time in the number of passenger groups.

► **Lemma 4 (Halving property).** *Let (p, f) be an optimal distance tariff, i.e., an optimal solution to model (1). Then one of the following is true:*

1. $\sum_{d \in D: r_d < p \cdot l_d + f} w_d \leq \frac{\sum_{d \in D} w_d}{2}$ and $\sum_{d \in D: r_d > p \cdot l_d + f} w_d \leq \frac{\sum_{d \in D} w_d}{2}$.
2. $f = 0$ and $\sum_{d \in D: r_d < p \cdot l_d + f} w_d > \frac{\sum_{d \in D} w_d}{2}$.

Proof. We apply the argument from [22, 23]. Assume $\sum_{d \in D: r_d > p \cdot l_d + f} w_d > \frac{\sum_{d \in D} w_d}{2}$. Then we can increase the base amount f by some $h \in \mathbb{R}_{>0}$ so that

$$\{d \in D : r_d > p \cdot l_d + f\} = \{d \in D : r_d > p \cdot l_d + (f + h)\}.$$

This however improves the objective function value because

$$\begin{aligned} & \sum_{d \in D} w_d |r_d - (p \cdot l_d + f + h)| \\ &= \sum_{d \in D: r_d > p \cdot l_d + f} w_d (r_d - p \cdot l_d - f - h) + \sum_{d \in D: r_d < p \cdot l_d + f} w_d (p \cdot l_d + f + h - r_d) \\ &= \sum_{d \in D} w_d |r_d - (p \cdot l_d + f)| + \underbrace{\underbrace{h}_{>0} \cdot \left(- \sum_{d \in D: r_d > p \cdot l_d + f} w_d + \sum_{d \in D: r_d < p \cdot l_d + f} w_d \right)}_{<0} \\ &< \sum_{d \in D} w_d |r_d - (p \cdot l_d + f)|. \end{aligned}$$

Hence, $(p, f + h)$ is a better solution, which is a contradiction to (p, f) being optimal.

The same argument works for the case that $\sum_{d \in D: r_d < p \cdot l_d + f} w_d > \frac{\sum_{d \in D} w_d}{2}$ with $h \in \mathbb{R}$ and $-f \leq h < 0$ as long as $f > 0$. However, if $f = 0$, it is not allowed to reduce the base amount f . This is the only situation in which it can be optimal that more than half of the passengers have a higher new ticket price than their (old) reference price. ◀

3 Integrality constraints

A common practical requirement is to have integer values for the resulting ticket prices. Note that by scaling the reference prices, we can ensure integer multiples of 10ct, 50ct, etc. Here, we not only want to ensure this for the ticket prices π_d of the actual set of passenger groups $d \in \mathcal{D}$, but also for all potential journeys, i.e., for all paths $J \in \mathcal{J}$ in the PTN. This can be guaranteed by requiring the integrality condition for the whole price list, i.e.,

$$\pi_i^{\text{km}} \in \mathbb{N}_0 \text{ for each positive kilometer distance } i \in \mathbb{N}.$$

These conditions may be added to the basis formulation (1):

$$\begin{aligned}
& \min_{p, f, \pi_d} \quad \sum_{d \in \mathcal{D}} w_d |r_d - \pi_d| \\
& \text{s.t.} \quad \pi_d = p \cdot l_d + f \quad \text{for all } d \in \mathcal{D} \\
& \quad \quad y_i = p \cdot i + f \quad \text{for all } i \in \mathbb{N} \\
& \quad \quad p, f \geq 0 \quad \text{for all } d \in \mathcal{D} \\
& \quad \quad y_i \in \mathbb{Z} \quad \text{for all } i \in \mathbb{N}.
\end{aligned} \tag{3}$$

We may replace $i \in \mathbb{N}$ by $i = \{1, 2, \dots, i_{\max}\}$ for some sufficiently large $i_{\max}$, but it is computationally not advantageous to have so many integer variables. This number can be reduced since it is in fact *equivalent* to require integrality only for p and f as the next lemma shows.

► **Lemma 5.** $p, f \in \mathbb{Z}$ if and only if $\pi_i^{\text{km}} \in \mathbb{Z}$ for all $i \in \mathbb{N}$.

Proof. If p and f are integer, then clearly $\pi_i^{\text{km}} = p \cdot i + f$ is integer for all natural numbers $i \in \mathbb{N}$. Vice versa, let $p \cdot i + f$ be integer for all $i \in \mathbb{N}$. For $i = 1$ and $i = 2$ we receive that $z_1 := p + f$ and $z_2 := 2p + f$ are both integer. Hence also $z_2 - z_1 = p$ is integer, and from this we get that $z_1 - p = f$ is also integer. ◀

The consequence is that (3) is equivalent to

$$\begin{aligned}
& \min_{p, f, \pi_d} \quad \sum_{d \in \mathcal{D}} w_d |r_d - \pi_d| \\
& \text{s.t.} \quad \pi_d = p \cdot l_d + f \quad \text{for all } d \in \mathcal{D} \\
& \quad \quad p, f \geq 0 \\
& \quad \quad p, f \in \mathbb{Z},
\end{aligned} \tag{4}$$

in which we only have two integer variables f and p instead of the $i_{\max}$ many variables $y_1, \dots, y_{i_{\max}}$. Clearly, the program can again be linearized as already seen for (2).

► **Corollary 6.** *The formulations (3) and (4) are equivalent.*

Two heuristic solutions that we have tested are the following:

f, p rounded As a heuristic we can also solve the program (1) without integrality constraints and round the optimal variables f^* and p^* to their closest integers. However, this is only a heuristic, although the rounding property of [11] is often satisfied for location or regression-like problems (see [24]), which have a similar structure as (1).

π rounded Another option is to round the prices π_d^* to the closest integer. Note however, that this does not result in a distance tariff since the resulting values $\text{round}(\pi_d^*)$ will not satisfy $\text{round}(\pi_d^*) = p \cdot l_d + f$ any more.

Both optimal integer solutions and the two heuristics are illustrated in Figure 1b.

4 Setting an upper price cap on the ticket prices

Implementing a distance tariff as in Notation 1, the ticket price increases unlimitedly with the distance. In practice, public transport tariffs often have a cap on the maximum ticket price per journey. Consider a path $J \in \mathcal{J}$. As long as the length of J is small, the price

11:8 Design of Distance Tariffs in Public Transport

function is an affine linear distance tariff (p, f) that becomes constant if the length of the path exceeds a threshold distance $L_{\max}$. Formally, let the price cap be $p_{\max}$. Then the price $\pi(l)$ for a path with length l is defined as a continuous piecewise linear function in l :

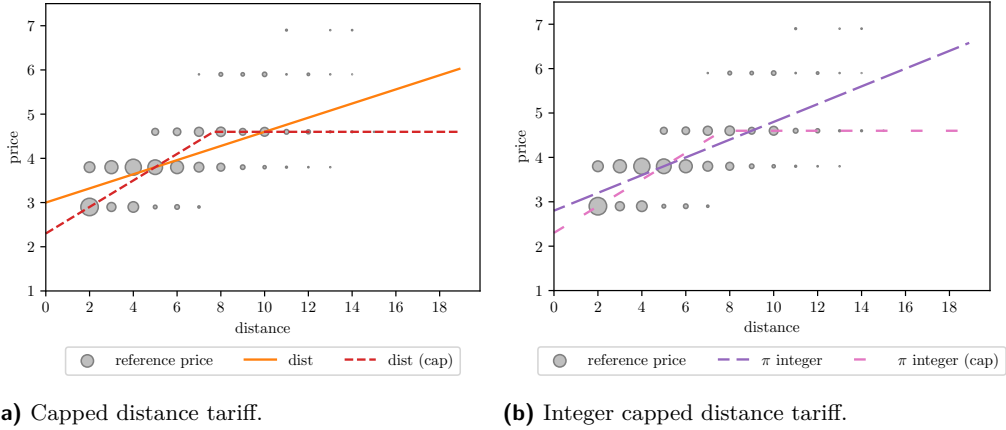
$$\pi(l) = \min\{p \cdot l + f, p_{\max}\}$$

which results in a threshold distance $L_{\max}$ for which $p \cdot L_{\max} + f = p_{\max}$, i.e., the price stays at $p_{\max}$ for all passengers traveling further than $L_{\max}$. In the following we investigate distance tariffs with such a cap on the ticket prices.

► **Notation 7.** Let $\mathcal{J}$ be the set of all paths in the PTN. A fare structure $\pi : \mathcal{J} \rightarrow \mathbb{R}$ is a capped distance tariff if the price for every path $J \in \mathcal{J}$ is defined as

$$\pi(J) := \min\{p \cdot l(J) + f, p_{\max}\},$$

where the parameters p and f describe the price per kilometer and the base amount of the distance tariff, and $p_{\max}$ is the cap on the ticket price. As before, $l(J)$ is the length of path J , measured by beeline or network distance. We denote a capped distance tariff π with price per kilometer p , base amount f and upper price limit $p_{\max}$ by $(p, f, p_{\max})$. If $p \neq 0$, its threshold distance $L_{\max}$ is given as $L_{\max} = \frac{p_{\max} - f}{p}$.



■ **Figure 2** Example data from Example 2 with optimal solutions. Each passenger group is marked as a point (l_d, r_d) showing its distance (rounded to full kilometers) and reference price. The size of the marker depends on the number of passengers w_d of the group. The optimized fare structures are plotted as functions of the distance.

Figure 2 shows an example of input data with an optimized capped distance tariff analogously to Figure 1. The horizontal part is the result of the upper price limit $p_{\max}$.

We can compute p , f and $p_{\max}$ with the following model:

$$\begin{aligned}
 \min_{p, f, p_{\max}, x_d^{\max}, \pi_d} \quad & \sum_{d \in \mathcal{D}} w_d |r_d - \pi_d| \\
 \text{s.t.} \quad & \pi_d \leq p \cdot l_d + f && \text{for all } d \in \mathcal{D} \\
 & \pi_d \leq p_{\max} && \text{for all } d \in \mathcal{D} \\
 & \pi_d \geq p \cdot l_d + f - M \cdot x_d^{\max} && \text{for all } d \in \mathcal{D} \\
 & \pi_d \geq p_{\max} - M \cdot (1 - x_d^{\max}) && \text{for all } d \in \mathcal{D} \\
 & p_{\max} \geq p \cdot l_d + f - M \cdot x_d^{\max} && \text{for all } d \in \mathcal{D} \\
 & p_{\max} \leq p \cdot l_d + f + M \cdot (1 - x_d^{\max}) && \text{for all } d \in \mathcal{D} \\
 & p, f, p_{\max} \geq 0 \\
 & x_d^{\max} \in \{0, 1\} && \text{for all } d \in \mathcal{D}.
 \end{aligned} \tag{5}$$

The additional variable $p_{\max}$ denotes the maximum price of the capped distance tariff, and the binary variables $x_d^{\max}$ indicate whether a passenger group $d \in \mathcal{D}$ has to pay the maximum price, or not. The first two constraints ensure that π_d is always smaller than the minimum of the affine distance tariff and the upper price limit. The next two constraints ensure that π_d is not smaller, but equal to $p \cdot l_d + f$ if $x_d^{\max} = 0$ and equal to $p_{\max}$ if $x_d^{\max} = 1$. Finally, the last two constraints ensure the relation between $p_{\max}$ and $p \cdot l_d + f$: If $x_d^{\max} = 0$, the constant part $p_{\max}$ must be larger than the affine part $p \cdot l_d + f$, and vice versa if $x_d^{\max} = 1$.

In the following we first bound the optimal value for $p_{\max}$. This then enables us to determine a reasonable constant for the big M parameter in model (5). Recall that $r_{\max} = \max_{d \in \mathcal{D}} r_d$.

► **Lemma 8.** *There is always an optimal solution $(p, f, p_{\max})$ to model (5) with $p \leq \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$ and $f \leq p_{\max} \leq r_{\max}$.*

Proof. Let $(p, f, p_{\max})$ denote an optimal capped distance tariff.

First, assume that $f > p_{\max}$. We then have $p_{\max} < p \cdot l_d + f$ for all $d \in \mathcal{D}$. Therefore, we can replace $(p, f, p_{\max})$ by $(0, p_{\max}, p_{\max})$, which yields the same optimal objective function value and is hence also an optimal solution. For the following let $f \leq p_{\max}$.

Second, assume that $p_{\max} > r_{\max}$. We replace $(p, f, p_{\max})$ by $(p, \min\{f, r_{\max}\}, r_{\max})$. For all passenger groups $d \in \mathcal{D}$ with $p \cdot l_d + f \leq r_{\max}$ (which only can occur if $f \leq r_{\max}$), the contribution to the objective function value does not change. For the remaining passenger groups, we have that $\min\{p \cdot l_d + f, p_{\max}\} > r_{\max} \geq r_d$ and hence obtain a reduction in the objective function value, which is a contradiction to $(p, f, p_{\max})$ being optimal. This yields that there is an optimal solution with $p_{\max} \leq r_{\max}$ and hence also with $f \leq r_{\max}$.

Third, assume that $p > \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$. We consider replacing $(p, f, p_{\max})$ by $(\max_{d \in \mathcal{D}} \frac{r_d}{l_d}, f, p_{\max})$. For all passenger groups $d \in \mathcal{D}$ with $p_{\max} \leq \max_{d' \in \mathcal{D}} \frac{r_{d'}}{l_{d'}} \cdot l_d + f < p \cdot l_d + f$ the contribution to the objective function value does not change because the price is determined by $p_{\max}$. For the remaining passenger groups, we have that $p_{\max} > \max_{d' \in \mathcal{D}} \frac{r_{d'}}{l_{d'}} \cdot l_d + f$. Therefore,

$$\begin{aligned}
 \min\{p \cdot l_d + f, p_{\max}\} &> \max_{d' \in \mathcal{D}} \frac{r_{d'}}{l_{d'}} \cdot l_d + f \quad \left(= \min \left\{ \max_{d' \in \mathcal{D}} \frac{r_{d'}}{l_{d'}} \cdot l_d + f, p_{\max} \right\} \right) \\
 &\geq \frac{r_d}{l_d} \cdot l_d + f = r_d + f \geq r_d.
 \end{aligned}$$

Thus, in this case the objective function value of $(\max_{d \in \mathcal{D}} \frac{r_d}{l_d}, f, p_{\max})$ is smaller than of $(p, f, p_{\max})$, which is a contradiction to $(p, f, p_{\max})$ being an optimal solution. This yields that there is an optimal solution with $p \leq \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$. ◀

► **Lemma 9.** *We consider model (5) with the additional constraints stated in Lemma 8. Then the parameter M can be chosen as $M := \max_{d \in \mathcal{D}} \frac{r_d}{l_d} \cdot \max_{d \in \mathcal{D}} l_d + r_{\max}$.*

Proof. Let $M := r_{\max} + \max_{d \in \mathcal{D}} \frac{r_d}{l_d} \cdot \max_{d \in \mathcal{D}} l_d$. Because of the additional constraints of Lemma 8, we have $p \leq \max_{d \in \mathcal{D}} \frac{r_d}{l_d}$ and $f \leq p_{\max} \leq r_{\max}$ in any feasible solution $(p, f, p_{\max})$. This yields $p \cdot l_d + f - M \leq 0$, $p_{\max} - M \leq 0$ and $p \cdot l_d + f + M \geq r_{\max}$. The big- M constraints in (5) are therefore redundant in case the big- M is active. ◀

The halving property as shown for model (1) in Lemma 4 does analogously hold for capped distance tariffs:

► **Lemma 10 (Halving property).** *Let $(p, f, p_{\max})$ be an optimal capped distance tariff, i.e., an optimal solution to model (5). Then one of the following is true:*

1. $\sum_{\substack{d \in \mathcal{D}: \\ r_d < \min\{p \cdot l_d + f, p_{\max}\}}} w_d \leq \frac{\sum_{d \in \mathcal{D}} w_d}{2}$ and $\sum_{\substack{d \in \mathcal{D}: \\ r_d > \min\{p \cdot l_d + f, p_{\max}\}}} w_d \leq \frac{\sum_{d \in \mathcal{D}} w_d}{2}$.
2. $f = 0$ and $\sum_{\substack{d \in \mathcal{D}: \\ r_d < \min\{p \cdot l_d + f, p_{\max}\}}} w_d > \frac{\sum_{d \in \mathcal{D}} w_d}{2}$.

Proof. The result can be shown analogously to Lemma 4. Instead of shifting the line $l \mapsto p \cdot l + f$, we shift the complete graph of $l \mapsto \min\{p \cdot l + f, p_{\max}\}$. ◀

5 Controlling the revenue and number of highly affected passengers

The objective function minimizing the sum of absolute deviations between reference prices and new prices implicitly controls the deviation of revenue and ticket price increases. However, in practice it may be required explicitly that the revenue is changed by a certain amount or that only a certain amount of passengers is affected by a high increase in the ticket price. These additional requirements are taken into consideration in Sections 5.1 and 5.2.

5.1 Controlling the changes in revenue

Under the assumption that all passengers pay the reference prices for their tickets, the revenue is given by $\sum_{d \in \mathcal{D}} w_d r_d$. When designing a new fare structure, there might be the requirement to obtain a certain revenue $R \in \mathbb{R}$. In practice the value R may be given in relation to the revenue gained by the reference prices as $R = \sum_{d \in \mathcal{D}} w_d r_d - \alpha_1$ with $\alpha_1 \in \mathbb{R}$ or as $R = \alpha_2 \cdot \sum_{d \in \mathcal{D}} w_d r_d$ with $\alpha_2 \in \mathbb{R}_{\geq 0}$. This bound can be acknowledged in the optimization process by adding the following linear constraint to the previous models (1)–(5) independent of the integrality and upper price limit specifications:

$$\sum_{d \in \mathcal{D}} w_d \pi_d \geq R. \quad (6)$$

Note that this additional constraint does in general not preserve previously shown properties of the solutions like the halving property.

Starting with model (1) or (5) without a constraint on the revenue and noticing that the revenue is lower than desired, one may be inclined to equally increase all ticket prices until the desired revenue is realized. Formally, this means that if $\sum_{d \in \mathcal{D}} w_d \pi_d < R$, we set $\Delta := R - \sum_{d \in \mathcal{D}} w_d \pi_d$ and increase the base amount f to $\tilde{f} := f + \frac{\Delta}{\sum_{d \in \mathcal{D}} w_d}$. This however does in general not lead to an optimal solution with respect to the objective of minimizing the sum of absolute deviations from reference prices as illustrated in Figure 3a. Thus, it is only a heuristic.

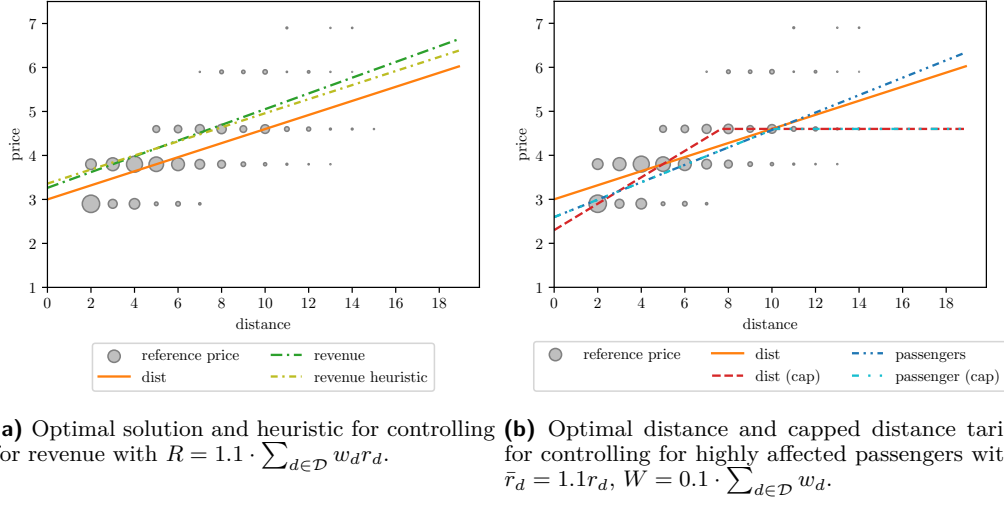


Figure 3 Example data from Example 2 with optimal solutions. Each passenger group is marked as a point (l_d, r_d) showing its distance (rounded to full kilometers) and reference price. The size of the marker depends on the number of passengers w_d of the group. The optimized fare structures are plotted as functions of the distance.

5.2 Controlling the number of highly affected passengers

While a high deviation between a reference price and a new ticket price is recognized and punished by the objective function, it is still possible that the price for a passenger group increases significantly. To prevent this, we can add a set of constraints to the previous models: For each passenger group $d \in \mathcal{D}$ let $\bar{r}_d$ be a threshold for the ticket price. Let $W \in \mathbb{R}_{\geq 0}$ be a limit on the number of passengers for which the new ticket price exceeds the threshold. In practice the threshold $\bar{r}_d$ may be chosen as $\bar{r}_d = r_d + \beta_1$ with $\beta_1 \in \mathbb{R}$ or as $\bar{r}_d = \beta_2 \cdot r_d$ with $\beta_2 \in \mathbb{R}_{\geq 0}$. The limit W may be given as $W = \gamma \cdot \sum_{d \in \mathcal{D}} w_d$ with $\gamma \in [0, 1]$. An example is illustrated in Figure 3b.

This restriction can be realized by incorporating the following set of constraints into the previous models (1)–(5) independent of the integrality and upper price limit specifications::

$$\pi_d \leq \bar{r}_d + M \cdot x_d \quad \text{for all } d \in \mathcal{D}, \quad (7)$$

$$\sum_{d \in \mathcal{D}} w_d x_d \leq W, \quad (8)$$

$$x_d \in \{0, 1\} \quad \text{for all } d \in \mathcal{D}, \quad (9)$$

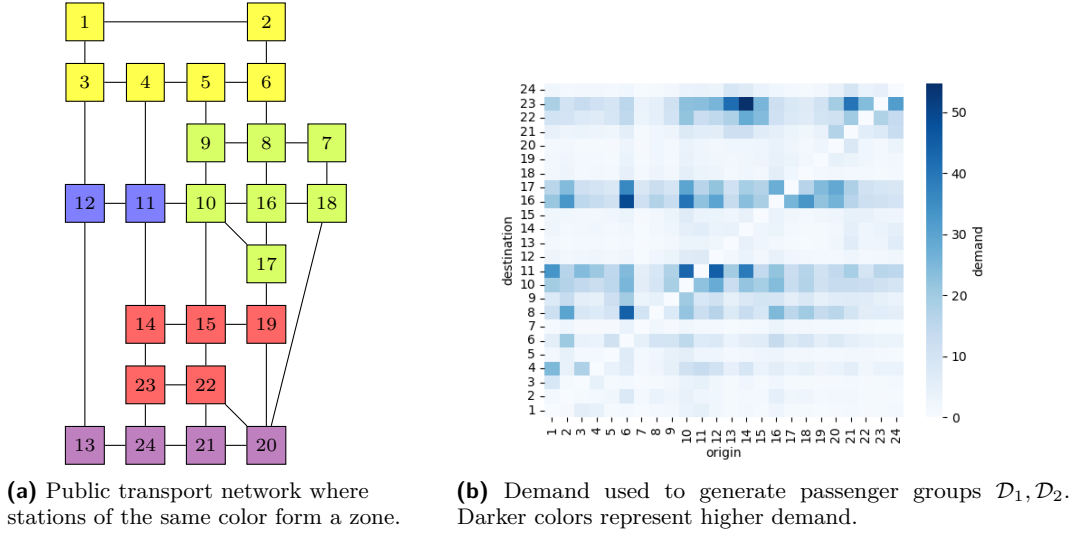
where we choose M as in Lemma 9. If $\pi_d > \bar{r}_d$, then the binary variable x_d is set to 1 and indicates that the new price exceeds the threshold. The total number of all passengers for which the threshold is exceeded is computed and limited by W . Note that this additional constraint does in general not preserve previously shown properties of the solutions like the halving property.

6 Experimental evaluation

In order to discuss the differences of the models, we conduct an experimental evaluation. For the input data we consider the data set `sioux_falls` provided in the open source software library LinTim [18, 19], see Figure 4. We generate two sets of passenger groups $\mathcal{D}_1, \mathcal{D}_2$ where for $\mathcal{D}_1$ the reference prices r_d are computed according to a zone tariff depicted in

11:12 Design of Distance Tariffs in Public Transport

Figure 4a and for $\mathcal{D}_2$ the reference prices r_d are computed based on a network distance tariff. Small perturbations are introduced by rounding the distances l_d of the passenger groups. For $\text{ratio_zone} \in \{0, 0.25, 0.5, 0.75, 1\}$, we create a set of passenger groups $\mathcal{D} = \{(l_d, r_d, \text{ratio_zone} \cdot w_d) : d \in \mathcal{D}_1\} \cup \{(l_d, r_d, (1 - \text{ratio_zone}) \cdot w_d) : d \in \mathcal{D}_2\}$. Thus, the reference prices for $\text{ratio_zone} = 0$ represent a perturbed distance tariff and for $\text{ratio_zone} = 1$ a perturbed zone tariff. By using two distance functions to determine l_d , $d \in \mathcal{D}$, in data set A and data set B, we get a total of 10 instances, detailed in Table 1. The models are implemented on a machine with Intel(R) Core(TM) i5-1335U CPU and 32GB RAM using Gurobi 12 [6].



■ **Figure 4** Public transport network and demand data for `sioux_falls`.

■ **Table 1** Input data for data set A and data set B, detailing the distance function used to compute l_d , the maximum distance over all passenger groups as well as the number of passenger groups depending on ratio_zone . Note that the distances l_d are rounded up to the nearest integer.

Data set	distance function	$\max_{d \in \mathcal{D}} l_d$	$\mathcal{D}$ for ratio_zone				
			0	0.25	0.5	0.75	1
data set A	rounded network	15	92	129	129	129	40
data set B	rounded beeline	153	128	183	183	183	59

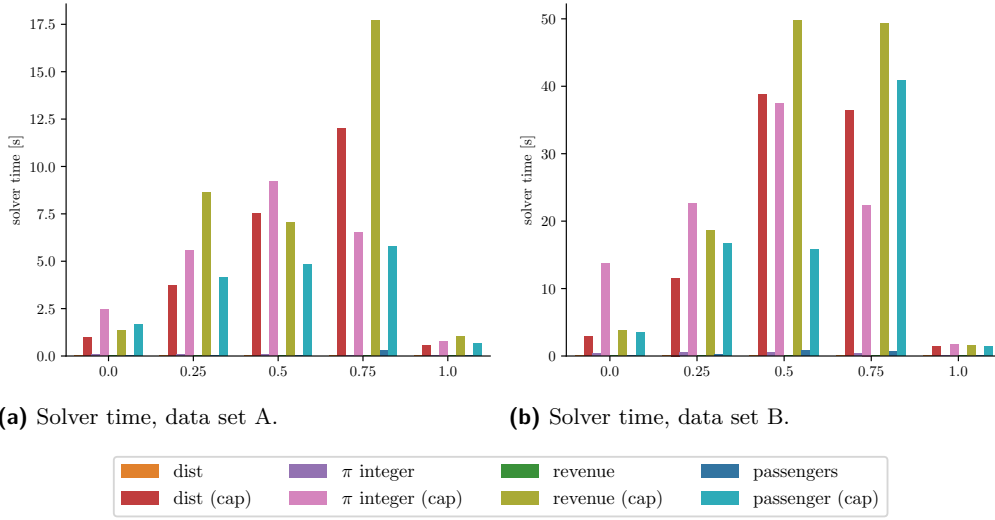
We implement and test 12 models discussed in this paper, see Table 2. In particular, we consider the basic distance tariff design models (dist), models with integer prices (integer) as well as controlling the revenue (revenue) and the number of highly affected passengers (passengers). For each of these models, a version with capped prices (cap) is implemented, as well as alternative models and heuristic approaches marked by *.

For controlling the revenue, we set $R = 1 \cdot \sum_{d \in \mathcal{D}} w_d r_d$, i.e., the revenue of the new solution has to be at least as high as the revenue generated by the reference prices. For controlling the number of highly affected passengers, we choose $\bar{r}_d = 1.1 r_d$, i.e., all passengers that have to pay more than 110% of the reference price are highly affected. Additionally, we restrict the number of highly affected passengers to $W = 0.1 \sum_{d \in \mathcal{D}} w_d$, i.e., at most 10% of the passengers may be highly affected. When integer prices are considered, we consider integer multiples of 10ct, both for the prices π and the parameters f, p .

■ **Table 2** Abbreviation and parameters of implemented models. Alternative formulations and heuristics are marked by * and are evaluated in Section 6.3.

Abbreviation	Model	Added parameters
dist	LP (2)	—
dist (cap)	MIP (5)	
π integer	MIP (3) (linearized)	multiples of 10ct
π integer (cap)	MIP (5) with $\pi_d \in \mathbb{Z}$	
π rounded*	LP (2), π rounded a posteriori	
f, p integer*	MIP (4) (linearized)	
f, p rounded*	LP (2), f, p rounded a posteriori	
revenue	LP (2) with (6)	$R = 1 \cdot \sum_{d \in \mathcal{D}} w_d r_d$
revenue (cap)	MIP (5) with (6)	
revenue heuristic*	LP (2), $\tilde{f} := f + \frac{\Delta}{\sum_{d \in \mathcal{D}} w_d}$ a posteriori	
passengers	MIP (2) with (7)-(9)	$\bar{r}_d = 1.1 r_d,$
passengers (cap)	MIP (5) with (7)-(9)	$W = 0.1 \sum_{d \in \mathcal{D}} w_d$

6.1 Solver time



■ **Figure 5** Solver time of the different models, grouped by `ratio_zone`.

As shown in Figure 5, the solver time of all models is low, with a maximal solver time of under 50 seconds. However, there is a clear difference between non-capped models and capped models. All non-capped models have solver times of under 2 seconds. For the capped models, however, the solver time increases to almost 50 seconds.

Additionally, the influence of the input structure and the corresponding number of passenger groups $|\mathcal{D}|$, see Table 1, on the solver time is evident. While a pure zone tariff as input (`ratio_zone` = 1) is the easiest to solve with solver times under 2 seconds also for models with an upper bound, the instances where both zone tariffs and distance tariffs are used to generate the input, i.e., $0 < \text{ratio_zone} < 1$, are hardest to solve. Furthermore, since data set A is a smaller data set, it has lower solver times than data set B.

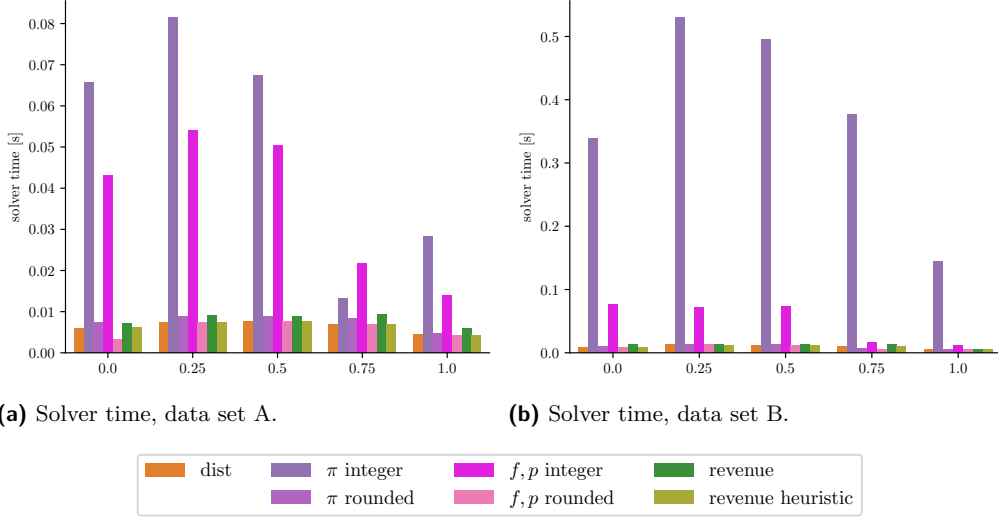


Figure 6 Solver time of the different alternative formulations and heuristics, grouped by `ratio_zone`.

Figure 6 details the solver time of the alternative formulations and heuristics. Note that for integer distances l_d , $d \in \mathcal{D}$, the models (3) and (4) are equivalent. However, modeling only f, p as integer variables clearly outperforms modeling all prices π_d , $d \in \mathcal{D}$, as integer variables. Note that the solver times of the revenue model and the heuristic revenue model are almost identical.

6.2 Evaluating different models

Solutions for the models described in Table 2 for `ratio_zone` = 0.25 are depicted Figure 7. Note that for other values of `ratio_zone` the solutions are structured similarly.

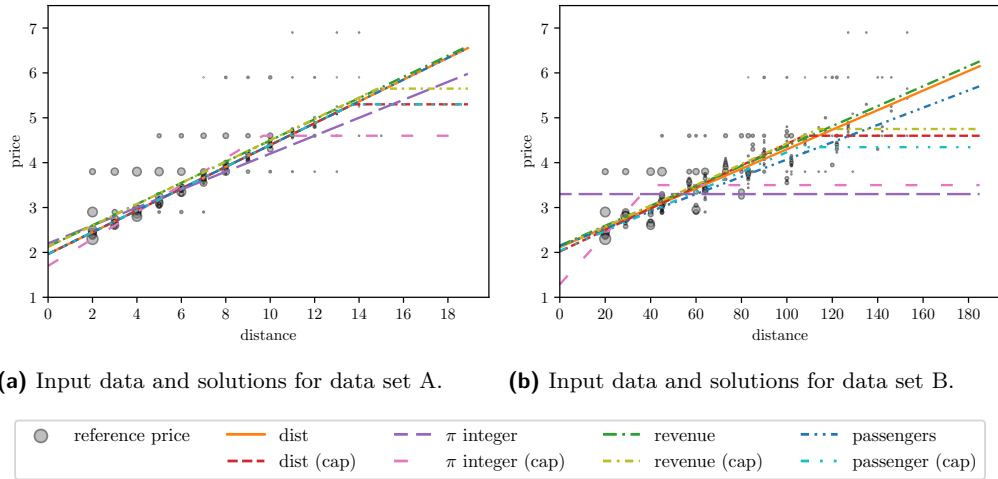


Figure 7 Input data and solutions for `ratio_zone`= 0.25. Every passenger group is marked as a point (l_d, r_d) showing its distance and reference price. The size of the marker depends on the number of passengers w_d of the group.

We observe that for data set A, for all models except π integer, the non-capped model and the capped model share the same base amount and price per kilometer f, p . Further evaluation shows that here the added flexibility of capped distance tariffs can hardly result in better solutions, see Figure 8. For data set B, however, introducing an upper limit results in new base amount and price per kilometer f, p for all models. Thus, the capped distance tariffs can better represent the reference prices of the passengers. For both data sets, the solutions for models with continuous prices are similar to each other. However, enforcing integer prices leads to considerably different solutions. For data set B, with a maximum distance of 153, enforcing integer prices results in a flat tariff, or a capped distance tariff that closely resembles a flat tariff. If integer prices are required here, it might be better to aggregate distances, e.g. to multiples of 5.

Evaluating the total deviation between prices and reference prices. Figure 8 details the objective value, i.e., the weighted absolute deviation from the reference prices for all considered models. In addition to the absolute objective value, a normalized version is depicted, where the objective value is represented in comparison to the objective value of the basic distance tariff (dist), i.e.,

$$\text{normalized objective}(\text{model}) = \frac{\text{objective}(\text{model}) - \text{objective}(\text{dist})}{\text{objective}(\text{dist})}. \quad (10)$$

Figure 8 shows that reference prices derived from a distance tariff (**ratio_zone**=0) can be approximated best but due to structure of the input data the reference prices cannot be met exactly. Furthermore, the absolute value of the total deviation increases with **ratio_zone**. Only for a zone-tariff-based input, i.e., **ratio_zone** = 1, the absolute objective values fall a little. This might be due to having fewer passenger groups $\mathcal{D}$.

By introducing an upper limit on the prices, the weighted absolute deviation can be reduced compared to the model without upper bound. In particular, multiple models outperform the basic distance tariff (dist). As observed in Figure 7, this effect is more pronounced for data set B than for data set A.

Note that for data set B, enforcing integer prices has a considerable effect on the solution quality, increasing the weighted absolute deviation to up to 180% of the corresponding deviation for the basic distance tariff. This is due to the high number of integer distance values paired with low prices, which often leads to flat tariffs that cannot approximate the input data well.

For all other models, the solution quality does not deviate by more than 25% compared to the deviation for the basic distance and the deviation is often considerably lower. Note that for data set B, enforcing the revenue to be at least as high as the revenue according to the reference prices can be done without noticeable losses in the objective value. This shows that imposing additional constraints still allows for tariffs that represent the reference prices well.

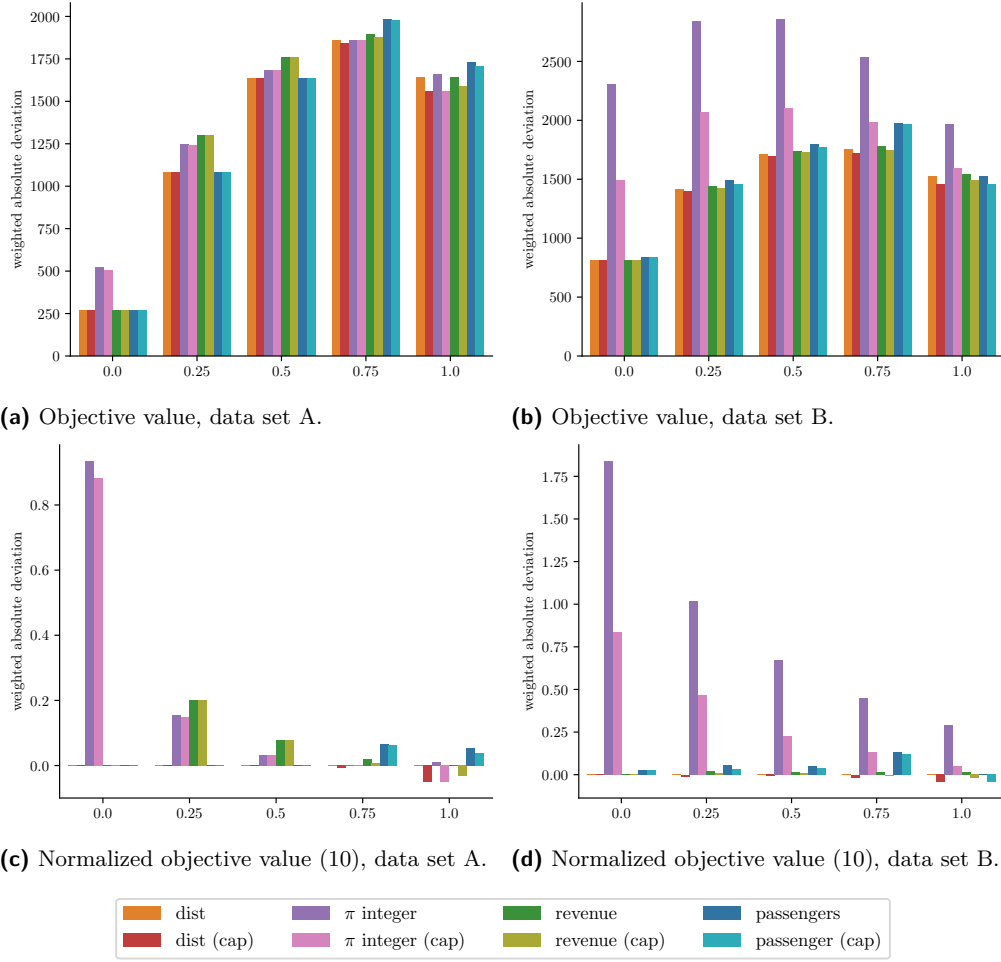


Figure 8 Evaluating the objective value (weighted absolute deviation from the reference prices) of the optimal solutions grouped by `ratio_zone`.

Evaluating the revenue. In addition to the objective value, we evaluate the revenue of the solutions in Figure 9. Note that the revenue is normalized by the revenue of the reference prices, i.e.,

$$\text{normalized revenue}(\text{model}) = \frac{\text{revenue}(\text{model}) - \sum_{d \in \mathcal{D}} w_d r_d}{\sum_{d \in \mathcal{D}} w_d r_d}. \quad (11)$$

Compared to the revenue generated by the reference prices, the revenue according to the new tariffs reduces by at most 7% and even increases slightly for some models. Note that when controlling for the revenue, the revenue does not decrease. Thus, simplifying the tariff to a distance based tariff with relevant practical constraints is possible without a high impact on the operator's revenue.

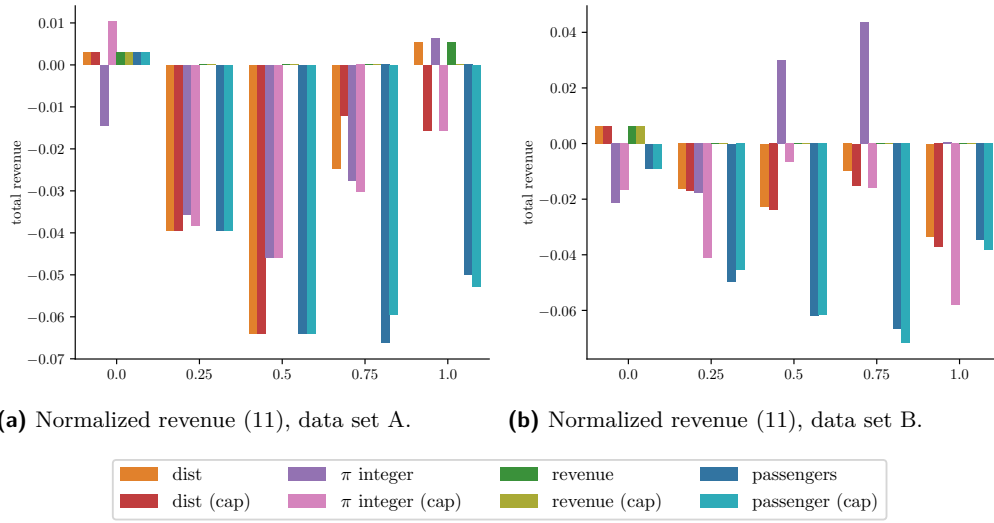


Figure 9 Evaluating the revenue value of the optimal solutions, normalized by reference revenue, grouped by `ratio_zone`.

6.3 Evaluating alternative models and heuristics

The alternative models and heuristics solution methods marked by * in Table 2 are evaluated in comparison to the original models.

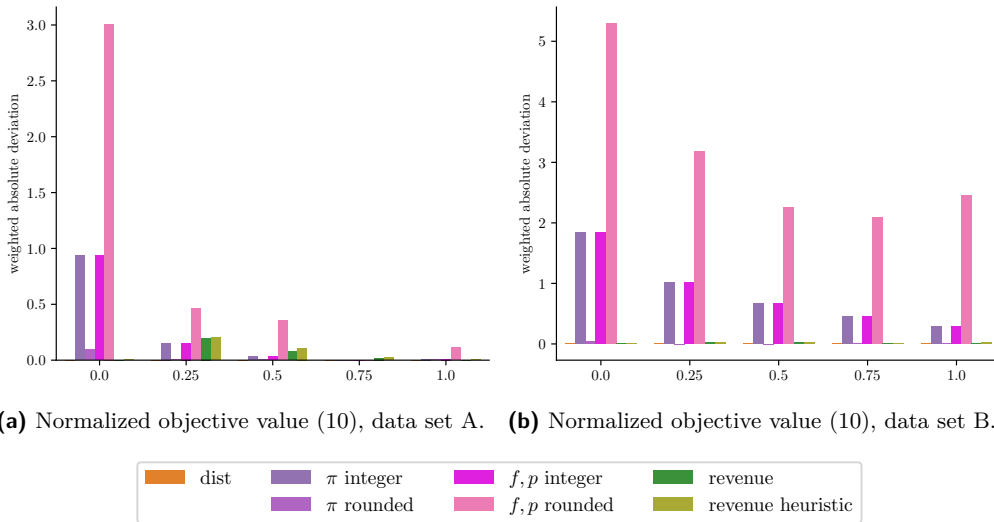


Figure 10 Evaluating the objective value (weighted absolute deviation from the reference prices) of the optimal solutions for alternative formulations and heuristics, grouped by `ratio_zone`.

Evaluating the total deviation between prices and reference prices. When considering integer prices, Figure 10 shows that the equivalent models (3) (π integer) and (4) (f, p integer) do indeed find equally good solutions. However, rounding the prices π or the base amount and price per kilometer f, p a posteriori results in very different solutions. While rounding π

does not lead to a distance tariff, the total deviation from the reference prices is always as good as the distance tariff (dist) and sometimes even results in slightly lower deviations. In contrast, rounding the base amount and price per kilometer f, p results in solutions with up to 5 times higher deviations from the reference prices compared to the basic distance tariff. This effect is even more pronounced for data set B, where there are more integer values for the considered distances. Note that the heuristic for revenue controlled tariffs performs almost as well as the exact solution method. However, a closer examination of the solutions shows that the resulting tariffs differ even though the objective values are almost identical.

Evaluating the revenue. When considering the revenue depicted in Figure 11, we observe again that rounding the base amount and price per kilometer f, p is an outlier compared to other models and can lead to significantly reduced revenues. The exact and heuristic models for revenue control are both able to avoid revenue losses. Note that while the heuristic model is designed to meet the reference price revenue exactly, the exact method can lead to slightly higher revenues with the same deviation from the reference prices.

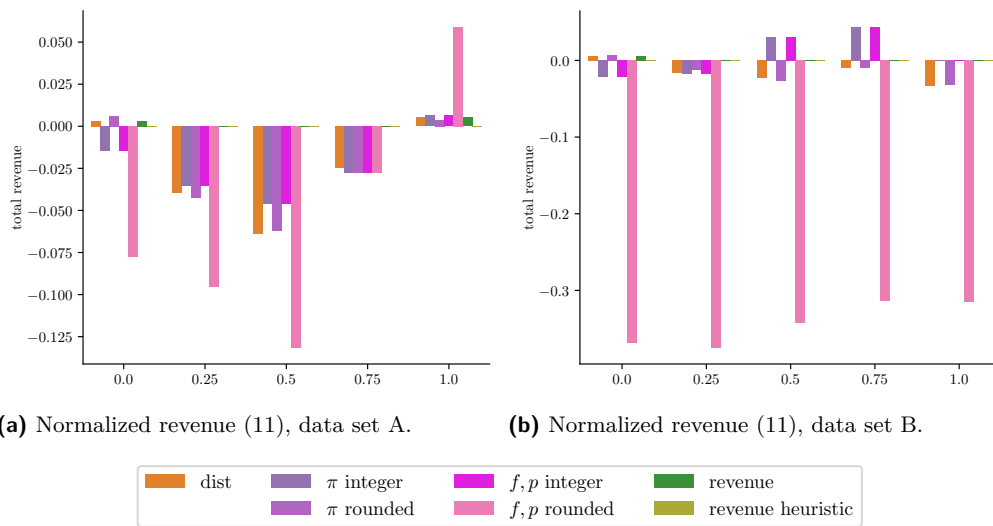


Figure 11 Evaluating the revenue value of the optimal solutions for alternative formulations and heuristics, grouped by `ratio_zone`.

7 Conclusion and further research

This paper provides models with several practically relevant specifications for changing a fare strategy to a distance tariff. In addition to a basic distance tariff composed of a price per kilometer and a base amount, also integrality requirements are discussed. A capped distance tariff additionally implements an upper price bound on the ticket prices. Furthermore, two constraints that can be added to the models and control the change in the revenue or the number of highly affected passengers, respectively, are suggested. The reported optimization models allow to explore and evaluate different options for distance tariffs with the possibility to add special requirements in the constraints. The experimental evaluation shows that practical constraints can be added when designing distance tariffs without sacrificing too much solution quality or revenue. In a confidential case study our partners found the results useful for their decisions on a distance tariff.

We anticipate further research in two different directions: First, the results are related to locating lines and other structures in computational geometry. In particular, for the capped distance tariff, we locate an “angled line” whose theoretical properties could be discussed along the lines of [25], not only for the median but also for the respective center problem.

Second, further research in tariff planning in public transport is interesting. We aim to strengthen the MIP formulations to solve large, realistic data sets by leveraging the structure of the reference prices. This is especially promising when the reference prices are derived from a zone tariff with few zones. Additionally, a bicriteria version as in [20] in which the revenue and the number of passengers using public transport are considered as two (conflicting) objective functions is a useful extension. Furthermore, passengers’ routes are influenced not only by their travel times but also by the tariff system. Since the routes of the passengers are crucial for planning lines and timetables, there is a need of integrating also tariff planning with these stages, and to discuss the differences between the sequential and the integrated approach [16, 17] also in this setting.

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