

Social Networks Through Time: Completeness of Temporal Network Logic

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Abstract

The framework of Temporal Network Logic (TNL) – first introduced in [29] and further developed in [30, 28] – was construed to model how a social network of interacting agents changes through time, with a language that could reflect both agents’ actions, such as posting information and following each other, as well as temporal operators. This framework allows, among other things, to detect social bots, by modelling given bot behaviours as logical formulas and checking these formulas against a given model. While such applications were the main focus in [29, 30, 28], the complete axiomatic system of this language has remained an open problem. In this paper, we focus on new logic-technical results and give a complete axiomatisation for TNL as well as a partial completeness result for its hybrid extension. The proofs are nonstandard and mix techniques from temporal and hybrid logics with “orthogonal” models.

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1 Introduction

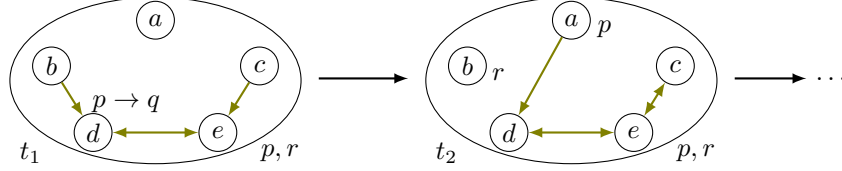
Temporal logics and multi-agent systems go hand in hand. Since the introduction of Alternating-time Temporal Logic (ATL) [1], much of the work in logics for multi-agent systems has been devoted to analysing how agents, their actions, strategies, and relations change through time. By multi-agent systems, we mean systems of multiple autonomous agents that may have diverging interests or diverging information but can interact, following [37], among others. By temporal logic, we mean logical frameworks that include modal operators such as *Past*, *Future*, *neXt-time*, *Yesterday*, *Since* or *Until* to reason about time. Temporal logics for multi-agent systems, in addition to ATL, include Epistemic-Temporal Logic (ETL) [26], for reasoning about agents’ knowledge of observable events through histories in time; Coalition Logic (CL) [27], used to model game-theoretic strategies of groups of agents;¹ and Strategy Logic (SL) [25] where strategies are treated as their own notions in the language, in contrast to ATL and CL where strategies are embedded in temporal operators. The BDI-architecture [31, 32] for modeling beliefs, desires and intentions in agents must also be mentioned as early work predating ATL in temporal reasoning of multi-agent systems.

A paradigm at the intersection of temporal logic and multi-agent systems concerns temporal logics to reason about *social networks*, which are graphs where the nodes represent agents and the relation between them represents a social relation (“follows”, “is friends

¹ It was proved in [20] that CL is in fact a fragment of ATL with only *neXt-time* as its temporal operator.



with”, etc.). Work in the field of temporal logics for social networks includes papers such as [24, 9, 16] and the recent work on *Temporal Network Logic (TNL)*, first introduced in [29] and further developed in [30, 28]. TNL is a temporal logic where the formulas are interpreted on a linear timeline with a beginning and no end. Within each time point, there is a social network, which lets us model how a network of agents changes through time. See Figure 1, taken from [28], for an example.



■ **Figure 1** A temporal network model.

This figure depicts the two first points of a temporal network model, modelling how a network of agents named from a to e develops from one step to another. The interpretation is that the model continues to the infinite. Agents can follow each other, denoted by an arrow: we can observe that, for instance, agent b follows agent d at t_1 . Agents can also post information in the network where the information posted is itself a logical formula. In the figure, posts are denoted by formulas next to the agent: for example, note that agent d posted $p \rightarrow q$ at t_1 . Propositional atoms are universally true or false at each time point, denoted in the figure by atoms on the right of the time point. Note that the absence of an atom means that its negation holds at that time point. The intuitive reading of the model is that it records snapshots of what is true in a network at given times; this means that we interpret posts that are there in one point but gone in the next as deleted from an agent’s profile. Similarly, we interpret that if an agent follows another in one point but not in the next, it has unfollowed the other. As an example, we observe that agent d has posted false information in t_1 , but deleted it in the next step: at t_1 agent d posted $p \rightarrow q$ while p is true but q is false, and at t_2 the post is gone.

The original work on TNL was developed to propose model checking as a detection method of *social bot* behaviour in temporal network models. Social bots are computer programs that act like humans on social media platforms and have been associated with certain behaviours such as sharing and amplifying misinformation online [36]. The proposal in [29] was to use the logical language of TNL to formalise bot behaviour; the global model checking problem was then interpreted as checking at what time points the behaviour occurred in the social network. In [30, 28], the language was extended with nominals and operators from hybrid logic [4]. In this extended hybrid-logical framework, formulas are checked against pairs of agents and time points, which meant that the output of the global model checking algorithm gave not only when, but also which agents in the network exhibited bot behaviour. For the most thorough explanation of TNL and its model checking procedure, we advise the reader to turn to [28, Chapter 4]. In the rest of the paper, we refer to this work for the original work on TNL.

This paper is about completeness of TNL, which until now has been an open problem, both in its standard form (for which we offer a complete axiomatics) and its extensions (for which we have partial completeness results). Proving completeness of the extended framework, including both temporal and “social” operators in the language, presents particular challenges, requiring techniques from *orthogonal frames* [6, 19] and *fusion logics* [14, 23]; we show completeness of a simplified bidimensional socio-temporal framework and highlight some of these difficulties towards eventually proving completeness of the full hybrid extension.

The paper is structured as follows. We begin in Section 2 by presenting Temporal Network Logic. We give an axiomatic system and show that it is weakly complete with respect to models of TNL. In Section 3 we introduce the hybrid extension of TNL, called Temporal Network Agent Logic (TNAL) and discuss steps towards proving completeness. These steps include the presentation of an intermediate logical framework that lies between TNL and TNAL, and a sketch to prove completeness for this logic. We conclude by sketching on-going work on an alternative framework with dynamic modalities instead of temporal operators.

2 Temporal Network Logic

We begin by presenting the syntax and semantics of Temporal Network Logic before introducing its axiomatic system.

2.1 Syntax and Semantics

► **Definition 1** (Syntax of TNL [28]). *Let $Prop$ be a finite set of atomic propositions and A be a finite set of agents. We define the well-formed formulas of the language of TNL, L_{TNL} , to be generated by the following grammar:*

$$\begin{aligned}\varphi &::= p \mid f_{a,b} \mid p_{a,\omega} \mid \neg\varphi \mid (\varphi \wedge \psi) \mid G\varphi \mid H\varphi \mid X\varphi \\ \omega &::= p \mid \neg\omega \mid (\omega \wedge \omega)\end{aligned}$$

where $p \in Prop$ and $a, b \in A$. We define propositional connectives like $\vee, \rightarrow$ and the formulas $\top, \perp$ as usual. Further, we define the duals $F := \neg G \neg$ and $P := \neg H \neg$.

Let Ω be the Boolean fragment of the language (i.e. the set of formulas given recursively by $\omega ::= p \mid \neg\omega \mid (\omega \wedge \omega)$). Intuitively, we read $f_{a,b}$ as “agent a follows agent b ”, $p_{a,\omega}$ as “agent a posted ω ” or “the post ω is on agent a ’s profile”. Note that an agent can only post information encoded in the Boolean fragment of the language. We read $G\varphi$ as “ φ is always the case in the future”, $H\varphi$ as “ φ is always the case in the past” and $X\varphi$ as “ φ is the case in the next time point”. The models of TNL are meant to emulate a discrete timeline with a beginning and no end; they are thus built from the natural numbers and their usual order $\mathbb{N}$.

► **Definition 2** (Temporal Network Frame and Model [28]). *A temporal network model is a tuple $M = (A, \mathbb{N}, <, (F_t)_{t \in \mathbb{N}}, (P_t)_{t \in \mathbb{N}}, V)$, where:*

- A is a finite set of agents;
- $(\mathbb{N}, <)$ is the set of natural numbers in their strict natural order;
- $P_t : A \rightarrow 2^\Omega$ is a function stating what each agent posts at each time $t \in \mathbb{N}$;
- $F_t \subseteq A^2$ is a “following” relation on the set of agents at each time $t \in \mathbb{N}$;
- $V : Prop \rightarrow 2^\mathbb{N}$ is a valuation.

We also define a pointed temporal network model (M, t) where M is a temporal network model and $t \in \mathbb{N}$ its distinguished point, at which the evaluation takes place.

We can now define the semantics of TNL.

► **Definition 3** (Semantics of TNL [28]). *Let $M = (A, \mathbb{N}, <, (F_t)_{t \in \mathbb{N}}, (P_t)_{t \in \mathbb{N}}, V)$ be a temporal network model on $\mathbb{N}$, $t \in \mathbb{N}$ be a time point, $a, b \in A$ be agents, and $\omega \in \Omega$ be a post. We inductively define the truth conditions for TNL as follows:*

$$\begin{aligned}
M, t &\models p \text{ iff } t \in V(p) \\
M, t &\models f_{a,b} \text{ iff } F_t ab \\
M, t &\models p_{a,\omega} \text{ iff } \omega \in P_t(a) \\
M, t &\models \neg\varphi \text{ iff } M, t \not\models \varphi \\
M, t &\models \varphi \wedge \psi \text{ iff } M, t \models \varphi \text{ and } M, t \models \psi \\
M, t &\models G\varphi \text{ iff } \forall s \in \mathbb{N} \text{ such that } t < s : M, s \models \varphi \\
M, t &\models H\varphi \text{ iff } \forall s \in \mathbb{N} \text{ such that } s < t : M, s \models \varphi \\
M, t &\models X\varphi \text{ iff } M, t+1 \models \varphi, \\
&\text{where } t+1 \text{ is the immediate successor of } t
\end{aligned}$$

2.2 Completeness

We continue by presenting the axiomatic system TNL and show that it is weakly complete with respect to temporal network frames on $\mathbb{N}$. As shown in [28], TNL is not compact and so therefore no axiomatisation for it can be strongly complete.

The logic TNL has the axioms and rules featured in Table 1.²

■ **Table 1** The axiomatic system TNL. Recall that $F := \neg G\neg$ and $P := \neg H\neg$.

(Taut)	All propositional tautologies	(HG)	$\varphi \rightarrow HF\varphi \wedge GP\varphi$
(K _G)	$G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$	(wc _G)	$G(G\varphi \rightarrow \psi) \vee G(G\psi \rightarrow \varphi)$
(K _X)	$X(\varphi \rightarrow \psi) \rightarrow (X\varphi \rightarrow X\psi)$	(wc _H)	$H(H\varphi \rightarrow \psi) \vee H(H\psi \rightarrow \varphi)$
(sd)	$\neg X\varphi \leftrightarrow X\neg\varphi$	(T _G)	$G\varphi \rightarrow GG\varphi$
(XG)	$G\varphi \leftrightarrow XG\varphi \wedge X\varphi$	(T _H)	$H\varphi \rightarrow HH\varphi$
(ind)	$X\varphi \wedge G(\varphi \rightarrow X\varphi) \rightarrow G\varphi$	(XH)	$XH\varphi \leftrightarrow \varphi \wedge H\varphi$
(K _H)	$H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$	(0 _H)	$H\perp \vee PH\perp$
(MP)	From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$, infer $\vdash \psi$.		
(Nec)	From $\vdash \varphi$, infer $\vdash X\varphi, \vdash G\varphi$ and $\vdash H\varphi$.		

Given a formula φ in the language of TNL we write $\models \varphi$ to denote that $M, t \models \varphi$ for every model M and every $t \in \mathbb{N}$; we write $\vdash_{\text{TNL}} \varphi$ (or simply $\vdash \varphi$) to denote that φ is derivable in the logic TNL. In this section we shall state:

► **Theorem 4** (Soundness and Weak Completeness). *For every formula φ in the language, $\models \varphi$ iff $\vdash \varphi$.*

Soundness ($\vdash \varphi$ implies $\models \varphi$) is routine; we shall devote the rest of this section to prove completeness ($\models \varphi$ implies $\vdash \varphi$). For this, it suffices to prove that any formula which is *consistent* (i.e. whose negation is not derivable) is made true in some temporal network model on $\mathbb{N}$ (for then if $\models \varphi$, it must be the case that $\vdash \varphi$, for otherwise its negation would be consistent and thus satisfied in some model).

The completeness proof we present in the sequel is non-standard: it is based on the completeness proof for *Temporal Logic with NeXt-time* [18], albeit we diverge from it substantially to offer a more simplified argument. The basic skeleton of this proof is as follows: starting from a consistent formula φ_0 , we take a finite set $\bar{T}$ of finite consistent sets of formulas, one

² Some of the axioms in Table 1 are redundant and included here for expositional clarity. In particular, the axioms T_G, T_H, wc_G, wc_H are derivable from the others (same proof as [18, Ch. 9]).

of which containing φ_0 , and we arrange the elements of $\bar{T}$ into a sequence resembling the natural numbers. After using this sequence for defining an appropriate valuation, we build a model $(\mathbb{N}, <, V)$ which we show satisfies all the formulas appearing in $\bar{T}$.

2.2.1 Maximal Consistent Sets

We say a set x of formulas in the language of TNL is *consistent* if there are no formulas $\varphi_1, \dots, \varphi_n \in x$ such that $\vdash_{\text{TNL}} \varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \perp$. A formula φ is *consistent* if the set $\{\varphi\}$ is consistent. A *maximal consistent set* is a consistent set having no consistent supersets.

The proof of the following result is standard and can be found in, e.g., [10].

► **Lemma 5.**

- (1) *Given a TNL-consistent set y , there exists some TNL-maximal consistent set x such that $y \subseteq x$.*
- (2) *A TNL-maximal consistent set x enjoys the following properties: (i) For all $\varphi \in L_{\text{TNL}}$, either $\varphi \in x$ or $\neg\varphi \in x$ (but not both); (ii) $\varphi \wedge \psi \in x$ iff $\varphi, \psi \in x$; (iii) $\text{TNL} \subseteq x$.*

Let us now fix a TNL-consistent formula φ_0 ; let us find a model for it in order to prove completeness. Let us fix as well some maximal consistent set x_0 such that $\varphi_0 \in x_0$, whose existence is guaranteed by the above Lemma. Let MCS be the set of TNL-maximal consistent sets; in it, we define the relation $x < y$ iff $\{\varphi : G\varphi \in x\} \subseteq y$; we then consider the set

$$T = \{y \in \text{MCS} : x_0 = y \text{ or } x_0 < y \text{ or } y < x_0\}.$$

► **Lemma 6.**

- (i) $x < y$ iff $\{\varphi : H\varphi \in y\} \subseteq x$.
- (ii) $<$ is transitive and connected (when restricted to T).
- (iii) If $x \in T$ and $G\theta \notin x$, then there is some $y \in T$ such that $x < y, \theta \notin y$.
- (iv) If $x \in T$ and $H\theta \notin x$, then there is some $y \in T$ such that $y < x, \theta \notin y$.
- (v) If $x \in T$, then the set $x^+ := \{\theta : X\theta \in x\}$ is also in T ; moreover $x < x^+$.

Proof. The proof of this result is quite standard; we sketch it here. Item (i) is proven using the (HG) axiom, noting that $\neg\varphi \in x < y$ implies $P\neg\varphi \in y$ implies $H\varphi \notin y$. For item (ii), transitivity is a consequence of (T_G) ; for connectedness, one first uses the (wc_G) and (wc_H) axioms to prove *weak connectedness* in both directions:

if either $y > x < z$ or $y < x > z$, then either $y < z$ or $z < y$ or $y = z$;

connectedness follows since all elements in T are related to x_0 . For item (iii) (resp. (iv)), one proves consistency of the set $\{\varphi : G\varphi \in x\} \cup \{\neg\theta\}$ (resp. $\{\varphi : H\varphi \in x\} \cup \{\neg\theta\}$) and extends it to some $y \in \text{MCS}$. For item (v), one uses the (sd) axiom to show that $\vdash \varphi \rightarrow \neg\psi$ implies $\vdash X\varphi \rightarrow \neg X\psi$, whence consistency follows; maximality is also a consequence of (sd) . ◀

2.2.2 The Filtrated Tree

Building a TNL model for φ_0 requires transforming the tree-like structure $(T, <)$ into a linear order isomorphic to $\mathbb{N}$. A way to achieve this is to construct a sequence by picking out elements from some finite set $\bar{T}$, which we will obtain by *filtrating* our tree T .³

³ For other uses of this technique, see, e.g. [18, 2].

Let us consider the set of formulas

$$\Delta := \text{subf } \varphi_0 \cup \{XG\theta, X\theta : G\theta \in \text{subf } \varphi_0\} \cup \{XH\theta : H\theta \in \text{subf } \varphi_0\} \cup \{H\perp, \perp\},$$

where $\text{subf } \varphi_0$ denotes the set of subformulas of φ_0 . Note that Δ is a finite set closed under subformulas.

We filtrate T through Δ in order to obtain the finite set

$$\bar{T} := \{\bar{x} : x \in T\}, \text{ where } \bar{x} := x \cap \Delta,$$

and we define two relations on $\bar{T}$ as follows:

$$\bar{x} <^+ \bar{y} \text{ iff, for some } z \in T, \bar{z} = \bar{x} \text{ and } \bar{z}^+ = \bar{y}.$$

$$\bar{x} <^* \bar{y} \text{ iff there exists some path } \bar{x} = \bar{x}_0 <^+ \dots <^+ \bar{x}_n = \bar{y}.$$

Finally, given $x \in T$, let

$$\delta_x := \bigwedge_{\theta \in \Delta \cap x} \theta \wedge \bigwedge_{\theta \in \Delta - x} \neg \theta.$$

be the *characteristic formula of x* . Note that, by definition, $\delta_x \in z$ iff the formulas of Δ that appear in x and in z are the same, i.e. iff $\bar{x} = \bar{z}$.

► **Lemma 7.**

- (1) If $x < y$, then $\bar{x} <^* \bar{y}$.
- (2) $<^*$ is connected and transitive.

Proof.

- (1) Let $x < y$; let us see that $\bar{x} <^* \bar{y}$. For this we consider the formula $\delta := \bigvee_{\bar{x} <^* \bar{z}} \delta_z$: note that $\delta \in y$ iff $\bar{x} <^* \bar{y}$. We prove that $G(\delta \rightarrow X\delta) \in x$: indeed, suppose not. Then there is some $z > x$ with $\delta \rightarrow X\delta \notin z$. Then $\delta \in z$ and $X\delta \notin z$. The fact that $\delta \in z$ implies that $\bar{x} <^* \bar{z}$. But then consider $\bar{z}^+$: since it also holds that $\bar{x} <^* \bar{z}^+$, then $\delta \in \bar{z}^+$ which implies $X\delta \in z$: a contradiction. We also note that $\delta \in x^+$ (because $\bar{x} <^* \bar{x}^+$) and thus $X\delta \in x$. But since $G(\delta \rightarrow X\delta) \in x$ and $X\delta \in x$, we may apply the induction axiom to conclude $G\delta \in x$, which implies $\delta \in y$, which implies $\bar{x} <^* \bar{y}$.
- (2) Transitivity: if there exists a $<^+$ -chain from $\bar{x}$ to $\bar{y}$, and one from $\bar{y}$ to $\bar{z}$, then there is one from $\bar{x}$ to $\bar{z}$. Connectedness: if $x \neq y$, then either $x < y$ or $y < x$, and thus by (1), $\bar{x} <^* \bar{y}$ or $\bar{y} <^* \bar{x}$. ◀

► **Lemma 8.** Let $\bar{x} <^* \bar{y}$.

- (1) If $G\theta \in \bar{x}$, then $G\theta, \theta \in \bar{y}$.
- (2) If $H\theta \in \bar{y}$, then $H\theta, \theta \in \bar{x}$.

Proof.

- (1) By induction on the length of the chain $\bar{x} <^+ \dots <^+ \bar{y}$. If this chain is of length 1, we have $\bar{x} <^+ \bar{y}$; this means that there is some $z \in T$ with $z \cap \Delta = x \cap \Delta, z^+ \cap \Delta = y \cap \Delta$. Since $G\theta \in z < z^+$, it follows that $\theta \in z^+$; since $\vdash G\theta \rightarrow GG\theta$, it follows that $GG\theta \in z$ and thus $G\theta \in z^+$; but then we have $\theta, G\theta \in z^+ \cap \Delta = \bar{y}$. Now, assume it is true for chains of length $n - 1$ and let $\bar{x} <^+ \dots <^+ \bar{x}_{n-1} <^+ \bar{y}$; we simply apply our induction hypothesis to obtain $G\theta \in \bar{x}_{n-1}$ and we are back to the length 1 case.
- (2) Analogous to (1). ◀

2.2.3 Initial and Final Elements

We say an element $\bar{x} \in \bar{T}$ is *final* iff $\bar{y} <^* \bar{x}$ for all $\bar{y} \neq \bar{x}$; we say it is *initial* iff $\bar{x} <^* \bar{y}$ for all $\bar{y} \neq \bar{x}$. The fact that $<^*$ is a connected and transitive relation on a finite set guarantees the existence of both initial and final elements. These elements need not be unique, except in the case of the initial element:

► **Lemma 9.** $\bar{x} \in \bar{T}$ is initial iff $H\perp \in \bar{x}$. As a consequence, there is only one initial element.

Proof. From left to right, let $H\perp \in x$ and let $\bar{y} \neq \bar{x}$. Suppose towards contradiction that $\bar{x} \not<^* \bar{y}$. Then by connectedness it holds that $\bar{y} <^* \bar{x}$ and thus $\perp \in \bar{y}$ by Lemma 8, against consistency. From right to left, suppose $\bar{x} <^* \bar{y}$ for all $\bar{y} \neq \bar{x}$. But then in particular if we take some $x_1 \in T$ with $H\perp \in x_1$, it holds that $\bar{x}_1 \neq \bar{x}$, and thus $\bar{x} <^* \bar{x}_1$, which in turn implies $\perp \in x$: contradiction. There cannot be two initial elements because if $\bar{x} \neq \bar{y}$ are both initial, then we would have $\bar{x} <^* \bar{y}$ and, since $H\perp \in y$, by Lemma 8 we obtain $\perp \in \bar{x}$: a contradiction. ◀

The final elements need not be unique. Let us show however that we can put them all in a very convenient order.

► **Definition 10.** An *F-path* is a path of the form $\bar{z}_1 <^+ \dots <^+ \bar{z}_q$ satisfying $\bar{z}_q <^+ \bar{z}_1$ and wherein all (and only) final elements appear.

► **Lemma 11.** There exists some *F-path*.

Proof. Let $\bar{z}_1, \dots, \bar{z}_q$ be the final elements in $\bar{T}$. Since all are final, we have

$$\bar{z}_1 <^* \dots <^* \bar{z}_q <^* \bar{z}_1,$$

ensuring the existence of a $<^+$ -path as described above. ◀

2.2.4 The Final Model

Recall φ_0 is a consistent TNL formula and that $x_0 \in T$ is a MCS such that $\varphi_0 \in \bar{x}_0$. Let $\bar{y}_0$ be the initial element, and let $\bar{z}_1 <^+ \dots <^+ \bar{z}_q$ be some *F-path* containing all and only final elements. Since $\bar{y}_0 <^* \bar{x}_0 <^* \bar{z}_1$, there is some sequence

$$\bar{y}_0 <^+ \bar{y}_1 <^+ \dots <^+ \bar{y}_n = \bar{x}_0 <^+ \bar{x}_1 <^+ \dots <^+ \bar{x}_m = \bar{z}_1.$$

We shall construct an infinite sequence consisting of the above path concatenated with infinite repetitions of the *F-path*:

$$\bar{y}_0 <^+ \dots <^+ \bar{x}_0 <^+ \dots <^+ \bar{z}_1 <^+ \dots <^+ \bar{z}_q <^+ \bar{z}_1 <^+ \dots <^+ \bar{z}_q <^+ \bar{z}_1 <^+ \dots$$

More explicitly, the sequence in question can be written as $(\bar{a}_t)_{t \in \mathbb{N}}$ where

$$\bar{a}_t = \begin{cases} \bar{y}_t & \text{if } t = 0, \dots, n; \\ \bar{x}_{t-n} & \text{if } t = n+1, \dots, n+m; \\ \bar{z}_j & \text{iff } t > m+n \text{ and } t - (m+n) \equiv j \pmod{q} \end{cases}$$

Note that, by construction, this sequence satisfies for all $t \in \mathbb{N}$:

- i. $\bar{a}_t <^+ \bar{a}_{t+1}$,
- ii. $\bar{a}_t <^* \bar{a}_n$ whenever $t < n$,
- iii. for all final $\bar{z}$, there is some $n > t$ such that $\bar{a}_n = \bar{z}$,
- iv. the only initial element is $\bar{a}_0$.

Now we have to make $\mathbb{N}$ into a model. We define $V(p) = \{t \in \mathbb{N} : p \in \bar{a}_t\}$. For $t \in \mathbb{N}, a \in A$, we define the relation and set:

$$F_t = \{(a, b) : f_{a,b} \in a_t \cap \Delta\}, P_t(a) = \{\omega \in \Omega : p_{a,\omega} \in a_t \cap \Delta\}.$$

Let $M = (A, \mathbb{N}, <, V, (P_t)_{t \in \mathbb{N}}, (F_t)_{t \in \mathbb{N}})$. We shall prove that this is the model we are looking for; but let us first make an observation about the sequence $(\bar{a}_t)_{t \in \mathbb{N}}$:

► **Lemma 12** (Filtrated Existence Lemma).⁴ Let $t \in \mathbb{N}$.

- (1) If $X\theta \in \Delta$, then $X\theta \in \bar{a}_t$ iff $\theta \in \bar{a}_{t+1}$.
- (2) If $G\theta \in \Delta - \bar{a}_t$, then $\theta \notin \bar{a}_s$ for some $s > t$.
- (3) If $H\theta \in \Delta - \bar{a}_t$, then $\theta \notin \bar{a}_s$ for some $s < t$.

Proof.

- (1) Let $X\theta \in \Delta$. Since by construction $\bar{a}_t <^+ \bar{a}_{t+1}$, we have that there is some $z \in T$ such that $\bar{z} = \bar{a}_t$, $\bar{z}^+ = \bar{a}_{t+1}$. But then $X\theta \in \bar{a}_t$ iff $X\theta \in \bar{z}$ iff $\theta \in \bar{z}^+$ iff $\theta \in \bar{a}_{t+1}$.
- (2) Let $G\theta \in \Delta$, $G\theta \notin \bar{a}_t$; let us assume towards contradiction that $\theta \in \bar{a}_s$ for all $s > t$. We can prove by a simple induction that, in this case, $G\theta \notin \bar{a}_s$ for all $s > i$: indeed, note that $X\theta \wedge XG\theta \rightarrow G\theta$ is derivable by the (XG) axiom, and since $G\theta \notin \bar{a}_t$ and $X\theta \in \bar{a}_t$ (the latter because $\theta \in \bar{a}_{t+1}$ by hypothesis), then we must have $XG\theta \notin \bar{a}_t$, and thus $G\theta \notin \bar{a}_{t+1}$; by repeating this process taking $t+1$ instead of i , we obtain $G\theta \notin \bar{a}_{t+2}$, and so on. But then take some $m > t$ such that $\bar{a}_m$ is final (such an m exists by construction of the sequence): since $G\theta \notin \bar{a}_m$, we have $G\theta \notin a_m$, which entails there is some $y > a_m$ with $\theta \notin y$. But, since (by Lemma 7) $\bar{a}_m <^* \bar{y}$, it must then be that $\bar{y}$ is final and thus by construction there is some $s \geq m$ with $\bar{y} = \bar{a}_s$, and thus we have $t < s$ and $\theta \notin \bar{a}_s$, in contradiction with our hypothesis.
- (3) Let $H\theta \in \Delta$, $H\theta \notin \bar{a}_i$; note that we must have $i > 0$, for $\bar{a}_0$ is an initial element, which means by Lemma 9 that $H\perp \in \bar{a}_0$, which implies $H\theta \in \bar{a}_0$, noting that $H\perp \rightarrow H\theta$ is derivable. Let us assume towards contradiction that $\theta \in \bar{a}_j$ for all $j < i$. In particular, $H\theta, \theta \in \bar{a}_0$, which combined with the (XH) axiom gives us $XH\theta \in \bar{a}_0$. But this means $H\theta \in \bar{a}_1$, and since $\theta \in \bar{a}_1$ by hypothesis, we obtain with the same reasoning $H\theta \in \bar{a}_2$. Repeating this process i times we reach the contradiction $H\theta \in \bar{a}_i$. ◀

► **Proposition 13** (Truth Lemma). For any formula $\varphi \in \Delta$ and any $t \in \mathbb{N}$,

$$M, t \models \varphi \text{ iff } \varphi \in \bar{a}_t.$$

Proof. Induction on φ . The cases $\varphi = p, \theta_1 \wedge \theta_2, \neg\theta$ are routine.

Cases $\varphi = f_{a,b}, \varphi = p_{a,\omega}$: by definition of F_t and $P_t(a)$.

Case $\varphi = X\theta$: By Lemma 12, $X\theta \in \bar{a}_t$ iff $\theta \in \bar{a}_{t+1}$ iff (by induction hypothesis) $t+1 \models \theta$ iff $t \models X\theta$.

Case $\varphi = G\theta$: If $G\theta \in \bar{a}_t$, then by Lemma 8 and noting that $\bar{a}_t <^* \bar{a}_s$ whenever $t < s$, we obtain $\theta \in \bar{a}_s$ for all $s > t$, which by hypothesis gives $s \models \theta$ for all $s > t$, and thus $t \models G\theta$. Conversely if $G\theta \notin \bar{a}_t$, then by Lemma 12, we have $\theta \notin \bar{a}_s$ for some $s > t$, and thus by induction $s \not\models \theta$ for some $s > t$, which implies $t \not\models G\theta$.

Case $\varphi = H\theta$: Analogous to the previous case: $H\theta \in \bar{a}_t$ implies $t \models H\theta$ by Lemma 8, and the converse holds by Lemma 12. ◀

This ends the proof of completeness, for we have started with a consistent formula $\varphi_0 \in x_0$ and we have proven that $M, t \models \varphi_0$, where $t \in \mathbb{N}$ is the index such that $\bar{a}_t = \bar{x}_0$.

⁴ A previous version of this proof was significantly more involved; proving this lemma allows us to greatly simplify the construction of the sequence used to build the valuation for the final model. We thank Rob Goldblatt for this idea.

3 Social Relations Through Time

In [28], a multi-dimensional extension of TNL is proposed, with a more expressive language which allows us to talk about the evolution of an agent's social network through time. This is achieved by moving the set of agents A to the semantics: formulas are now read with respect to a pair (a, t) , where a is an agent and t is a time point, thus we can add formulas to our language which say, e.g., “every follower of a will satisfy p at some future time”. In this section, we present *Temporal Network Agent Logic* (TNAL) as it appears in [28], discuss a possible completeness proof and the foreseeable challenges thereof, and make a first step towards such proof by giving the complete axiomatics for a (modified) fragment of it.

3.1 Temporal Network Agent Logic

► **Definition 14** (Syntax of TNAL [28]). *The language of Temporal Network Agent Logic (henceforth, TNAL) is L_{TNAL} , built from a countable set Prop of propositional variables, a countable set Nom of nominals, and a countable set SVar of state variables with the following grammar:*

$$\begin{aligned}\varphi &::= p \mid n \mid x \mid f_{n,n} \mid p_{n,\omega} \mid \neg\varphi \mid (\varphi \wedge \psi) \mid G\varphi \mid H\varphi \mid X\varphi \mid \Box\varphi \mid \Box^{-1}\varphi \mid @_n\varphi \mid \downarrow x.\varphi \\ \omega &::= p \mid \neg\omega \mid (\omega \wedge \omega),\end{aligned}$$

where $p \in \text{Prop}$, $x \in \text{SVar}$, $n \in \text{Nom}$.

A formula $\Box\varphi$ (with dual $\Diamond\varphi$) is meant to be read “ φ holds at all of the agents that the current agent is following”, whereas $\Box^{-1}\varphi$ can be interpreted as “all those who follow the current agent φ ”. In TNAL, formulas are evaluated at pairs of agents and time points; thus, the follower relation can be interpreted in the language with a modal box. The operators $@_n\varphi$ and $\downarrow x.\varphi$, standing respectively for “agent n φ ” and “the current agent φ ”, are borrowed from *hybrid logic* [11]: each *nominal* n is a *name* that will have to be assigned to an agent by the model, whereas each *state variable* x will also refer to an agent but this reference can be determined within the formula via the $\downarrow x.$ operator. Using this language, one can express things like “All of my followers follow n ” ($\Box\Diamond^{-1}n$), “ n and m follow each other” ($@_n\Diamond m \wedge @_m\Diamond n$), n follows the current agent $\downarrow x.@_n\Diamond x$.

► **Definition 15** (Models of TNAL [28]). *Temporal network agent models (henceforth TNAL-models) are tuples $M = (A, \mathbb{N}, <, (\rightsquigarrow_t)_{t \in \mathbb{N}}, (P_t)_{t \in \mathbb{N}}, V)$*

- A is a finite set of agents;
- $(\mathbb{N}, <)$ are the natural numbers in their usual order;
- for each $t \in \mathbb{N}$, $\rightsquigarrow_t$ is a “follow” relation on the set A ;
- for each t , $P_t : A \rightarrow 2^\Omega$ is a function;
- $V : \text{Prop} \cup \text{Nom} \rightarrow 2^{A \times \mathbb{N}}$ is a valuation function satisfying the following requirement: for each $t \in \mathbb{N}$, $n \in \text{Nom}$, there exists a unique $a \in A$ such that $(a, t) \in V(n)$.

Given $a \in A$, $t \in \mathbb{N}$ we say that $n \in \text{Nom}$ names a at time t if a is the unique agent such that $(a, t) \in V(n)$; we write $\text{name}_t(n) = a$ to express this.

► **Definition 16** (Semantics of TNAL [28]). *Formulas are evaluated with respect to a temporal network agent model $M = (A, \mathbb{N}, <, (\rightsquigarrow_t)_{t \in \mathbb{N}}, (P_t)_{t \in \mathbb{N}}, V)$ and a triple (a, t, g) , where $a \in A$, $t \in \mathbb{N}$ and $g : \text{SVar} \rightarrow A$ is an assignment function, as follows:*

$M, a, t, g \models p$ iff $(a, t) \in V(p)$;	$M, a, t, g \models \Box\varphi$ iff $M, b, t, g \models \varphi$ whenever $a \rightsquigarrow_t b$;
$M, a, t, g \models n$ iff $a = \text{name}_t(n)$;	$M, a, t, g \models \Box^{-1}\varphi$ iff $M, b, t, g \models \varphi$ whenever $b \rightsquigarrow_t a$;
$M, a, t, g \models f_{nn'}$ iff $\text{name}_t(n) \rightsquigarrow_t \text{name}_t(n')$;	$M, a, t, g \models G\varphi$ iff $\forall s > t : M, s, a, g \models \varphi$;
$M, a, t, g \models x$ iff $g(x) = a$;	$M, a, t, g \models H\varphi$ iff $\forall s < t : M, s, a, g \models \varphi$;
$M, a, t, g \models p_{n\omega}$ iff $\omega \in P_t(\text{name}_t(n))$;	$M, a, t, g \models X\varphi$ iff $M, a, t+1, g \models \varphi$;
$M, a, t, g \models \neg\varphi$ iff $M, a, t, g \not\models \varphi$;	$M, a, t, g \models @_n\varphi$ iff $M, \text{name}_t(n), t, g \models \varphi$;
$M, a, t, g \models \varphi_1 \wedge \varphi_2$ iff $M, a, t, g \models \varphi_i, i = 1, 2$;	$M, a, t, g \models \downarrow x.\varphi$ iff $M, a, t, g_a^x \models \varphi$,
$M, a, t, g \models x$ iff $g(x) = a$;	where $g_a^x(x) = a, g_a^x(y) = g(y)$ for $y \neq x$.

In [28], a model checking algorithm for the TNAL framework is employed to check formulas against pairs (a, t) in the model instead of just time points t as in the non-hybrid TNL. The output of the global model checking algorithm given in [28] is the set of all pairs of agents and time points where the given formula is true; in effect, it gives in addition to at what time, also which agents exhibited the behaviour, characterised by the formula. Finding the complete logic of TNAL models is a matter of ongoing investigation. In the following subsection we offer partial results, in the form of completeness of a simplified fragment of TNAL (taking us on a journey through transfer properties of fusion logics [14, 23] and orthogonal frames [6]), we explain why standard techniques like filtration do not quite work in this framework and highlight some of the most interesting technical challenges of this framework.

3.2 Time and Friendship: A Complete Logic

Let us present a fragment of the language TNAL containing simply the temporal and “follower” modality, and define models and a semantics for it. We shall call this framework *Temporal Social Logic*, or TSL. We have to make some concessions: first, with the set of agents A no longer being part of the syntax, and in the absence of nominals, we will have to work without the $p_{a,\omega}$ and $f_{a,b}$ atomic formulas. Secondly, our models no longer have a temporal beginning and will now be based on the temporal structure on $\mathbb{Z}$, for keeping $\mathbb{N}$ would present important technical challenges, as we will later highlight. This change may be problematic if one is conceptually committed to the beginning of time, but we argue it is a harmless change for most use cases. Thirdly, we do not demand that the set of agents A be finite; finding this framework’s version of the *finite model property* is still a pending task. We make one further concession: we consider *symmetric* social relations; by doing so, we model the simplest case wherein the “follower” relation acts more like a “friendship” relation and the $\Box^{-1}$ operator becomes redundant. We note that the proof below works the same way if one removes symmetry, by simply removing the (*sym* $_{\Box}$) axiom.

► **Definition 17** (Syntax and Semantics of TSL). *Given a countable set Prop of propositional variables, we let L_{TSL} be the set of formulas defined inductively as follows:*

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box\varphi \mid G\varphi \mid H\varphi \mid X\varphi,$$

with $p \in \text{Prop}$. Models for TSL are tuples $M = (A, \mathbb{Z}, <, (\rightleftharpoons_t)_{t \in \mathbb{Z}}, V)$, where A is a set of agents, $(\mathbb{Z}, <)$ is the set of integers in its natural order, $\rightleftharpoons_t \subseteq A^2$ is a symmetric relation for each t , and $V : \text{Prop} \rightarrow 2^{A \times \mathbb{Z}}$ is a valuation.

Formulas of L_{TSL} are read on a model M with respect to a pair $(a, t) \in A \times \mathbb{Z}$ as follows:

$M, a, t \models p$ iff $(a, t) \in V(p)$;	$M, a, t \models G\varphi$ iff $M, a, s \models \varphi$ for all $s > t$;
$M, a, t \models \varphi_1 \wedge \varphi_2$ iff $M, a, t \models \varphi_i$ for $i = 1, 2$;	$M, a, t \models H\varphi$ iff $M, a, s \models \varphi$ for all $s < t$;
$M, a, t \models \neg\varphi$ iff $M, a, t \not\models \varphi$;	$M, a, t \models X\varphi$ iff $M, a, t+1 \models \varphi$;
	$M, a, t \models \Box\varphi$ iff $M, b, t \models \varphi$ for all $b \rightleftharpoons_t a$.

We shall show that the complete logic for this framework is simply the fusion of the logics corresponding to the “temporal” and “social” components of the framework. We thus further divide the language into two fragments, only to then combine the resulting logics.

3.2.1 The Temporal Fragment

The *temporal fragment* of this language, L_T , is comprised of formulas of the shape

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg \varphi \mid G\varphi \mid H\varphi \mid X\varphi.$$

Note that this, minus the $p_{a,\omega}$ and $f_{a,b}$ atoms, is almost identical to the language of TNL. And so are its models: a T-model is a structure $(\mathbb{Z}, <, V)$, where $V : Prop \rightarrow 2^{\mathbb{Z}}$ is a valuation and formulas are otherwise read as in Definition 3. The logic of these models based on $\mathbb{Z}$ is exactly like TNL minus the axiom establishing a beginning. Indeed, let $TNL_{\leftrightarrow}$ be the least normal L_T -modal logic having the same axioms and rules as TNL except (0_H) . We have:

► **Proposition 18.** $TNL_{\leftrightarrow}$ is the logic of T-models.

Proof. Very similar to the proof of completeness of TNL; see the Appendix for a sketch. ◀

3.2.2 The Social Fragment

The *social fragment* of this language is L_S , given by $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg \varphi \mid \Box \varphi$, with $p \in Prop$. Models for L_S (henceforth *S-models*) are of the shape $(A, \rightleftharpoons, V)$, where A is a set of agents and $\rightleftharpoons$ is a symmetric relation. The sound and complete logic of these models is the normal modal logic whose axioms are

$$(K_{\Box}) \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \quad (sym_{\Box}) \varphi \rightarrow \Box\Diamond\varphi,$$

with Modus Ponens (from φ and $\varphi \rightarrow \psi$, infer ψ) and Necessitation (from φ , infer $\Box\varphi$) as inference rules. Let us call this logic SL. The proof of the following result is standard in modal logic (see, e.g., [10]) and not included:

► **Proposition 19.** SL is the logic of (finite) S-models.

3.3 Combining Both Fragments

The sound and complete logic of TSL models is the fusion of the above logics, i.e., the least normal modal logic containing the axioms of TNL and SL, as shown in Table 2. Let us refer to this logic as TSL.

Let us prove this is indeed the complete logic of TSL models. For this, we will have to visit the proofs of some well-known transfer results for fusion logic; we start by describing a new class of frames:

► **Definition 20 (Sproutings).** A sprouting⁵ is a birelational model $(X, <, \rightleftharpoons, V)$ satisfying:

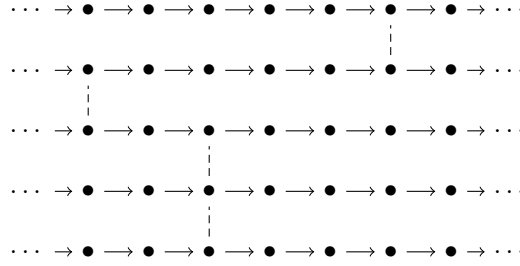
- (S1) Every connected component⁶ of the $<$ relation (henceforth a timeline) is a T-frame (i.e. isomorphic to $(\mathbb{Z}, <)$).
- (S2) Every connected component of the $\rightleftharpoons$ relation is an S-frame.

⁵ We borrow this nomenclature from Fine and Schurz [14].

⁶ Connected component: equivalence class of the least equivalence relation containing $<$.

■ **Table 2** The logic TSL.

(Taut)	All propositional tautologies	(wc _G)	$G(G\varphi \rightarrow \psi) \vee G(G\psi \rightarrow \varphi)$
(K _G)	$G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$	(wc _H)	$H(H\varphi \rightarrow \psi) \vee H(H\psi \rightarrow \varphi)$
(K _X)	$X(\varphi \rightarrow \psi) \rightarrow (X\varphi \rightarrow X\psi)$	(T _G)	$G\varphi \rightarrow GG\varphi$
(sd)	$\neg X\varphi \leftrightarrow X\neg\varphi$	(T _H)	$H\varphi \rightarrow HH\varphi$
(XG)	$G\varphi \leftrightarrow XG\varphi \wedge X\varphi$	(XH)	$XH\varphi \leftrightarrow \varphi \wedge H\varphi$
(ind)	$X\varphi \wedge G(\varphi \rightarrow X\varphi) \rightarrow G\varphi$	(K _□)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(K _H)	$H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$	(sym _□)	$\varphi \rightarrow \Box\Diamond\varphi$
(HG)	$\varphi \rightarrow HF\varphi \wedge GP\varphi$		
(MP)	From $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$, infer $\vdash \psi$		
(Nec)	From $\vdash \varphi$, infer $\vdash \Box\varphi, \vdash X\varphi, \vdash G\varphi$ and $\vdash H\varphi$		



■ **Figure 2** A finite sprouting with five timelines. The dashed lines represent the $\equiv$ relation.

(S3) Given a closed loop of timelines $t_0 \equiv t_1 \equiv \dots \equiv t_n \equiv t_0$, there exist unique elements $x_i \in t_i$ such that $x_0 \equiv \dots \equiv x_n \equiv x_0$, where we define

$$t \equiv t' \text{ iff } x \equiv x' \text{ for some } x \in t, x' \in t'.$$

Note that this entails that two timelines are connected by $\equiv$ at no more than one point, and, while points are allowed to be reflexive, no two points in the same timeline are connected by $\equiv$. A sprouting is said to be finite (resp. countable) if it has a finite (resp. countable) number of timelines. Given a sprouting $(X, <, \equiv, V)$, and given $x \in X$, let

$$t_x := \{y \in X : y < x \text{ or } y > x \text{ or } y = x\}$$

denote the timeline of x , and let

$$x^+ := \min\{y \in X : x < y\}, \quad x^- := \max\{y \in X : y < x\}$$

denote respectively the successor and predecessor of x . Note that, by definition, every point of X has a successor and predecessor, and $y \in t_x$ iff there is some n such that either $y = x^{+n}$ or $y = x^{-n}$ (where the $\pm n$ superindices denote the concatenation of the $\pm$ operators n times).

We read formulas of TSL on sproutings as follows:

► **Definition 21** (Semantics for Sproutings.). Given a sprouting $S = (X, <, \equiv, V)$ and formula φ in the language of TSL, we say φ is true at $x \in X$, denoted $S, x \models \varphi$, if:

$$\begin{array}{ll}
 S, x \models p & \text{iff } x \in V(p); \\
 S, x \models \neg\varphi & \text{iff } S, x \not\models \varphi; \\
 S, x \models \varphi \wedge \psi & \text{iff } S, x \models \varphi \text{ and } S, x \models \psi; \\
 S, x \models G\varphi & \text{iff } S, y \models \varphi \text{ for all } y > x; \\
 S, x \models H\varphi & \text{iff } S, y \models \varphi \text{ for all } y < x; \\
 S, x \models X\varphi & \text{iff } S, x^+ \models \varphi; \\
 S, x \models \Box\varphi & \text{iff } S, y \models \varphi \text{ for all } y \equiv x.
 \end{array}$$

As stated earlier, our goal here is to use these auxiliary structures in order to prove that TSL is the complete logic of TSL models. This will be done in two steps: first, we shall show that TSL is the logic of countable sproutings. Then we prove that countable sproutings and TSL models satisfy the same formulas, whence completeness follows.

Let us show these results in order. The first is already proven in the literature: it appears (albeit hidden) in the proof of a well-known result on fusion logics (see e.g. [14, 15, 23]).

► **Proposition 22** (TSL is the logic of sproutings). *If a formula φ is TSL-consistent, then there is a countable sprouting satisfying φ .*

Proof sketch. This result is proven in [14, 23]; see the Appendix for a proof sketch. ◀

All that remains to finish our completeness proof is to transform a countable sprouting into a regular TSL model.

► **Proposition 23** (Sproutings and TSL models satisfy the same formulas.). *If a formula φ_0 is satisfied on a countable sprouting, then there is some TSL model satisfying φ_0 .*

Proof. Let $S = (X, <, \rightleftharpoons, V)$ be a countable sprouting, and let $x_0 \in X$ such that $S, x_0 \models \varphi_0$. Let t_0 be the timeline with $x_0 \in t_0$. We may assume without loss of generality that, for any two timelines t_i and t_j , there is some path $t_i \rightleftharpoons \dots \rightleftharpoons t_j$ (for otherwise we could just take the submodel of S consisting of timelines reachable by t_0 , and a trivial induction shows this submodel also satisfies φ). We shall now take a function $f : X \rightarrow \mathbb{Z}$ that assigns an integer to each point in X satisfying the following properties, for all $x, y \in X$:

$$(F1) \ f(x^+) = f(x) + 1, f(x^-) = f(x) - 1; \quad (F2) \ x \rightleftharpoons y \text{ implies } f(x) = f(y).$$

In the Appendix, we show that:

► **Lemma 24.** *There is some function $f : X \rightarrow \mathbb{Z}$ satisfying (F1) and (F2).*

Take some such function f ; note that by (F1), the restriction of f to any timeline is an order isomorphism. For any timeline t and integer n , let $f_t^{-1}(n)$ denote the unique $x \in t$ such that $f(x) = n$. Finally, consider the tuple $M = (A, \mathbb{Z}, <, (\rightleftharpoons_n)_{n \in \mathbb{Z}}, V^*)$, where:

- A is the set of timelines of S ;
- For each $n \in \mathbb{Z}$, $t_i \rightleftharpoons_n t_j$ iff there are $x \in t_i, y \in t_j$ such that $x \rightleftharpoons y$ and $f(x) = f(y) = n$.
- $V^*(p) = \{(t, n) : f_t^{-1}(n) \in V(p)\}$.

This is a TSL model (indeed, each $\rightleftharpoons_n$ relation is symmetric); therefore the following Lemma (proven in the Appendix) finishes our proof:

► **Lemma 25.** *For any formula ψ , $S, x \models \psi$ iff $M, (t_x, f(x)) \models \psi$.*

This proves that M validates φ_0 at $(t_0, 0)$, as we desired to show. ◀

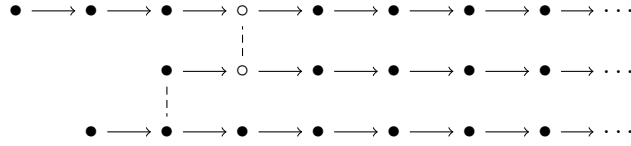
3.4 Towards a Complete Logic for TNAL

In the present and previous sections we have axiomatised fragments of L_{TNAL} ; finding a complete logic for the whole framework remains an open problem. It is not an easy problem: many parts of this framework are non-standard and require rather esoteric proof techniques. The proof we showed in this section illustrates some of these difficulties, while shedding light on some interesting properties of our models. We highlight some of these below:

Finite Agents. The TNAL framework is reasonably defined to have finitely many agents; in this first foray into its completeness proof we have allowed for up to countably many. The reason is that this framework is resistant to the usual technique of *filtration*: models for TNAL are infinite bidimensional models which are countably infinite in one dimension (the timeline) yet finite in the other (the agents); this is in contrast to frameworks such as [22], where the agents are defined at the syntax level. Now, of course, any filtration in our models would undesirably make the whole model finite, and any careless attempt to unravel our finite filtrated model back into a timeline risks adding back infinite agents. We do not know of many bidimensional frameworks with such a cardinality requirement. Some novel filtration technique will most likely have to be developed for this.

Timelines. Famously, models based on timelines require care in their completeness proofs, since a canonical model does not usually have the desired structure of a model: being countable (for the set), being irreflexive (for $<$) are not modally definable, and the functionality of the successor relation is usually not preserved through filtration. The technique of “pre-filtrating” the canonical model and constructing a sequence from the finite filtrated set (see [18], [2] and Section 2 of this paper) does not translate well into our bidimensional framework. For this technique to work, some non-trivial adaptations are in order.

The Beginning of Time. As we noted earlier, removing the requirement that our timelines have a beginning simplifies the proof. Here is how: Suppose we take the framework TSL_0 , defined exactly as TSL except with models based on $\mathbb{N}$. One would expect the logic of this framework to be the fusion of TNL with SL. However, if one tries to prove this using the same techniques in this section, one can easily get sproutings with the following kind of shape:



Our numbering trick from Prop. 23 does not work here: there is no way to number the elements in this sprouting so that (i) each timeline is properly numbered and (ii) the two white dots in the drawing have the same number assigned. It is certainly not straightforward to avoid these models wherein an agent finds herself at time t yet has a friend who finds himself at time $s \neq t$. This is not a problem in $\mathbb{Z}$, where given a discrete sequence infinite in both directions, we can take any element to be its zero, and adjust the numbering of its neighbour timelines accordingly. Is there some trick to redefine sproutings in the framework based on $\mathbb{N}$ so that this numbering works? We dare to conjecture there is not: a crucial difference between the sociotemporal models based on $\mathbb{Z}$ and those based on $\mathbb{N}$ is that, in the former, no meaningful theorems express any interaction between the social and temporal dimensions (recall that TSL has no axioms combining social and temporal modalities), whereas in the latter some meaningful multimodal formulas will be theorems of the logic: for instance, the formula $H\perp \rightarrow \Box H\perp$, expressing “if I am at time $t = 0$, then all my friends are at time $t = 0$ ” is clearly valid in TNAL (and thus any sociotemporal subframework thereof). We thus expect the logic of sociotemporal models based on $\mathbb{N}$ to include some interaction axioms between the modalities H, G, X on one hand and $\Box$ on the other. This would take us away from the theory of fusion logics and require a different proof technique. It is most

curious how this seemingly simple change in the structure of the models creates a significant difference in expressive power: in the models based on $\mathbb{N}$, our agents apparently have a way to keep track of time, something impossible in models based on $\mathbb{Z}$.⁷

Hybrid Operators. Languages with hybrid modalities have historically been resistant to proofs based on canonical models; see for example [34], where a completeness proof based on tree sequent calculus is given for Epistemic Friendship Logic [35]. A recent technique [7] involves considering maximal consistent sets of *named formulas*, i.e., formulas preceded by the $@_n$ modality; noting that the $@_n$ operator is self-dual ($\neg @_n \varphi \leftrightarrow @_n \neg \varphi$ is valid in TNAL), one has that, if x is a maximal consistent set and n is a fixed nominal, then $x_n = \{\varphi : @_n \varphi \in x\}$ is also a maximal consistent set. The aforementioned technique uses this fact to encode the name of an agent directly onto the canonical model.

4 Conclusion and Future Work

In this paper, we solved the open problem of completeness of Temporal Network Logic and discussed steps towards proving completeness of its hybrid extension Temporal Network Agent Logic. These steps included introducing the intermediate logic Temporal Social Logic as well as its completeness proof. The aim for future work is to build on techniques from the completeness proof for TSL to develop a proof for the full hybrid extension of TNL. As discussed in the previous section, this is a task that will likely require mixing techniques from fusion logics [14, 23], hybrid logics [11, 7], temporal logics [18, 2] and orthogonal frames [6], and perhaps coming up with a few novel ones.

Decidability remains an open problem; we conjecture TNAL to be undecidable due to the undecidability of the hybrid logic $\mathcal{H}(@, \downarrow)$ [3] and certain epistemic-temporal logics [22, 39].

In on-going work, we are also modelling the bot behaviour that one observes in specific multi-agent systems, not in temporal models as in this paper, but by using the techniques coming from Dynamic Epistemic Logic (DEL) [8, 40]. This allows us to model the dynamics of posting-events and (un)following social actions in tandem, using two types of events each of which comes with its own dynamic operators. Such models would be two-dimensional, similar to those that have operators involving both behaviour change and social network changes in [38, 5]. DEL is particularly well suited to make the dynamic mechanisms of this type of information change explicit in such multi-agent systems. Moreover, the DEL framework will make it easier to include also propositions about the epistemic information states of the agents in the language and to allow for ‘posts’ that go beyond the Boolean fragment of the logic. In DEL, learning a piece of incoming information from a highly trusted source or from a source that is considered to be unreliable, will lead to a different result in the updated information state of the learner. These information states can then feature as the precondition for an agent to follow or unfollow another agent in the network. Formally, such a setting will require us to model the temporal flow of the model as a sequence of step-wise generated models, each of which is the result of a specific event that acts as a model-transformer on the model of the previous step. Such sequences of event-generated models are typically only future oriented in standard DEL but extensions exist so that one can also look back into the past using the language and framework described in [33].

⁷ Note as well that a formula such as $P(p \wedge \neg q) \wedge P(q \wedge \neg p)$ can only be true at timepoints $t \geq 2$ in models based on $\mathbb{N}$, yet can be true anywhere in $\mathbb{Z}$!

Another venue for future work includes other extensions to the framework for reasoning about further phenomena on social networks. A path left to explore is to add agents' opinions to the network; this could let us model that certain posts by one agent could trigger opinion change in another. Existing work using logic to model opinion and opinion change in social networks includes [13, 12, 17, 21].

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A

 Appendix

Proof sketch of Prop. 18. This proof is almost identical to the proof of completeness for TNL in the Section 2, with the following changes:

- There is no longer a unique initial element in the filtrated tree $\bar{T}$. Instead, we have multiple initial elements $\bar{i}_1, \dots, \bar{i}_p$ and we can arrange them into an *I-path*, $\bar{i}_1 <^+ \dots <^+ \bar{i}_p$, where $\bar{i}_p <^+ \bar{i}_1$. The existence of such a path is proven just like in Lemma 11.
- Given $\bar{x}_0 \in \bar{T}$ such that $\varphi_0 \in \bar{x}_0$, we build a sequence of elements of $\bar{T}$ much like in Section 2.2.4, except it is now isomorphic to $\mathbb{Z}$, i.e. infinite in both directions: first we take a finite $<^+$ -path that goes from some initial element (say $\bar{i}_p$) to some final element (say $\bar{f}_1$), going through $\bar{x}_0$; we then plug a infinite repetition of an *F-path* to its right, and an infinite repetition of the *I-path* to the left:

$$\dots <^+ \bar{i}_p <^+ \underbrace{\bar{i}_1 <^+ \dots <^+ \bar{i}_p}_{I\text{-path}} <^+ \dots <^+ \bar{x}_0 <^+ \dots <^+ \underbrace{\bar{f}_1 <^+ \dots <^+ \bar{f}_q}_{F\text{-path}} <^+ \bar{f}_1 <^+ \dots$$

- We assign integers to the elements of this sequence, assigning 0 to $\bar{x}_0$ and numbering the rest according to their relative position to $\bar{x}_0$; we use this to build a valuation on $(\mathbb{Z}, <)$ and the rest of the proof proceeds like in Lemma 12 and Proposition 13, noting one no longer needs the (0_H) axiom to prove item (3) of Lemma 12, which is instead proven exactly like item (2). ◀

Proof sketch of Prop. 22. This result is proven in [14] and [23] for sproutings based on arbitrary complete normal modal logics L_1 and L_2 . The notion of “sprouting” in the present paper is simplified to fit our particular logics, but generally a *sprouting* is a birelational frame (W, R_1, R_2) than can be built as the union of L_1 -frames and L_2 -frames in a way such that the different L_i -frames have no points in common and, for $i \neq j$, any of the L_i -frames and any of the L_j -frames have at most one point in their intersection.

One shows that, for any formula φ which is consistent in the fusion of both logics $L_1 \oplus L_2$, there is a sprouting satisfying φ . This is used to prove a weaker result: namely that the fusion $L_1 \oplus L_2$ is complete with respect to frames (W, R_1, R_2) where (W, R_i) is an L_i -frame for $i = 1, 2$.

Let us see a sketch of this construction for our particular pair of logics:

Given a formula φ in the language of TSL, a *T-constituent* of φ is any maximal subformula of φ which is preceded by one of the temporal operators G, H or X (or their duals); whereas an *S-constituent* is a maximal subformula preceded by $\Box$ (or $\Diamond$).

The *T-projection* of φ , denoted φ^T , is the formula in the language of TNL resulting from substituting the *S*-constituents of φ with “surrogate” propositional variables; its *S-projection*, φ^S , is the formula in the language of SL obtained by replacing the *T*-constituents with surrogates.

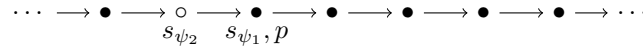
(For example, take the formula $\varphi = F(p \wedge \Diamond Xq) \wedge \Diamond Pr$. Its *S*-constituents are $\psi_1 = \Diamond Xq$ and $\psi_2 = \Diamond Pr$; its *T*-projection can be written as $\varphi^T = F(p \wedge s_{\psi_1}) \wedge s_{\psi_2}$, where s_{ψ_1} and s_{ψ_2} are fresh “surrogate” variables.)

It holds that, if φ is a consistent formula in the fusion logic TSL, then both its projections φ^T and φ^S are consistent in their respective fragments (note that if, e.g., φ^T were inconsistent in TNL, then $\neg\varphi^T$ would be a theorem of TSL since $\text{TNL} \subseteq \text{TSL}$, and so would $\neg\varphi$, obtained from φ^T via uniform substitution; likewise for ψ^S).

The process of building the sprouting that satisfies φ is done in a number of steps no larger than the modal depth of φ ; it goes as follows:

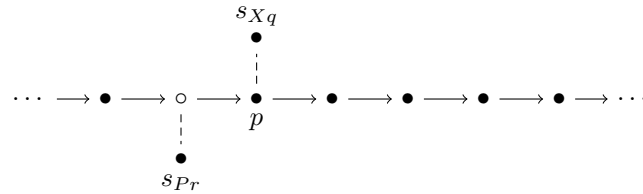
- First, one constructs a T-model satisfying the *T*-projection φ^T . Some of the points in this model will satisfy the “surrogate” variables of this formula.

Using our example formula $\varphi = F(p \wedge \Diamond Xq) \wedge \Diamond Pr$ to illustrate this, we have the following model satisfying φ^T at the white point:



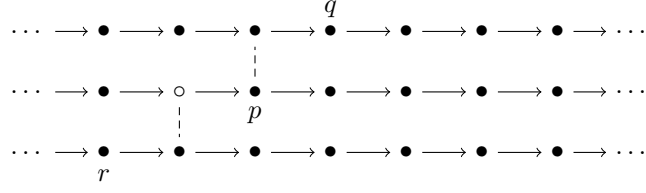
- We now take each of these points satisfying some surrogate variable s_{ψ} and we make it into the root of a finite S-model satisfying ψ^S . Note that the (possibly infinitely many) models obtained in this step have no points in common amongst themselves and at most one point in common with the model built in the previous step.

In our example, we compute the projections, $\psi_1^S = (\Diamond Xq)^S = \Diamond s_{Xq}$ and $\psi_2^S = (\Diamond Pr)^S = \Diamond s_{Pr}$, and we make a finite S-model for ψ_2^S sprout from the white point and one for ψ_1^S sprout from its successor:



- Again, some of the points in these S-models will satisfy some surrogate variables p_{θ}^T . For each of these, we construct a T-model satisfying θ^T at some point and we “plug it” into the point satisfying p_{θ}^T . Again, we note that the models obtained in this step are disjoint among themselves, have no points in common with any T-models obtained in previous steps, and each of them has at most one point in common with at most one S-model obtained in the previous step.

In our running example:



- We repeat this process, alternating T-models and S-models that hang off each other, until the formula is exhausted.

Some caveats are necessary to make sure this construction does not backfire: for instance, the authors of [23] work with partial valuations and have some restrictions on the types of models one can use in the alternating sequence, whereas those of [14] circumvent this by skipping the step-by-step construction and using Zorn's Lemma to postulate the existence of a maximal sprouting; likewise, one has to be careful with Boolean combinations of surrogate variables. We point the reader to the aforementioned texts to fill out the finer details.

What is relevant for our purposes is that, at the end of this construction, we are left with a birelational model $M = (X, <, \rightleftharpoons, V)$ that satisfies φ and, by construction, is a countable sprouting. ◀

Proof of Lemma 24. We construct f by defining it first on $D_0 = t_0$ and successively augmenting its domain to the sets $D_1, D_2, \dots$, where

$$D_{n+1} = D_n \cup \{x : t \rightleftharpoons t_x \text{ for some } t \in D_n\}.$$

(In other words, D_n is the union of all timelines separated from t_0 by a $\rightleftharpoons$ -path of length at most n ; note that every timeline is a subset of some D_n , and that $X = \bigcup_{n \in \mathbb{N}} D_n$).

- First, assign $f(x_0) = 0$. Then, for every element $x \in t_0$, assign a number corresponding to the relative position of this element to x_0 in the timeline (i.e. $f(x_0^+) = 1, f(x_0^{++}) = 2, f(x_0^-) = -1$, etc.)
- Once all the elements of D_n have a number assigned, take a timeline $t \subseteq D_{n+1} \setminus D_n$; there is some timeline $t' \subseteq D_n$ such that $t' \rightleftharpoons t$; there is thus a unique pair of points $x \in t, x' \in t'$ such that $x \rightleftharpoons x'$. Set $f(x) := f(x')$ and assign values to the rest of the elements in t according to their relative position to x ($f(x^+) = f(x) + 1, f(x^-) = f(x) - 1, \dots, f(x^{\pm n}) = f(x) \pm n$).

Note that this assignment is well-defined on account of property (S3) of the definition of *sprouting*: if $t'' \subseteq D_n$ is another timeline such that $t'' \rightleftharpoons t$, then, remembering that all timelines have a symmetric $\rightleftharpoons$ -path to t_0 , we take a closed loop

$$t_0 \rightleftharpoons \dots \rightleftharpoons t' \rightleftharpoons t \rightleftharpoons t'' \rightleftharpoons \dots \rightleftharpoons t_0$$

and we apply (S3) to conclude that t'' and t are also connected at point x and that $f(x') = f(x'')$, where x'' is the unique point in t'' such that $x'' \rightleftharpoons x$.

By this inductive process, we are left with a function $f : X \rightarrow \mathbb{Z}$ which, by construction, satisfies (F1) and (F2). ◀

Proof of Lemma 25. Simple induction on ψ ; the case $\psi = p$ follows from the definition of V^* , whereas the cases $\psi = \neg\theta, \psi = \theta_1 \wedge \theta_2$ are trivial.

Case $\psi = G\theta$: $S, x \models G\theta$ iff $S, y \models \theta$ for all $y > x$, iff (by induction hypothesis) $M, (t_y, f(y)) \models \theta$ for all $y > x$, iff (noting that for all $m > f(x)$ there is some $y > x$ with $f(y) = m, t_y = t_x$), iff $M, (t_x, m) \models \theta$ for all $m > f(x)$, iff $M, (t_x, f(x)) \models G\theta$.

Cases $\psi = H\theta, \psi = X\theta$: analogous to the previous case.

Case $\psi = \Box\theta$: $S, x \models \Box\theta$ iff $S, y \models \theta$ for all $y \rightleftharpoons x$, iff (by hypothesis) $M, (t_y, f(y)) \models \theta$ for all $y \rightleftharpoons x$ iff (remembering that $f(x) = f(y)$ when $x \rightleftharpoons y$) $M, (t_y, f(x)) \models \theta$ whenever $t_y \rightleftharpoons_{f(x)} t_x$, iff $M, (t_x, f(x)) \models \Box\theta$. ◀