

Refined Integer Programs and Polyhedral Results for the Target Visitation Problem

Sven Mallach 

Chair of Management Science, University of Siegen, Germany
University of Bonn, Germany

Abstract

The Target Visitation Problem (TVP) combines the Traveling Salesman Problem and the Linear Ordering Problem, and thus serves as a natural model for route planning applications where both the travel costs and the order of the sites to visit matter. More precisely, in addition to the costs that apply for the selected links connecting two subsequently visited sites, the relative urgency of visiting one site before another is quantified and taken into account. In this article, we present refined integer linear programming formulations for the TVP, along with clarifications and extensions regarding the description of the polytopes associated with their feasible solution sets by a minimal set of linear equations and facet-defining inequalities. The practical effectiveness of exploiting the proposed improvements by means of a branch-and-cut algorithm is demonstrated in a computational study. In addition, we report the optimal values for some previously unsolved instances.

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1 Introduction

In the Target Visitation Problem (TVP), the task is to determine an ordering (permutation) of a set of target locations (sites) that *maximizes* the *difference* between the *total sum of pairwise rewards* $p_{ij} \in \mathbb{R}$, $i \neq j$, obtained when ranking a site i (anywhere) before another site j , and the *total sum of pairwise costs* $c_{ij} \in \mathbb{R}$ paid when ranking sites i and j *consecutively*. Considering a permutation as a site-traversal order, the rewards p_{ij} may be interpreted as the strength of the preference of visiting site i before site j while the costs c_{ij} may reflect e.g. the traveling distance from i to j . Figure 1 displays which rewards and costs effectively contribute to the objective value of an example ordering of five sites, and a more formal problem definition is given at the beginning of Section 2. As this illustrates, the TVP is a combination (and generalization) of the Linear Ordering Problem (LOP) and the (Weighted) Asymmetric Hamiltonian Path Problem (AHPP) in a complete graph. With only slight modifications, one may also consider it as a combination of the LOP and the (Weighted) Asymmetric Traveling Salesman Problem (ATSP) [4, 5]. By choosing sufficiently high rewards, it may further serve as a proxy for the ATSP with precedence constraints. Moreover, each of these related combinatorial optimization problems is well-known to be strongly $\mathcal{NP}$ -hard whence so is the TVP [5, 6].

The TVP is a natural model for route planning problems where both the total travel costs and the order of the visited sites are of (potentially competing) interest. A common scenario is that the demands of supply (urgencies) and the supply times between locations need to be juxtaposed in opposition, like in the seminal application with unmanned aerial vehicles (drones) in [4]. This includes particularly the scheduling of rescue or relief missions



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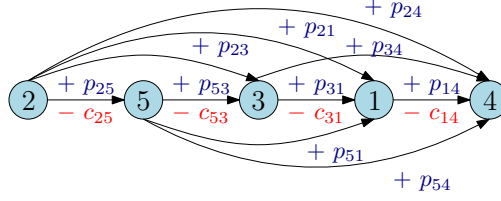
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■ **Figure 1** An example permutation (Hamiltonian path) of five sites indicating which rewards and costs contribute to its respective objective function value in terms of the TVP.

in disaster areas, the planning of town-cleaning or snow-plowing services [5], as well as further applications in transportation where the route sequence is supposed to reflect visiting preferences beyond the shortest distance or travel time, for instance to integrate an additional route-internal pickup and delivery as considered e.g. in [2].

Since its introduction in 2004 by Grundel and Jeffcoat [4] who addressed the problem with a local search approach, the TVP has mainly been studied in the context of integer programming and polyhedral combinatorics by Hildenbrandt, especially in his PhD thesis [5] and a related article [6]. In the dissertation, he also evaluated several heuristics. Furthermore, there is a semidefinite programming relaxation for the TVP by Hungerländer [7] as well as further heuristics by Blázsik et al. [2] and Arulselvan et al. [1].

In this work, we first present three integer programming (re)formulations for the TVP which improve on their relatives from [5, 6] in terms of continuous relaxation strength, preciseness in characterizing TVP solutions by means of compact constraint sets, and solution times. An accompanying analysis of the impact of certain equation and inequality classes provides a tidied-up dominance relation between candidate constraints that enforce a consistent permutation or link different variable subsets. Then, we present revised and extended results regarding the description of the polytopes associated with the proposed formulations, i.e. the convex hull of the vectors describing their feasible solutions, by means of a minimal set of linear equations and facet-defining inequalities. In an experimental study, we demonstrate the practical effects of our refinements in terms of solving the formulations with a branch-and-cut algorithm, and we report the optimal values for some previously unsolved instances. Besides that, we draw relations to the Asymmetric Betweenness Problem and the Quadratic Linear Ordering Problem which allow to translate certain structural results between canonical integer programming formulations for these problems and for the TVP.

This paper is organized as follows. In Section 2, we first provide a more formal definition of the TVP, we recall the two primary integer programs proposed in [5, 6], and we briefly address further aspects of closely related work. We then present our refined formulations in Section 3, along with accompanying results on their relaxation strength and the necessity of certain constraint sets. Our polyhedral results are the subject of Section 4, and we report on our computational experiments in Section 5. Finally, a brief conclusion is given in Section 6.

2 Preliminaries and Related Work

Let $n \in \mathbb{N}$ be the number of target sites to be visited in the TVP and let Π_n be the set of all permutations of $[n] := \{1, \dots, n\}$. A permutation $\pi \in \Pi_n$ is here treated as a bijective function that maps each site $i \in [n]$ to its position $\pi(i) \in [n]$, and where we conversely denote

by $\pi^{-1}(i)$ the site that is placed at the position $i \in [n]$. The objective function of the TVP may then be formally written as

$$\max_{\pi \in \Pi_n} \sum_{i=1}^{n-1} \sum_{j=i+1}^n p_{\pi^{-1}(i)\pi^{-1}(j)} - \sum_{i=1}^{n-1} c_{\pi^{-1}(i)\pi^{-1}(i+1)},$$

or likewise as

$$\max_{\pi \in \Pi_n} \sum_{i,j \in [n]: \pi(i) < \pi(j)} p_{ij} - \sum_{i,j \in [n]: \pi(j) = \pi(i)+1} c_{ij}.$$

As will become apparent in the following, the latter variant is particularly suited to be implemented by means of common binary decision variables in an integer linear programming (ILP) formulation. Since such formulations and accompanying polyhedral considerations are the central subject of this work, we mainly concentrate the exposition of the prevalent literature on the seminal (and so far, in all conscience, not otherwise resumed) research by Hildenbrandt [5, 6]. Specifically, he presents several integer programs for the TVP, two of which are in the focus of his investigations as well as of this paper.

The first, major, model by Hildenbrandt is called TVP-HP, and it combines the central ingredients of known formulations for the AHPP and the LOP in a canonical way.

$$\max \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_{ij}y_{ij} + p_{ji}(1 - y_{ij})) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}x_{ij} \quad (\text{TVP-HP})$$

$$\text{s.t.} \quad \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_{ij} = n - 1 \quad (1)$$

$$\sum_{j=1, j \neq i}^n x_{ij} \leq 1 \quad \text{for all } i \in [n] \quad (2)$$

$$\sum_{i=1, i \neq j}^n x_{ij} \leq 1 \quad \text{for all } j \in [n] \quad (3)$$

$$y_{ij} + y_{jk} - y_{ik} \leq 1 \quad \text{for all } i, j, k \in [n] : i < j < k \quad (4)$$

$$-y_{ij} - y_{jk} + y_{ik} \leq 0 \quad \text{for all } i, j, k \in [n] : i < j < k \quad (5)$$

$$x_{ij} - y_{ij} \leq 0 \quad \text{for all } i, j \in [n] : i < j \quad (6)$$

$$x_{ji} + y_{ij} \leq 1 \quad \text{for all } i, j \in [n] : i < j \quad (7)$$

$$x_{ij} \geq 0 \quad \text{for all } i, j \in [n] : i \neq j \quad (8)$$

$$x_{ij} \in \{0, 1\} \quad \text{for all } i, j \in [n] : i \neq j \quad (9)$$

$$y_{ij} \in \{0, 1\} \quad \text{for all } i, j \in [n] : i < j \quad (10)$$

In this light, TVP-HP involves (AHPP) variables x_{ij} , for $i, j \in [n]$, $i \neq j$, where $x_{ij} = 1$ if $\pi(j) - \pi(i) = 1$ (i.e., site j is visited immediately after site i) and $x_{ij} = 0$ otherwise, as well as (LOP) variables y_{ij} , for $i, j \in [n]$, $i < j$, where $y_{ij} = 1$ if $\pi(i) < \pi(j)$ (i.e., site j is visited anytime after site i) and $y_{ij} = 0$ if $\pi(j) < \pi(i)$. Assuming that the AHPP variables take on binary values, the constraints (1)–(3) reflect that $n - 1$ immediate successor relations (“edges”, when considering the permutation as a path) are established while each site has at most one successor and at most one predecessor. Likewise, assuming that the LOP variables take on binary values, the three-di-cycle inequalities (4) and (5) are known to be sufficient in

order to uniquely determine a permutation $\pi \in \Pi_n$ [3]. In [5, 6], Hildenbrandt points out that the linking of the two variable sets by means of inequalities (6) and (7) excludes disconnected subpaths (respectively subtours) from the feasible set, i.e., they ensure that the binary values of the AHPP variables are in one-to-one correspondence with a directed Hamiltonian path. In the original presentation of TVP-HP in these references, the constraints (7) are nevertheless missing despite they are hence necessary. At the same time, they are included in the list of facet-defining inequalities for the related polytope that Hildenbrandt describes explicitly for $n = 4$ and they also have been recognizably included when carrying out his computational experiments with TVP-HP.

The second model in Hildenbrandt's focus is an extended formulation, called TVP-E.

$$\begin{aligned}
 \max \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_{ij}y_{ij} + p_{ji}(1 - y_{ij})) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}x_{ij} & (\text{TVP-E}) \\
 \text{s.t.} \quad & (1)-(6), (8)-(10) \\
 & b_{jik} + b_{kji} + b_{jki} + y_{ij} = 1 \quad \text{for all } i, j, k \in [n] : i < j, k \neq \{i, j\} & (11) \\
 & x_{ij} - b_{ijk} - b_{kij} \leq 0 \quad \text{for all } i, j, k \in [n] : i \neq j \neq k \neq i & (12) \\
 & b_{ijk} \geq 0 \quad \text{for all } i, j, k \in [n] : i \neq j \neq k \neq i & (13) \\
 & b_{ijk} \in \{0, 1\} \quad \text{for all } i, j, k \in [n] : i \neq j \neq k \neq i & (14)
 \end{aligned}$$

Specifically, he obtains TVP-E from TVP-HP by extending the latter in two steps. First, he introduces the additional variables b_{ijk} , for all $i, j, k \in [n]$, $|\{i, j, k\}| = 3$, with the interpretation that $b_{ijk} = 1$ if $\pi(i) < \pi(k) < \pi(j)$ (i.e., k is ranked between i and j while i is ranked before j) and $b_{ijk} = 0$ otherwise. Second, he appends the constraint sets (11) and (12) to link these new variables to the AHPP respectively LOP variables. Thereby, the intuition behind the inequalities (12) is that if site j is visited immediately after site i , then any other site k must be visited either before or after both i and j . Notably, the constraints (11) (only) ensure that, if $\pi(j) < \pi(i)$, then any other site k must be placed either before, in between, or after this accordingly ordered pair. The reverse direction, i.e., $b_{ijk} + b_{kji} + b_{jki} = 1$ if $\pi(i) < \pi(j)$, is however missing.

Indeed, solutions where all the six b -variables associated with a fixed triple of sites $i, j, k \in [n]$, $i < j < k$, are zero, are thus verifiably feasible for TVP-E (in addition to those where these variables receive values that are consistent with the ordering expressed in the LOP variables). Consequently, a feasible solution to the TVP unintendedly corresponds to more than one feasible solution of TVP-E (which we will cure in Section 3). Since a feasible TVP solution is correctly (and unambiguously) encoded in the integral AHPP- and LOP-variables though, and since the b -variables do not affect the objective function, one may still derive optimum solutions from TVP-E. Apart from that, the inequalities (7) are again not included in the original presentation of TVP-E in [5, 6], and in this case they are indeed negligible as they are implied by (11) and (12).

Even despite the unintended additional solutions, the upper bound on the optimal value obtained when solving the continuous relaxation (i.e., the linear programming, or short LP, relaxation that results from neglecting all integrality restrictions) of TVP-E is provably at least as strong as the one obtained when solving the relaxation of TVP-HP, and is experimentally observed in [5, 6] to be usually significantly stronger. Nevertheless, TVP-E has not been further considered as an option for a branch-and-cut approach in these references, due to the reported high solution times of its relaxation. As will be demonstrated in the following sections, these solution times can however be improved by a careful reformulation.

Another observation that rather appears as a detail in [5], and that will play an important role for the refinement of TVP-HP, is that the three-di-cycle inequalities (4), (5) can be extended in the here given common context with the AHPP-variables. These *extended three-di-cycle inequalities* can be written as follows:

$$y_{ij} + y_{jk} - y_{ik} + x_{ji} \leq 1 \text{ for all } i, j, k \in [n] : i < j < k \quad (15)$$

$$-y_{ij} - y_{jk} + y_{ik} + x_{ij} \leq 0 \text{ for all } i, j, k \in [n] : i < j < k \quad (16)$$

$$y_{ij} + y_{jk} - y_{ik} + x_{kj} \leq 1 \text{ for all } i, j, k \in [n] : i < j < k \quad (17)$$

$$-y_{ij} - y_{jk} + y_{ik} + x_{jk} \leq 0 \text{ for all } i, j, k \in [n] : i < j < k \quad (18)$$

$$y_{ij} + y_{jk} - y_{ik} + x_{ik} \leq 1 \text{ for all } i, j, k \in [n] : i < j < k \quad (19)$$

$$-y_{ij} - y_{jk} + y_{ik} + x_{ki} \leq 0 \text{ for all } i, j, k \in [n] : i < j < k \quad (20)$$

For completeness, we remark that Hildenbrandt describes a further model that combines the notion of a subsequently visited pair of sites and other sites ranked (somewhere) after the respective pair, as well as a formulation based on distance variables. However, these formulations were found to be significantly inferior to TVP-HP and TVP-E in [5], whence we refer to this reference for further details. Finally, adapted formulations for the case where the TVP is rather considered as a combination of the LOP and the ATSP are stated in [1, 5].

3 Three Refined Integer Programming Formulations for the TVP

In this section, our efforts are directed to derive integer programming formulations for the TVP that fulfill two requirements. The first one is that the feasible solutions of a formulation are in one-to-one correspondence with those of the TVP, and, to achieve compactness and strength at the same time, the second one is that a formulation solely consists of equations and inequalities which take part in a minimal description of the polytope that is given as the convex hull of the (incidence) vectors describing these solutions. The latter condition subdivides into the two necessities that each equation must be part of a minimum equation system for the respective polytope and that every inequality must induce a facet of it. In total, we present three according formulations which are subject to an experimental comparison in Section 5. Besides that, the accompanying analysis of the impact of certain equation and inequality classes serves as an intermediate step to the polyhedral results in Section 4.

As a first formulation, we propose TVP-XY, which is obtained from TVP-HP by replacing the three-di-cycle inequalities (4) and (5) with their extended pendants (15)–(20).

$$\begin{aligned} \max \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_{ij}y_{ij} + p_{ji}(1 - y_{ij})) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}x_{ij} \\ \text{s.t.} \quad & (1)–(3), (6)–(10), (15)–(20) \end{aligned} \quad (\text{TVP-XY})$$

This comparably small change is motivated as follows. In [5, 6], Hildenbrandt defines the polytope

$$P_{\text{TVP}}^n := \text{conv} \left\{ (x, y) \in \{0, 1\}^{2\binom{n}{2} + \binom{n}{2}} : (x, y) \text{ satisfies } (1)–(7) \right\}$$

(related to TVP-HP) whose vertices are in one-to-one correspondence with the incidence vectors of the feasible solutions to the TVP, expressed in AHPP- and LOP-variables. Also, the equation (1) has been identified as a minimum equation system for P_{TVP}^n , and the inequalities (2), (3), (6) and (7) have been identified as facets of P_{TVP}^n , $n \geq 4$. However, the inequalities (4) and (5) do not define facets of P_{TVP}^n , $n \geq 4$, whereas the inequalities (15)–(20) do (facet class

29 in [5]). Their exchange thus ensures the fulfillment of the second requirement mentioned above. In addition, we propose and explicitly state TVP-XY to foster its consideration and evaluation as a fundamental formulation on its own, which is also supported by our computational results in Section 5. As opposed to that, in [5, 6], the replacement of (4) and (5) by (15)–(20) is rather treated as optional, and the resulting computational effect, especially regarding the upper bounds obtained from the continuous relaxation, cannot be sufficiently deduced respectively distinguished from other effects that are subject to TVP-HP simultaneously.

While all constraint classes of TVP-XY take part in a minimal description of P_{TVP}^n , it is noteworthy that (as opposed to TVP-HP) TVP-XY is yet not irreducible in the sense that its feasible solutions remain in one-to-one correspondence with those of the TVP when removing inequalities (2) and (3). This is formally recorded in Proposition 1. In other words, these inequalities are kept in TVP-XY only to strengthen its continuous relaxation. Moreover, by construction, the latter provides an upper bound on the optimal value that is at least as strong as the one of the continuous relaxation of TVP-HP.

► **Proposition 1.** *Suppose that (x, y) is binary and satisfies (1), (6), (7), and (15)–(20). Then, (x, y) is feasible for TVP-XY, i.e., it satisfies (2), (3) as well.*

Proof. By the integrality of y and the extended three di-cycle inequalities (15)–(20), the position of each site $i \in [n]$ is given by $\pi(i) = 1 + \sum_{k \in [n], k < i} y_{ki} + \sum_{k \in [n], i < k} (1 - y_{ik})$. W.l.o.g., let i be the site at position $\pi(i) \in \{1, \dots, n-1\}$, and let k be its successor, i.e., $\pi(k) = \pi(i) + 1$. Then $y_{ik} = 1$, and for any other site $j \in [n]$, $j \neq \{i, k\}$, either $y_{ij} = 0$ (respectively $y_{ji} = 1$ if $j < i$) or $y_{jk} = 0$ (respectively $y_{kj} = 1$ if $k < j$), but not both. If $i < j < k$, it thus follows directly from (16) that $x_{ij} = 0$, otherwise the same result is established by another instance of (15)–(20) w.r.t. the matching index order. The only variable that appears in (2) and that is not implied to be zero like this is x_{ik} , hence (2) is satisfied. Finally, if $\pi(i) = n$, then $x_{ij} = 0$ for all $j \in [n]$, $j \neq \{i, k\}$, due to (6) and (7), so (2) is satisfied again. The proof for (3) is analogous. ◀

Our second and third formulations, called TVP-XYB and TVP-XYBR, respectively, relate to TVP-E, i.e., reflect an extended formulation with asymmetric betweenness variables. Since TVP-E does not fulfill the first requirement stated at the beginning of this section, we first alter it to TVP-XYB which reads as follows:

$$\max \quad \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_{ij}y_{ij} + p_{ji}(1 - y_{ij})) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}x_{ij} \quad (\text{TVP-XYB})$$

$$\text{s.t.} \quad b_{kij} + b_{ikj} + b_{ijk} + b_{kji} + b_{jki} + b_{jik} = 1 \text{ for all } i, j, k \in [n] : i < j < k \quad (21)$$

$$b_{ijk} + b_{kij} + b_{ikj} - y_{ij} = 0 \text{ for all } i, j, k \in [n] : i < j, k \neq \{i, j\} \quad (22)$$

$$(1)-(3), (8)-(10), (12), (13)$$

Specifically, the addition of the equations (21) ensures that for each triple of sites i, j, k , $i < j < k$, exactly one of their six possible orderings is enforced. In addition, equations (22) ensure $b_{ijk} + b_{kij} + b_{ikj} = 1$ if $\pi(i) < \pi(j)$ which was the missing part in TVP-E. When combined with (21), they imply equations (11), and that all b -variables are enforced to be binary if all the y -variables are (see also Theorem 4 below) whence we omit (14).

Since the feasible solutions of TVP-XYB are thus in one-to-one correspondence with those of the TVP, we now define the associated polytope given by the convex hull of the respective incidence vectors involving components for all three variable classes. To emphasize that it is actually not precisely the same polytope that Hildenbrandt defined in [5, 6] related to TVP-E, which he called P_{TVE}^n , we call the here considered polytope P_{TVE}^n which is:

$$P_{\text{TVE}}^n := \text{conv} \left\{ (x, y, b) \in \{0, 1\}^{2\binom{n}{2} + \binom{n}{2} + 6\binom{n}{3}} : (x, y, b) \text{ satisfies (1)–(3), (12), (21)–(22)} \right\}$$

Since P_{TVE}^n is defined via integral vectors, the (extended) three-di-cycle inequalities could be part of its definition but may also be omitted as a consequence of Theorem 3 below. For the same reason and in fulfillment of the second requirement stated at the beginning of this section, they are also not part of TVP-XYB. The equations (21) are added to TVP-XYB because they are part of a minimum equation system for P_{TVE}^n , $n \geq 4$, as is shown in Theorems 7, 8 (Corollary 9), and Theorem 11 in Section 4. In addition, it will become visible from the proofs of the Theorems 2 and 3 that they have, in combination with equations (11), or likewise with equations (22), a crucial impact on the strength of TVP-XYB, respectively its continuous relaxation. Further, the inequalities (2), (3), and (12) are all facet-defining for P_{TVE}^n (with a special exception regarding (2) and (3) for $n = 4$), see also Theorems 13 and 14 in Section 4. However, inequalities (2), (3) are again dispensable in terms of retaining the one-to-one correspondence with the feasible solutions of the TVP, since an analogue of Proposition 1 is readily at hand for TVP-XYB given Theorems 2 and 3 whose proofs do not rely on (2) and (3). So again these inequalities are kept in TVP-XYB only to strengthen its continuous relaxation.

► **Theorem 2.** *Let (x, y, b) be feasible for the continuous relaxation of TVP-XYB and $n \geq 4$. Then, for all $i, j \in [n]$, $i < j$, $x_{ij} - y_{ij} \leq 0$ and $x_{ji} + y_{ij} \leq 1$.*

Proof. We have $x_{ij} \leq b_{ijk} + b_{kij} \leq y_{ij}$ where the first relation follows from (12) and the second follows from (22). Moreover, (22) and (21) together imply the equation (11). Combining the latter with (12) again, we obtain $x_{ji} \leq b_{jik} + b_{kji} \leq 1 - y_{ij}$. ◀

► **Theorem 3.** *Let (x, y, b) be feasible for the continuous relaxation of TVP-XYB and $n \geq 4$. Then (x, y, b) satisfies the extended three-di-cycle inequalities (15)–(20).*

Proof. Fix some $i, j, k \in [n]$ where $i < j < k$. We prove the statement explicitly for (15) and (16) while the other cases are analogous. To this end, we first combine the instances of (22) respectively (11), depending on the index order, as follows:

$$\begin{array}{rclcl} b_{kij} + b_{ijk} & + & b_{ikj} & = & y_{ij} \\ & & b_{ijk} + b_{jki} & + & b_{jik} & = & y_{jk} \\ \hline b_{kij} & + & b_{jki} & + & b_{kji} & = & 1 - y_{ik} \\ \hline 2b_{kij} + 2b_{ijk} + 2b_{jki} + b_{ikj} + b_{jik} + b_{kji} & = & 1 + y_{ij} + y_{jk} - y_{ik} \end{array}$$

Substituting, in this sum, for the reordering $b_{ikj} + b_{jik} + b_{kji} = 1 - b_{kij} - b_{ijk} - b_{jki}$ of (21), we obtain the “base equation” $b_{kij} + b_{ijk} + b_{jki} = y_{ij} + y_{jk} - y_{ik}$.

Then, we substitute for b_{jki} based on accordingly resolving the equation $b_{jik} + b_{kji} + b_{jki} + y_{ij} = 1$ (which is (11) for $i < j$ and k) and rearranging terms. This gives

$$\underbrace{b_{kij} + b_{ijk} + 1 - y_{ij}}_{\leq y_{ij}} = y_{ij} + y_{jk} - y_{ik} + \underbrace{b_{jik} + b_{kji}}_{\geq x_{ji}}$$

Due to (12), the right-hand side of this equation is at least as large as $y_{ij} + y_{jk} - y_{ik} + x_{ji}$, which is the left-hand side of (15), and clearly, the left-hand side of the equation is smaller than or equal to one.

To show that (16) is satisfied as well, we first negate the base equation and substitute again for b_{jki} as above. This gives

$$\underbrace{b_{jik} + b_{kji} - 1 + y_{ij}}_{\leq 1 - y_{ij}} = -y_{ij} - y_{jk} + y_{ik} + \underbrace{b_{kij} + b_{ijk}}_{\geq x_{ij}}.$$

Again by (12), we have $-b_{kij} - b_{ijk} \leq -x_{ij}$, i.e., the right-hand side of the equation is at least as large as $-y_{ij} - y_{jk} + y_{ik} + x_{ij}$, which is the left-hand side of (16), while the left-hand side of the equation is clearly non-positive. ◀

A further impression on the strength of the equations (21) and (22), or (21) and (11), becomes apparent when considering a natural relation between TVP-XYB (TVP-E) and formulations for the Asymmetric Betweenness Problem as well as the Quadratic Linear Ordering Problem [8]. This relation is given by the correspondence $b_{ijk} = y_{ij}y_{jk}$ for all $i, j, k \in [n] : i \neq j \neq k \neq i$, that applies to every feasible solution (here, y_{ij} is to be replaced by $1 - y_{ji}$ if $j < i$ and y_{jk} is to be replaced by $1 - y_{kj}$ if $k < j$). In this respect, the equations (21) and (22), respectively (21) and (11), imply that the b -variables are equal to the associated product term when the y -variables take on binary values. This follows directly from the following result (a proof can be derived from an analogue result in [8]):

► **Theorem 4.** *Let (x, y, b) be a feasible solution to the continuous relaxation of TVP-XYB. Then, for all $i, j, k \in [n]$, $|\{i, j, k\}| = 3$, it holds that $b_{ijk} \leq y_{ij}$, $b_{ijk} \leq y_{jk}$, and $y_{ij} + y_{jk} - b_{ijk} \leq 1$*

Finally, we derive our third formulation TVP-XYBR from TVP-XYB which has less variables and is more amenable to a cutting plane approach. To this end, we employ the equations (21) and (22) to project out the variables b_{ijk} , b_{kji} , b_{jki} , and b_{jik} . More precisely, for each selection of $i, j, k \in [n]$, $i < j < k$, we first build the intermediate set of linearly independent equations which imply (21) and (22), see also [6, 8]:

$$\begin{aligned} b_{kij} + b_{ikj} + b_{ijk} - y_{ij} &= 0 & \text{for all } i, j, k \in [n] : i < j < k \\ b_{jik} + b_{ikj} + b_{ijk} - y_{ik} &= 0 & \text{for all } i, j, k \in [n] : i < j < k \\ b_{jik} + b_{jki} + b_{ijk} - y_{jk} &= 0 & \text{for all } i, j, k \in [n] : i < j < k \\ b_{kji} + b_{jki} + b_{ijk} + y_{ij} &= 1 & \text{for all } i, j, k \in [n] : i < j < k \end{aligned}$$

Then, we resolve them for the aforementioned variables.

$$\begin{aligned} b_{ijk} &= -b_{kij} - b_{ikj} + y_{ij} & \text{for all } i, j, k \in [n] : i < j < k \\ b_{jik} &= b_{kij} - y_{ij} + y_{ik} & \text{for all } i, j, k \in [n] : i < j < k \\ b_{jki} &= b_{ikj} - y_{ik} + y_{jk} & \text{for all } i, j, k \in [n] : i < j < k \\ b_{kji} &= 1 - y_{jk} - b_{kij} - b_{ikj} & \text{for all } i, j, k \in [n] : i < j < k \end{aligned}$$

TVP-XYBR is then obtained from TVP-XYB by eliminating equations (21) and (22), by substituting each of the aforementioned variables with the respective right-hand side in the remaining constraints of TVP-XYB, and by adding inequalities to enforce the non-negativity of these right-hand sides which gives (22a)–(22d).

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_{ij}y_{ij} + p_{ji}(1 - y_{ij})) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}x_{ij} & (\text{TVP-XYBR}) \\
\text{s.t.} \quad & \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_{ij} = n - 1 \\
& \sum_{j=1, j \neq i}^n x_{ij} \leq 1 & \text{for all } i \in [n] \\
& \sum_{i=1, i \neq j}^n x_{ij} \leq 1 & \text{for all } j \in [n] \\
& b_{kij} + b_{ikj} - y_{ij} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (22a) \\
& -b_{kij} + x_{ij} - y_{ik} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (22b) \\
& -b_{ikj} + x_{ik} - y_{jk} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (22c) \\
& b_{kij} + b_{ikj} + y_{jk} \leq 1 & \text{for all } i, j, k \in [n] : i < j < k & (22d) \\
& x_{ij} + b_{ikj} - y_{ij} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (12a) \\
& x_{ik} - b_{ikj} - b_{kij} + y_{ij} - y_{ik} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (12b) \\
& x_{ji} + y_{ij} + y_{jk} - y_{ik} + b_{ikj} \leq 1 & \text{for all } i, j, k \in [n] : i < j < k & (12c) \\
& x_{jk} + y_{ik} - y_{jk} - y_{ij} + b_{kij} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (12d) \\
& x_{ki} - b_{kij} - b_{ikj} + y_{ik} - y_{jk} \leq 0 & \text{for all } i, j, k \in [n] : i < j < k & (12e) \\
& x_{kj} + b_{kij} + y_{jk} \leq 1 & \text{for all } i, j, k \in [n] : i < j < k & (12f) \\
& x_{ij} \geq 0 & \text{for all } i, j \in [n] : i \neq j \\
& b_{ikj}, b_{kij} \geq 0 & \text{for all } i, j, k \in [n] : i < j < k \\
& x_{ij} \in \{0, 1\} & \text{for all } i, j \in [n] : i \neq j \\
& y_{ij} \in \{0, 1\} & \text{for all } i, j \in [n] : i < j
\end{aligned}$$

4 Polyhedral Results for P_{TVE}^n

In this section, we investigate the polytope P_{TVE}^n defined in Section 3 and provide fundamental results concerning its (minimal) description by linear equations and inequalities. As a byproduct, these results also clarify some statements in [5, 6] about the polytope P_{ETV}^n considered there.

Our starting point is the following result by Hildenbrandt:

► **Lemma 5** (Lemma 4.11 in [5]). *For $n \geq 4$, the equations (1), (21), and (22) are linearly independent.*

Deviating from the presentation in [5, 6], we however find that, if $n = 4$, then there is another set of equations that is valid for all incidence vectors of feasible solutions to the TVP consistently expressed in x -, y -, and b -variables.

► **Theorem 6.** *The following equations are valid for P_{TVE}^4 :*

$$\sum_{\substack{j \in [4] \\ j \neq i}} (x_{ij} + x_{ji}) - \sum_{\substack{j \in [4] \\ i < j}} y_{ij} - \sum_{\substack{j \in [4] \\ j < i}} (1 - y_{ji}) + \sum_{\substack{j \in [4] \\ j \neq i}} \sum_{\substack{k \in [4] \\ i \neq k \neq j}} b_{ijk} = 1 \quad \text{for all } i \in [4] \quad (23)$$

Proof. Let $\langle i_1, i_2, i_3, i_4 \rangle$ be a permutation of $\{1, 2, 3, 4\}$. Then for the site i_j at the j -th position,

- the number of path-neighbors is $\sum_{\ell \in [4], \ell \neq i_j} (x_{i_j \ell} + x_{\ell i_j}) = k$ with $k = 2$ if $j \in \{2, 3\}$ and $k = 1$ if $j \in \{1, 4\}$,
- the negated total number of successor sites is $-\sum_{\ell \in [4], i_j < \ell} y_{i_j \ell} - \sum_{\ell \in [4], \ell < i_j} (1 - y_{\ell i_j}) = j - 4$,
- the number of successor site-pairs is $\sum_{\ell \in [4], \ell \neq i_j} \sum_{m \in [4], i_j \neq \{m, \ell\}} b_{i_j \ell m} = k$ with $k = 0$ if $j \in \{3, 4\}$, $k = 1$ if $j = 2$, and $k = 3$ if $j = 1$.

Therefore, for any fixed $j \in \{1, 2, 3, 4\}$, the left-hand side of (23) evaluates to 1. ◀

The following theorem further establishes that the equations (1), (21), (22) do not constitute a minimum equation system for P_{TVE}^4 , as there is a linearly independent selection of equations (23) that is also linearly independent from those.

► **Theorem 7.** *For $n = 4$, the previously known equations (1), (21), (22), and three of the four new equations (23) are linearly independent.*

Proof. Consider the following excerpt of the equation system for the variables x_{12}, x_{41}, x_{42} and x_{43} (which only appear in (1) and (23)). Here $(*)$ means that the respective entries may be neglected for the argumentation.

Eq.	x_{12}	...	x_{41}	x_{42}	x_{43}	...
(1)	1	1 ... 1	1	1	1	$(*) = n - 1$
(21)	0	0 ... 0	0	0	0	$(*) = 1$
(22)	0	0 ... 0	0	0	0	$(*) = 0$
(23) ₁	1	$(*)$	1	0	0	$(*) = 1$
(23) ₂	1	$(*)$	0	1	0	$(*) = 1$
(23) ₃	0	$(*)$	0	0	1	$(*) = 1$

The 3x3 identity submatrix for the variables (columns) x_{41}, x_{42} and x_{43} gives linear independence for (23)₁–(23)₃ among each other as well as from all rows except the first one, while the column for x_{12} shows that the first row cannot be combined from (23)₁–(23)₃ as well. ◀

The instances of (1), (21), (22), and (23) considered in the proof of Theorem 7 amount to $1 + 12 + 4 + 3 = 20$ equations in total. Since there are $4 \cdot 3 + \binom{4}{2} + 6\binom{4}{3} = 42$ variables in TVP-XYB, Theorem 7 implies that $\dim P_{\text{TVE}}^4 \leq 42 - 20 = 22$ (while the dimension of P_{ETV}^4 was stated to be 25 in [5, 6] which cannot be the case irrespective of its different definition).

► **Theorem 8.** *The dimension $\dim P_{\text{TVE}}^4$ of P_{TVE}^4 is equal to 22.*

Proof. Since there are 24 permutations of four sites, P_{TVE}^4 has 24 vertices. Let M be the matrix having the incidence vectors of these vertices as its rows. Then $\dim P_{\text{TVE}}^4$ is equal to the affine rank of M , denoted $\text{arank}(M)$. We first determine that $\text{rank}(M) = 23$, i.e., 23 of the 24 vertices are linearly independent. Moreover, $(0, 0, 0) \notin P_{\text{TVE}}^4$, so P_{TVE}^4 is spanned by 22 affinely independent vectors, and $\text{arank}(M) = \text{rank}(M) - 1 = 22$. ◀

► **Corollary 9.** *For $n = 4$, a minimum equation system for P_{TVE}^4 is given by (1), (21), (22), and three of the four equations (23).*

To complete the picture, we verified that the following set of inequalities from appendix A.2 in [5] and (12) together induce all the facets of P_{TVE}^4 :

$$\sum_{\ell=1, \ell \neq i}^4 -x_{i\ell} - b_{jki} - b_{kji} \leq -1 \text{ for all } i, j, k \in [4] : i \neq j < k \neq i \quad (24)$$

$$\sum_{\ell=1, \ell \neq i}^4 -x_{\ell i} - b_{ijk} - b_{ikj} \leq -1 \text{ for all } i, j, k \in [4] : i \neq j < k \neq i \quad (25)$$

$$x_{ik} + x_{ij} + x_{jk} + b_{kji} + b_{ikl} - b_{ijk} \leq 1 \quad \text{for all } i, j, k, \ell \in [4] : |\{i, j, k, \ell\}| = 4 \quad (26)$$

$$-x_{ji} - x_{j\ell} + x_{\ell k} + b_{kji} + b_{lik} + b_{lji} - b_{ljk} - b_{lkj} \leq 0 \quad \text{for all } i, j, k, \ell \in [4] : |\{i, j, k, \ell\}| = 4 \quad (27)$$

$$-x_{ji} - x_{j\ell} - x_{ki} - x_{kj} - x_{k\ell} + b_{kji} + b_{lji} - b_{ljk} \leq -1 \text{ for all } i, j, k, \ell \in [4] : |\{i, j, k, \ell\}| = 4 \quad (28)$$

$$x_{ji} + x_{ki} + x_{kj} + x_{k\ell} + x_{\ell i} - b_{jil} + b_{j\ell k} - b_{k\ell i} - b_{\ell i k} + b_{\ell j k} - b_{\ell k i} \leq 1 \quad \text{for all } i, j, k, \ell \in [4] : |\{i, j, k, \ell\}| = 4 \quad (29)$$

In total, one obtains 144 inequalities. The inequality class “2” listed in appendix A.2 of [5] is not facet-defining for P_{TVE}^4 (the dimension of their associated faces is 10).

The following lemma will simplify the proof of the subsequent theorem, and it follows directly from according results for the HPP polytope [10] and the LOP polytope [3] of order n , respectively.

► **Lemma 10.** *For $n \geq 4$, there is no valid equation for P_{TVE}^n that has non-zero coefficients for x -variables only and is linearly independent from (1), or that has non-zero coefficients for y -variables only.*

► **Theorem 11.** *For $n \geq 5$, the equations (1), (21) and (22) constitute a minimum equation system for P_{TVE}^n .*

Proof. W.l.o.g., we consider the reduced space of the $d(n) := n(n-1) + \binom{n}{2} + 2\binom{n}{3}$ variables associated with TVP-XYBR, respectively with the projection $\bar{P}_{\text{TVE}}^n$ of P_{TVE}^n onto the x , y , b_{kij} and b_{ikj} variables (which eliminates the equations (21) and (22), as described in Section 3).

Let M_n be the $n! \times d(n)$ zero-one matrix whose rows consist of the vertices of $\bar{P}_{\text{TVE}}^n$, i.e., the incidence vectors of all permutations $\pi \in \Pi_n$ w.r.t. the aforementioned variables. Moreover, let $e_{n!}$ be the all-ones vector with $n!$ components. In the following, the first $n(n-1)$ columns of M_n are referred to as $X = (X_{ij})$, the following $\binom{n}{2}$ columns as $Y = (Y_{ij})$, and the final $2\binom{n}{3}$ columns are referred to as $B = [(B_{kij})(B_{ikj})]$, i.e, $M_n = [X \ Y \ B]$.

Suppose that every (vertex) $(x, y, b) \in \bar{P}_{\text{TVE}}^n$ satisfies the equation $\alpha^T x + \beta^T y + \gamma^T b = \delta$.

Then $(\alpha, \beta, \gamma, \delta)$ is a solution to the system $M_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta e_{n!}$, and conversely, if $(\alpha, \beta, \gamma, \delta)$ is a solution to this system, then $\alpha^T x + \beta^T y + \gamma^T b = \delta$ is satisfied by every $(x, y, b) \in \bar{P}_{\text{TVE}}^n$.

Now in order to prove the theorem, by Lemma 10, it suffices to show that there does not exist an equation that is valid for $\bar{P}_{\text{TVE}}^n$, and that either (a) links a y -variable with either x - or b -variables or both, or that (b) links only b -variables among each other, or b - with x -variables.

We first show this for the case $n = 5$, and for convenience, we write the system $M_5 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta e_{5!}$ more explicitly as:

$$\sum_{i=1}^5 \sum_{j=1, j \neq i}^5 \alpha_{ij} X_{ij} + \sum_{i=1}^4 \sum_{j=i+1}^5 \beta_{ij} Y_{ij} + \sum_{i=1}^3 \sum_{j=i+1}^4 \sum_{k=j+1}^5 (\gamma_{kij} B_{kij} + \gamma_{ikj} B_{ikj}) = \delta e_{5!} \quad (*)$$

We may prove (a) by showing that *all* solutions to $(*)$ have $\beta = \mathbf{0}$. To this end, consider $(*)$ as the set of constraints of a linear program in the variables $\alpha, \beta, \gamma, \delta$, where we may w.l.o.g. add the restriction $\delta \geq 0$ since the other variables are free in their sign. Moreover, for all $i, j \in [n]$, $i < j$, consider the (objective) function $f_{ij}(\beta) := e_{ij}^T \beta$, where e_{ij} is the unit vector of dimension $\binom{n}{2}$ (here dimension 10) with the 1-entry in the component associated with β_{ij} . In other words, β_{ij} is the only variable obtaining an objective coefficient of one while all other variables receive a zero coefficient.

By solving the so defined LPs with objective $\max f_{ij}(\beta)$ and with objective $\min f_{ij}(\beta)$ for all $i, j \in [5]$, $i < j$, we obtain that the optimal value is always $f_{ij}^*(\beta) = 0$. This certifies

that truly no solution to $M_5 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta e_{5!}$ with $\beta \neq \mathbf{0}$ exists, as such a solution was rewarded by a positive or negative objective value in at least one of the considered LPs (if it existed, such an LP would actually be unbounded).

To prove (b), we follow the same approach except for choosing the (objective) functions $f_{kij}(\gamma) := e_{kij}^T \gamma$ and $f_{ikj}(\gamma) := e_{ikj}^T \gamma$ for all $1 \leq i < j < k \leq 5$ (e has now dimension $2\binom{n}{3}$, here 20). Each of the 40 LPs obtained in total for maximization and minimization is again feasible with objective value zero. Thus, there is also no solution to $M_5 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta e_{5!}$ with $\gamma \neq \mathbf{0}$.

Finally, we complete the proof by showing that if the claim holds for $n = \ell$, $\ell \geq 5$, then it holds as well for $n = \ell + 1$. For the purpose of deriving a contradiction, suppose that for $n = \ell + 1$ there exists a solution to $M_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta e_n!$ where $\beta \neq \mathbf{0}$ ($\gamma \neq \mathbf{0}$). W.l.o.g., let $\beta_{ij} \neq 0$ ($\gamma_{ikj} \neq 0$), and choose a site $v \notin \{i, j\}$ ($v \notin \{i, j, k\}$). Suppose that we fix v to the last (or first) position. Then certain columns of X , Y , and B , which do not include Y_{ij} (B_{ikj}) can be fixed to zero or one as well. Eliminating these variables, the remaining ones correspond (after renumbering) to a TVP with $n - 1$ sites while the resolved equation still has $\beta_{ij} \neq 0$ ($\gamma_{ikj} \neq 0$) and is feasible for all vertices of $\bar{P}_{\text{TVE}}^{n-1}$ – a contradiction. $\blacktriangleleft$

We thus conclude the dimension stated by Hildenbrandt in [5, 6] (for P_{ETV}^n) holds true for P_{TVE}^n and $n \geq 5$.

► **Corollary 12.** For $n \geq 5$, $\dim P_{\text{TVE}}^n = 2\binom{n}{2} + \binom{n}{2} + 6\binom{n}{3} - 1 - (n-2) \cdot \binom{n}{2} - \binom{n}{3} = \frac{n(n-1)}{2} \cdot \frac{(2n+5)}{3} - 1$.

In the remainder of this section, we provide the justifications for our decisions in Section 3 to keep the inequalities (2) and (3), as well as the inequalities (12), in our proposed formulations. While we prove the latter ones to be facet-defining for P_{TVE}^n , $n \geq 4$, we emphasize the special case that (2) and (3) do not define facets of P_{TVE}^4 (given the additional equations), but do define facets for P_{TVE}^n , $n \geq 5$.

► **Theorem 13.** For $n \geq 5$, the inequalities (2) and (3) define facets of P_{TVE}^n .

Proof. Let $S_i^n = \{(x, y, b) \in P_{\text{TVE}}^n : \sum_{j=1, j \neq i}^n x_{ij} = 1\}$ be the face of P_{TVE}^n induced by (2) w.r.t. $i \in [n]$. It is easy to see that, for each $i \in [n]$, S_i^n is spanned by all the $n! - (n-1)!$ incidence vectors that belong to the permutations of $[n]$ where i is ranked at any but the last position, as these satisfy the corresponding instances of (2) with equality.

Specifically, for $n = 5$, we have $\dim P_{\text{TVE}}^5 = 49$, and it may be verified for each $i \in \{1, \dots, 5\}$, that the matrix M having the aforementioned $5! - 4! = 96$ incidence vectors¹ as its rows has rank 49. Therefore, and since $(\mathbf{0}, \mathbf{0}, \mathbf{0}) \notin S_i^5$, $\dim(S_i^5) = \text{rank}(M) - 1 = 48$, so all the instances of (2) define facets of P_{TVE}^5 .

Given this basis, let the statement of the theorem be true for dimension n , but suppose, for the purpose of showing a contradiction, that the inequality (2) w.r.t. some $i \in [n+1]$ does not define a facet of P_{TVE}^{n+1} . Then this implies (for a proof see e.g. the standard textbook [9]) the existence of an equation $\alpha^\top x + \beta^\top y + \gamma^\top b = \delta$ that is linearly independent from the equations (1), (21), and (22) forming the minimum equation system according to Theorem 11 as well as from $\sum_{j=1, j \neq i}^{n+1} x_{ij} = 1$, and that is valid for all incidence vectors in S_i^{n+1} . Among these incidence vectors, there are $n!$ many that correspond to the permutations where $i \in [n+1]$ is ranked first. Suppose now that we fix i to the first position by setting variables in (x, y, b) accordingly, and thus obtain a new equation from $\alpha^\top x + \beta^\top y + \gamma^\top b = \delta$ that is still feasible for the $n!$ considered incidence vectors. Then, however, after re-indexing, this equation must also be feasible for all vertices of P_{TVE}^n , in terms of the remaining TVP without the fixed $i \in [n+1]$, which is a contradiction to the minimality of the equation system according to Theorem 11.

The proof for the inequalities (3) is analogous. ◀

► **Theorem 14.** For $n \geq 4$, the inequalities (12) define facets of P_{TVE}^n .

Proof. Let $S_{ijk}^n = \{(x, y, b) \in P_{\text{TVE}}^n : x_{ij} - b_{kij} - b_{ijk} = 0\}$ be the face of P_{TVE}^n induced by (12) w.r.t. $i, j, k \in [n]$, $|\{i, j, k\}| = 3$.

Each S_{ijk}^n is spanned by all the incidence vectors belonging to the permutations of $[n]$ except those where i is ranked non-immediately before j and k is ranked either before i or after j .

Specifically, for $n = 4, 5, 6$, we verified explicitly that the matrices having these incidence vectors as their rows have rank 22, 49, and 84, respectively, whence inequalities (12) define facets of P_{TVE}^4 , P_{TVE}^5 and P_{TVE}^6 .

Given this basis, let the statement of the theorem be true for dimension n , but suppose, for the purpose of showing a contradiction (starting from $n = 5$), that the inequality (12) w.r.t. some $i, j, k \in [n+2]$, $|\{i, j, k\}| = 3$, does not define a facet of P_{TVE}^{n+2} . Then this implies the existence of an equation $\alpha^\top x + \beta^\top y + \gamma^\top b = \delta$ that is linearly independent from the equations (1), (21), and (22), forming the minimum equation system according to Theorem 11, as well as from the equation $x_{ij} - b_{kij} - b_{ijk} = 0$, and that is valid for all incidence vectors in S_{ijk}^{n+2} . Among these incidence vectors, there are $n!$ many that correspond to the permutations where $i \in [n+2]$ is ranked first and where $j \in [n+2]$ is ranked last since then $x_{ij} = b_{kij} = b_{ijk} = 0$. Suppose now that we fix i to the first and j to the last position by setting variables in (x, y, b) accordingly, and thus obtain a new equation from $\alpha^\top x + \beta^\top y + \gamma^\top b = \delta$ that is still feasible for the $n!$ considered incidence vectors. Then, however, after re-indexing, this equation must also be feasible for all vertices of P_{TVE}^n , i.e. regarding the remaining TVP without i and j , which is a contradiction to the minimality of the equation system according to Theorem 11. ◀

¹ For $n = 4$, there are only $4! - 3! = 18$ such incidence vectors whence these inequalities do not define facets of P_{TVE}^4 .

5 Computational Experiments

Given the formulations from Section 3 and the according changes when compared to their relatives, we provide an impression on the computational effects when they are solved with a branch-and-cut algorithm implemented with a state-of-the-art ILP solver. Specifically, TVP-XY involves a larger number of constraints than TVP-HP while it provides a stronger continuous relaxation, and it is of interest whether this translates into a net improvement regarding the ability to solve the ILP. The continuous relaxation of TVP-XYB is stronger than the one of TVP-E while TVP-XYB is also more compact at the same time, but the solution times for the relaxation of TVP-XYB are still high when compared to TVP-XY. Here, the primary question is whether the variant TVP-XYBR, which consists of inequalities only and is thus particularly suited to be solved by a cutting plane approach, can be solved quickly enough such that the strong upper bounds provided by its relaxation make it truly competitive to TVP-XY.

To address these questions, we employ the publicly available TVP instances that have been used in [5, 6] as well. They involve between 26 and 45 sites and follow the naming scheme “XX_OOO_N_ID” where “XX” is a label that defines lower and upper bounds on the ratio of the objective values of an optimal LOP solution f_{LOP} and an optimal TSP solution f_{TSP} , “OOO” encodes the number of random interchanges of preference values when creating the instance, “N” is the number of sites, and finally “ID” is a running index. Let $r = f_{\text{LOP}}/f_{\text{TSP}}$. The label “XX” is “ER” if $\frac{1}{2} \leq r \leq \frac{3}{2}$, “LB” if $\frac{3}{2} < r \leq 3$ and “LD” if $3 < r$. The label “OOO” is “CFO” if the preference values were left unchanged, “MCO” if $\frac{1}{4}N^2$ random interchanges were made among them, and “BCO” if $\frac{1}{2}N^2$ interchanges took place. For further details on how the distance and preference values were derived, we refer to [5].

To solve the LP relaxations and ILPs, we use Gurobi (here in version 12.0.1), restricted to a single thread and with the additional parameter settings `Seed=1`, and `PreCrush=1`, as well as `LazyConstraint=1` and `MIPGap=10-6` in the ILP case. For each formulation, the respective inequality classes of cubic cardinality are dynamically separated by enumeration, and all violated inequalities found are passed to Gurobi. More precisely, they are manually added when solving only the LP relaxation and then trigger a warmstart from the respective solution, whereas when solving the ILP, they are handed over via the callback mechanism both for the branch-and-bound node relaxations and as lazy constraints. For TVP-HP, these separated constraints are the three-di-cycle inequalities (4)–(5), whereas for TVP-XY the extended three-di-cycle inequalities (15)–(20) are separated. For TVP-XYB the inequalities (12) are separated, and for TVP-XYBR, first the inequalities (22a)–(22d) and, if none of these are violated, then inequalities (12a)–(12f) are separated. When solving the continuous relaxations, any of these dynamically added inequalities is also removed again if its left-hand side, evaluated for the respective current solution, is at least by 10^{-4} smaller than its right-hand side and if it was not added just in the previous iteration. In case of the ILP runs, the time limit for each instance was set to 15 minutes and the test system is equipped with an AMD Ryzen 9 9900X processor, 96 GB RAM, and Ubuntu 24.04.2 LTS.

In Table 1, we list those instances where at least one of the models was solved to proven optimality within this time frame. The first two columns display the name and the value of an optimal solution (OPT) for the respective instance. Then, for each of the formulations there is a block of columns. The first column “ILP [s/%]” shows either the time needed to solve the ILP (in seconds) or, if the time limit was reached, in parenthesis the remaining gap between the global upper bound (UB) obtained on the optimal value at termination, and the

actual optimal value, calculated as $100 \frac{UB - OPT}{|OPT|}^2$. The second column “LP Bound” shows the upper bound provided by the continuous relaxation of the respective formulation. This column is omitted for TVP-XYBR since the bound is the same as with TVP-XYB. Finally, the column “LP [s]” displays the time, in seconds, needed to solve the continuous relaxation.

■ **Table 1** Computational results for the formulations TVP-HP, TVP-XY, TVP-XYB, and TVP-XYBR, and all instances from the testbed that could be solved with at least one formulation (if there is only one, then it is always TVP-XY) within 15 minutes.

Inst.	OPT	TVP-HP			TVP-XY			TVP-XYB			TVP-XYBR		
		ILP [s / %]	LP Bound	LP [s]	ILP [s / %]	LP Bound	LP [s]	ILP [s / %]	LP Bound	LP [s]	ILP [s / %]	LP [s]	
ER_CFO_30_1	-7300	67.80	91.17	0.01	32.96	-3469.20	0.03	141.50	-4688.47	3.32	156.14	0.62	
ER_CFO_30_2	-11803	(15.37)	480.50	0.01	146.10	-2014.12	0.02	702.56	-4678.21	6.28	506.85	0.69	
ER_CFO_35_1	-3198	(133.79)	10015.30	0.02	198.31	4827.20	0.03	(33.97)	3421.84	8.52	(61.87)	0.73	
ER_CFO_40_1	-3000	611.08	4178.37	0.02	50.70	920.76	0.07	192.11	-377.33	37.82	891.44	1.97	
ER_CFO_40_3	-10693	(8.02)	-4182.40	0.02	229.77	-7060.67	0.09	(9.63)	-8059.99	40.61	(12.88)	2.42	
ER_CFO_40_5	-11827	362.53	-3812.54	0.03	84.46	-6195.17	0.07	175.79	-8842.04	50.02	754.11	2.29	
LB_CFO_26_1	24774	23.52	34948.17	0.01	5.23	31959.62	0.01	11.43	27488.75	0.52	15.77	0.24	
LB_CFO_26_2	8358	200.40	18201.33	0.01	41.84	14504.30	0.03	130.55	12758.91	1.25	239.95	0.34	
LB_CFO_26_3	5078	168.14	14713.45	0.01	57.75	12026.90	0.03	171.24	9308.24	0.82	224.32	0.29	
LB_CFO_30_1	23715	177.81	34448.29	0.01	15.70	30543.88	0.01	34.31	28517.17	1.77	153.08	0.40	
LB_CFO_30_2	4312	428.42	16138.97	0.01	61.85	12236.89	0.03	159.81	8654.11	3.89	316.27	0.79	
LB_CFO_35_1	985	715.51	11581.64	0.01	137.84	8193.62	0.03	700.37	4811.02	10.16	557.10	1.30	
LB_CFO_35_2	1932	651.90	9299.48	0.02	51.46	5106.41	0.05	491.06	4592.81	11.87	608.75	1.06	
LB_CFO_40_1	21829	(10.36)	31801.50	0.01	523.47	27562.06	0.05	(8.38)	25560.57	34.85	(11.38)	2.04	
LD_CFO_26_1	43677	259.21	58741.25	0.01	60.86	54214.19	0.03	147.17	51832.37	1.98	166.88	0.44	
LD_CFO_35_1	163453	(2.64)	179161.00	0.01	315.20	175567.50	0.03	(1.91)	169349.05	10.72	(1.65)	1.46	
ER_MCO_26_1	-7639	202.71	1331.96	0.03	42.79	-1009.25	0.04	198.17	-3325.27	4.31	219.96	0.57	
ER_MCO_30_1	-4532	(76.33)	4702.52	0.06	220.51	1279.46	0.07	(20.14)	-750.60	14.81	(27.70)	1.56	
ER_MCO_30_2	-2314	(234.52)	9249.03	0.03	176.35	5055.46	0.06	469.36	3656.83	7.96	797.65	1.37	
ER_MCO_30_3	-8787	169.94	-2938.36	0.03	12.96	-5613.54	0.09	23.48	-6647.23	11.31	78.62	1.23	
ER_MCO_30_4	-4024	(123.85)	10089.52	0.03	139.20	3815.65	0.07	(8.69)	1185.79	11.36	(32.11)	1.48	
ER_MCO_35_1	-8356	(23.89)	80.99	0.07	184.45	-3568.12	0.18	802.98	-4616.94	46.55	(18.99)	3.34	
ER_MCO_40_1	-17246	837.55	-10134.67	0.15	80.42	-12866.08	0.32	296.25	-15584.65	149.71	708.72	8.43	
LB_MCO_26_1	1826	220.24	14497.19	0.02	4.97	11609.81	0.03	93.48	9157.03	1.87	72.41	0.59	
LB_MCO_30_1	695	(612.54)	13666.96	0.04	36.96	8225.06	0.07	117.71	4468.28	10.98	303.50	1.36	
LB_MCO_35_1	16527	(40.00)	33236.17	0.06	558.08	25867.61	0.14	(21.00)	21617.76	31.93	(15.05)	3.28	
LB_MCO_35_2	11561	(48.40)	25844.41	0.07	83.88	18658.23	0.12	320.10	14749.52	32.49	899.98	3.46	
LD_MCO_30_1	246422	106.19	260607.05	0.04	13.10	249724.71	0.08	76.80	248297.25	11.25	111.31	1.37	
LD_MCO_40_1	743577	(1.20)	768067.58	0.11	560.92	758705.45	0.15	(0.54)	752834.49	57.97	(0.77)	3.89	
ER_BCO_26_1	-5600	61.42	4381.20	0.03	3.15	1553.00	0.03	8.95	-4187.86	4.76	92.02	1.00	
ER_BCO_26_2	-221	85.83	11930.25	0.02	8.59	8055.90	0.04	31.61	1851.60	2.50	62.86	0.62	
ER_BCO_35_1	-16328	(6.86)	-4788.40	0.11	86.49	-7697.59	0.21	(4.55)	-12726.68	62.74	(5.33)	4.74	
ER_BCO_35_2	-6224	(78.43)	4369.33	0.07	81.02	-563.81	0.21	665.95	-1285.94	47.19	619.63	3.10	
LB_BCO_30_1	2402	(132.43)	15073.67	0.04	48.23	8971.46	0.09	271.74	5459.97	9.79	143.91	1.68	
LB_BCO_35_1	4213	(173.42)	17762.29	0.08	350.10	11388.41	0.22	(62.71)	9463.72	31.58	(42.74)	2.46	

The results demonstrate that TVP-XY significantly improves over TVP-HP and also remains the overall winner in these experiments as it is the only formulation that Gurobi could solve to optimality for all of the displayed instances. In particular, the employed branch-and-cut approach leads to only slightly increased times to solve the relaxation of TVP-XY despite the larger set of constraints taken into account. Compared to TVP-XY, the upper bounds provided by the LP relaxation of TVP-XYB are even again significantly better, but the solution times remain high. TVP-XYBR then achieves a tremendous reduction of these relaxation solution times, which is however still not sufficient to translate into a superior ILP performance on the instance set considered. The competitiveness of TVP-XYBR might still be improved by tuning the separation procedures as well as the selection of cutting planes, or by employing further cutting planes, but this is out of the scope of this paper.

² We use the optimal value as a reference to compute the gap, rather than the best feasible solution found, because the major difficulty in the solution process is to improve the upper bound. As opposed to that, Gurobi found good and optimal solutions relatively quickly even though we did not implement any problem-specific heuristic. Using the lower bounds instead may thus lead to misleading values in the rare cases where the best solution found at the occasion of the time limit was not yet (near-)optimal.

Finally, we report in Table 2 optimal values of previously unsolved instances that we could determine using longer runs with TVP-XY.

■ **Table 2** Optimal solution values for four instances that remained unsolved in [5, 6].

Instance	OPT
LB_CFO_45_1	19877
LB_CFO_45_2	1082
LD_CFO_45_2	133647
LB_MCO_45_1	23301

6 Conclusion

In this paper, we have shown that the seminal integer programming formulations for the TVP based on AHPP, LOP, and asymmetric betweenness variables, can be refined to exclusively employ linear constraints which participate in a minimal description of the polytope that is defined by the convex hull of their feasible solutions. We then demonstrated that this leads to provably strengthened continuous relaxations as well as improved computational results when the reformulations are solved with a branch-and-cut algorithm. Further, our investigations revealed that results on the structure of the polytope associated with the formulation that is extended with asymmetric betweenness variables needed to be revised. On the one hand, this is motivated because the vertices of the respective polytope considered in the seminal works [5, 6] are not in one-to-one correspondence with the feasible solutions to the TVP. On the other, we found inconsistencies in the description of this polytope by means of a minimal set of linear constraints. Addressing this, we found a new set of equations which is valid if there are exactly four sites, and we clarified the dimension as well as the set of facet-defining inequalities in this case. Moreover, we proved that the previously assumed closed form to compute the dimension holds true when there are at least five sites and that the classical in- and out-degree inequalities from the AHPP are then always facet-defining.

The so far established results build a fundament for further research regarding polyhedral relaxations for the TVP. Particularly, it would be expedient to achieve further insights about the facial structure of the polytopes associated with the TVP with at least five sites, also because the knowledge of further classes of facet-defining inequalities may allow for a further strengthening of branch-and-cut algorithms for the TVP.

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