Report from Dagstuhl Seminar 12451

The Constraint Satisfaction Problem: Complexity and Approximability

Edited by

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Abstract

During the past two decades, an impressive array of diverse methods from several different mathematical fields, including algebra, logic, analysis, probability theory, graph theory, and combinatorics, have been used to analyze both the computational complexity and approximabilty of algorithmic tasks related to the constraint satisfaction problem (CSP), as well as the applicability/limitations of algorithmic techniques. The Dagstuhl Seminar 12451 "The Constraint Satisfaction Problem: Complexity and Approximability" was aimed at bringing together researchers using all the different techniques in the study of the CSP, so that they can share their insights. This report documents the material presented during the course of the seminar.

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1 Executive Summary

Johan Håstad
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The *constraint satisfaction problem*, or CSP in short, provides a unifying framework in which it is possible to express, in a natural way, a wide variety of computational problems dealing with mappings and assignments, including satisfiability, graph colorability, and systems of equations. The CSP framework originated 25-30 years ago independently in artificial intelligence, database theory, and graph theory, under three different guises, and it was realised only in the late 1990s that these are in fact different faces of the same fundamental problem. Nowadays, the CSP is extensively used in theoretical computer science, being a mathematical object with very rich structure that provides an excellent laboratory both for classification methods and for algorithmic techniques, while in AI and more applied areas of computer science this framework is widely regarded as a versatile and efficient way of

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modelling and solving a variety of real-world problems, such as planning and scheduling, software verification and natural language comprehension, to name just a few. An instance of CSP consists of a set of variables, a set of values for the variables, and a set of constraints that restrict the combinations of values that certain subsets of variables may take. Given such an instance, the possible questions include (a) deciding whether there is an assignment of values to the variables so that every constraint is satisfied, or optimising such assignments in various ways, or (b) finding an assignment satisfying as many constraints as possible. There are many important modifications and extensions of this basic framework, e.g. those that deal with soft or global constraints.

Constraint satisfaction has always played a central role in computational complexity theory; appropriate versions of CSPs are classical complete problems for most standard complexity classes. CSPs constitute a very rich and yet sufficiently manageable class of problems to give a good perspective on general computational phenomena. For instance, they help to understand which mathematical properties make a computational problem tractable (in a wide sense, e.g. polynomial-time solvable or non-trivially approximable, fixed-parameter tractable or definable in a weak logic). It is only natural that CSPs play a role in many high-profile conjectures in complexity theory, exemplified by the Dichotomy Conjecture of Feder and Vardi and the Unique Games Conjecture of Khot.

The recent flurry of activity on the topic of the seminar is witnessed by two previous Dagstuhl seminars, titled "Complexity of constraints" (06401) and "The CSP: complexity and approximability" (09441), that were held in 2006 and 2009, respectively. This seminar was a follow-up to the 2009 seminar. Indeed, the exchange of ideas at the 2009 seminar has led to new ambitious research projects and to establishing regular communications channels, and there is a clear potential of a further systematic interaction that will keep on cross-fertilizing the areas and opening new research directions. The 2012 seminar brought together forty four researchers from different highly advanced areas of constraint satisfaction and involved many specialists who use universal-algebraic, combinatorial, geometric and probabilistic techniques to study CSP-related algorithmic problems.

The seminar included two substantial tutorials: one on the classification of the complexity of constraint languages via methods of logic and universal algebra (given by A. Krokhin from Durham U, UK and R. Willard from Waterloo U, CA), and the other on the approximability of CSP (given by P. Austrin from KTH Stockholm, SE). Other participants presented, in 28 further talks, their recent results on a number of important questions concerning the topic of the seminar.

Concluding Remarks and future plans. The seminar was well received as witnessed by the high rate of accepted invitations and the great degree of involvement by the participants. Because of the multitude of impressive results reported during the seminar and the active discussions between researchers with different expertise areas, the organisers regard this seminar as a great success. With steadily increasing interactions between such researchers, we foresee a new seminar focussing on the interplay between different approaches to studying the complexity and approximability of the CSP. Finally, the organisers wish to express their gratitude to the Scientific Directors of the Dagstuhl Centre for their support of the seminar.

Description of the Topics of the Seminar

Classical computational complexity of CSPs. Despite the provable existence of intermediate (say, between P and NP-complete, assuming $P \neq NP$) problems, research in computational complexity has produced a widely known informal thesis that "natural problems are almost always complete for standard complexity classes". CSPs have been actively used to support and refine this thesis. More precisely, several restricted forms of CSP have been investigated in depth. One of the main types of restrictions is the *constraint language* restriction, i.e., a restriction on the available types of constraints. By choosing an appropriate constraint language, one can obtain many well-known computational problems from graph theory, logic, and algebra. The study of the constraint language restriction is driven by the CSP *Dichotomy Conjecture* of Feder and Vardi which states that, for each fixed constraint language, the corresponding CSP is either in P or NP-complete. There are similar dichotomy conjectures concerning other complexity classes (e.g. L and NL). Recent breakthroughs in the complexity of CSP have been made possible by the introduction of the universal-algebraic approach, which extracts algebraic structure from the constraint language and uses it to analyse problem instances. McKenzie's talk surveyed classes of algebras that arise in this context and Pinsker related this approach with infinite-valued CSPs. The algebraic approach has been applied to prove the Dichotomy Conjecture in many important special cases (e.g. Bulatov's dichotomy theorems for 3-valued and conservative CSPs), but the general problem remains open. A powerful universal-algebraic theory of absorption has been developed in the last couple of years by Barto and Kozik, specifically motivated by CSP classification questions. This theory has already produced several spectacular classification results resolving long-standing problems (including a characterization of CSPs of bounded width, i.e. solvable by local propagation algorithms), and there is a clear sense that there is much more to come from it. Kozik presented new results on CSPs in NL that are based on the absorption theory.

Algebraic approaches to studying exact exponential and sublinear algorithms for CSPs were presented by Jonsson and Yoshida, respectively.

The complexity of Valued CSPs, which are a significant generalisation of Max CSP, was considered in the talks by Huber, Kolmogorov, Thapper, and Živný. Very strong result were reported, especially the full description of tractable cases by Thapper and Živný. Raghavendra presented results that might lead to closer interchange of ideas between algebraic and probabilistic approaches to CSPs.

The algebraic approach to the complexity of counting solutions for CSPs, with many results, was presented by Bulatov, Dyer, Goldberg, and Jerrum, while Lu reported recent progress on classifying the complexity of Holant problems.

Approximability of CSPs. The use of approximation algorithms is one of the most fruitful approaches to coping with NP-hardness. Hard optimization problems, however, exhibit different behavior with respect to approximability, making it an exciting, and by now, well-developed but far from fully understood, research area. The CSP has always played an important role in the study of approximability. For example, it is well known that the famous PCP theorem has an equivalent reformulation in terms of inapproximability of a certain CSP; moreover, the recent combinatorial proof of this theorem by Dinur in 2006 deals entirely with CSPs. The first optimal inapproximability results by Håstad in 2001 were about certain CSPs, and they led to the study of a new hardness notion called *approximation resistance* (which, intuitively, means that a problem cannot be approximated beyond the approximation ratio given by picking an assignment uniformly at random, even on almost satisfiable instances). Many CSPs have been classified as to whether they are approximation resistant but there is not even a reasonable conjecture for a full classification. Håstad, Huang, and K. Makarychev presented new results on approximation resistance.

In a related development, Guruswami and Zhou have discussed, in 2010, a "hybrid" form of tractability for CSPs, where classical tractability is combined with good approximability on almost satisfiable instances, and they conjecture that CSPs of bounded width have this desirable property. This conjecture was proved by Barto and Kozik in 2012 (and presented by Barto at the seminar), with further results in this direction presented by Dalmau.

Arguably, the most exciting development in approximability in the past five to six years is the work around the *unique games conjecture* (UGC), which was introduced by Khot in 2002. It states that, for CSPs with a certain constraint language over a large enough domain, it is NP-hard to tell almost satisfiable instances from those where only a small fraction of constraints can be satisfied. This conjecture (if true) is known to imply optimal inapproximability results for many classical optimization problems. Moreover, if the UGC is true then, as shown by Raghavendra in 2008, a simple algorithm based on semidefinite programming provides the best possible approximation for all CSPs (though the exact quality of this approximation is unknown). In 2010, Arora *et al.* gave a sub-exponential time algorithm for unique games CSPs, which is based on a new graph decomposition method. This does not give strong evidence in favor or against the conjecture, but it shows that there are important new algorithmic ideas to be discovered. Y. Makarychev presented an asymptotically optimal (modulo UGC) approximation algorithm for the general Max CSP.

Parameterized complexity of CSPs. A different way to cope with NP-hardness is provided by parameterized complexity, which relaxes the notion of tractability as polynomial-time solvability to allow non-polynomial dependence on certain problem-specific parameters. A whole new set of interesting questions arises if we look at CSPs from this point of view. Most CSP dichotomy questions can be revisited by defining a parameterized version; so far, very little work was done in this direction compared to investigations in classical complexity. Interestingly, some of the most tantalizing open problems in parameterized algorithmics (e.g. the fixed-parameter tractability of the Biclique problem) are directly related to complexity of CSPs, and Marx's talk contained an overview of such problems. A new research direction (often called "parameterizing above the guaranteed tight bound") led to unexpected positive results for Max r-SAT by Alon *et al.* in 2010. In this direction, the basic question is to decide the fixed-parameter tractability of the following type of problems: if we know that a random assignment satisfies at least E clauses/constraints in expectation (and hence such an assignment is easy to find), find an assignment that satisfies at least $E+k$ clauses/constraints. Gutin presented recent results in this direction.

Along with the constraint language restriction, another important restriction of CSPs that has been thoroughly investigated is the *structural* restriction, where the way in which the immediate interaction between variables in instances is restricted. In this direction, the notions of (hyper)graph decompositions and treewidth turned out to be particularly important. These notions are core concepts of parameterized algorithmics, and so, it is not surprising that parameterized complexity is an important tool in characterizing structural restrictions that lead to tractable CSPs. In particular, many known classification results with respect to classical complexity in this direction (e.g. Grohe, 2007) use tools from parameterized complexity. Scarcello and Szeider described their new results in this direction.

Logic and the complexity of CSP. Starting from earlier work by Kolaitis ad Vardi, concepts and techniques from logic have provided unifying explanations for many tractable CSPs. This has led to the pursuit of classifications of CSP with respect to *descriptive complexity*, i.e. definability in a given logic. Logics considered in this context include first order logic and its extensions, finite-variable logics, the logic programming language Datalog and its fragments. Kozik's talk described a contribution in this direction.

The CSP can be recast as the problem of deciding satisfiability of existential conjunctive formulas. Natural extensions of this framework that allow counting or universal quantifiers were considered in the talks by Martin and Chen, respectively. Atserias' talk related proof complexity, CSPs, and semidefinite programming.

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3 Overview of Talks

3.1 Semi-Algebraic Proofs, Gaussian Elimination, and CSPs with Short Proofs of Unsatisfiability

Albert Atserias (UPC – Barcelona, ES)

Despite impressive recent progress in obtaining conditional results, one of the big remaining mysteries is why semi-definite programming appears to be the optimal polynomial-time algorithm for approximating constraint satisfaction problems. The lack of a complete understanding is illustrated by the fact that a small generalization of semi-definite programming, the low-degree sum-of-squares method, remains still a candidate algorithm that could beat the UG-optimal Goemans-Williamson bound for max-cut. This raises the obvious question: how powerful low-degree sum-of-squares methods, or more generally low-degree semi-algebraic proofs, really are? A first observation we offer is that low-degree semi-algebraic dag-like proofs, unlike their tree-like versions, are able to simulate both Gaussian elimination over prime fields and bounded-width constraint propagation. Time permitting, we put the question in the more general context of characterizing which CSPs have polynomial-size proofs of unsatisfiability in a given proof system, and offer a general decidable criterion that, unfortunately, so far applies only to width-1 or linear programming.

3.2 Approximability of Constraint Satisfaction Problems: A Tutorial

Per Austrin (KTH Stockholm, SE)

This talk is intended to be a tutorial on the approximability of constraint satisfaction problems, where the goal is to find an assignment satisfying as many constraints as possible.

The tutorial will cover:

1. use of linear and semidefinite programming to obtain non-trivial approximation guarantees
2. the PCP theorem and the Unique Games Conjecture, and the ideas that allow us to derive optimal hardness of approximation results from them
3. qualitative notions of approximability such as approximation resistance and robust approximation

3.3 Robust Satisfiability of Constraint Satisfaction Problems

Libor Barto (Charles University – Prague, CZ)

Joint work of Barto, Libor; Kozik, Marcin
Main reference L. Barto, M. Kozik, "Robust satisfiability of constraint satisfaction problems," in Proc. of the 44th Symp. on Theory of Computing, STOC'12, pp. 931–940, ACM, 2012.
URL http://dx.doi.org/10.1145/2213977.2214061

An algorithm for a constraint satisfaction problem is called robust if it outputs an assignment satisfying almost all constraints given an almost satisfiable instance. Guruswami and Zhou conjectured that CSP over a fixed constraint language admits and efficient robust algorithm if and only if the CSP has bounded width. We confirm their conjecture. The proof is based on an interesting connection between semidefinite programming relaxations and Prague strategies.

3.4 Counting CSPs and Datalog Fixed Points

Andrei A. Bulatov (Simon Fraser University – Burnaby, CA)

Joint work of Bulatov, Andrei A.; Dalmau, Victor; Thurley, Marc

The problem of counting independent sets in bipartite graphs (#BIS) is important in the study of the approximation complexity of counting CSPs. In their 2003 paper Dyer et al. proved that one of the problems interreducible (with respect to approximation preserving reductions) with #BIS is the problem of counting fixed points of a linear Datalog program. We try to determine for which relational structures B the problem #CSP(B) can be expressed as the problem of finding the number of fixed points.

3.5 Meditations on Quantified Constraint Satisfaction

Hubie Chen (Universidad del País Vasco – Donostia, ES)

Main reference H. Chen, "Meditations on Quantified Constraint Satisfaction," in R.L. Constable, A. Silva, (eds.), "Logic and Program Semantics", LNCS 7230, pp. 35–49, Springer, 2012.
URL http://dx.doi.org/10.1007/978-3-642-29485-3_4
URL http://arxiv.org/abs/1201.6306

The quantified constraint satisfaction problem (QCSP) is the generalization of the CSP where universal quantification is permitted in addition to existential quantification: one is given a structure and a sentence built from atoms, conjunction, and the two quantifiers, and the problem is to decide if the sentence holds on the structure. As with the CSP, one obtains a family of problems by defining, for each structure B, the problem QCSP(B) to be the QCSP where the structure is fixed to be B.

We discuss the research program of trying to classify the complexity of QCSP(B) for each finite structure B. We overview the use of universal-algebraic notions and techniques in this research program, and present conjectures concerning when various complexity behaviors occur. We attempt to emphasize open issues and potential research directions, and promise to present concrete open questions.

A protagonist of the talk is the growth rate of the number of elements needed to generate the powers $A^1, A^2, \ldots$ of an algebra A: showing an at-most polynomial growth rate can (essentially) be translated to a complexity upper bound for a structure with algebra A.

3.6 Robust Approximability with Polynomial Loss.

Victor Dalmau (Univ. Pompeu Fabra – Barcelona, ES)

Joint work of Dalmau, Victor; Krokhin, Andrei
Main reference V. Dalmau, A. Krokhin, "Robust Satisfiability for CSPs: Hardness and Algorithmic Results," submitted for journal publication.
URL http://www.dur.ac.uk/andrei.krokhin/papers/robust1.pdf

An algorithm for a constraint satisfaction problem, CSP(H), is called robust if it outputs an assignment satisfying at least a $(1 - f(\epsilon))$-fraction of constraints for each $(1 - \epsilon)$-satisfiable instance (i.e, such that at most a ϵ-fraction of constraints needs to be removed to make the instance satisfiable), where $f(\epsilon) \to 0$ as $\epsilon \to 0$. Barto and Kozik have shown that CSP(H) admits a robust polynomial algorithm if and only if H has bounded width, confirming a conjecture of Guruswami and Zhou. In the present talk we shall describe some additional requirements that guarantee that CSP(H) has a robust algorithm with polynomial loss, namely, such that $f(\epsilon) = O(\epsilon^{1/k})$ for some k.

3.7 The Complexity of Approximating Conservative Counting CSPs

Martin Dyer (University of Leeds, GB)

Joint work of Dyer, Martin; Chen, Xi; Goldberg, Leslie Ann; Jerrum, Mark; Lu, Pinyan; McQuillan, Colin; Richerby, David

We consider the complexity of approximation for the weighted counting constraint satisfaction problem #CSP(F). We study #CSP(F) in the conservative case, where F contains all unary functions. A classification was known for the Boolean domain, which we extend to problems with general finite domain. If F has a property called weak log-modularity, we show that #CSP(F) is in FP. Otherwise, #CSP(F) is as hard to approximate as #BIS. This is the problem of counting independent sets in a bipartite graph, and is believed to be intractable. We further classify the #BIS-hard problems. If F has a property called weak log-supermodularity, we show that #CSP(F) is as easy as Boolean log-supermodular weighted #CSP. Otherwise, #CSP(F) is NP-hard to approximate. Finally, we show that there is a trichotomy for the binary case. Then #CSP(F) is either in FP, or is equivalent to #BIS, or is NP-hard to approximate.

3.8 Approximate Weighted Boolean #CSPs

Leslie Ann Goldberg (University of Liverpool, GB)

Joint work of Bulatov, Andrei; Dyer, Martin; Goldberg, Leslie Ann; Jerrum, Mark; McQuillan, Colin
Main reference A.A. Bulatov, M. Dyer, L.A. Goldberg, M. Jerrum, C. McQuillan, "The expressibility of functions on the Boolean domain, with applications to Counting CSPs," arXiv:1108.5288v4 [cs.CC].
URL http://arxiv.org/abs/1108.5288

Motivated by a desire to understand the computational complexity of (weighted) counting CSPs, we have developed a notion of functional clones (analogous to relational clones) and have studied the landscape of these clones. This was described in Mark Jerrum's talk. In this talk, we give the applications to weighted counting CSPs. We give a complexity classification for the case in which constraints are functions from Boolean tuples to efficiently-computable non-negative reals. For every finite set F of constraint functions, we show that either (1) Approximate counting CSPs are tractable when constraints are taken from F and from any finite set of unary constraint functions, or (2) There is a finite set of unary constraint functions for which this approximation is difficult (subject to complexity-theoretic assumptions which will be described). If there is a function in F which is not log-supermodular, then the approximation problem is NP-hard (without any additional assumptions).

3.9 CSPs Parameterized Above Tight Lower Bounds

Gregory Z. Gutin (RHUL – London, GB)

Results on Max CSPs such as MaxSat and MaxLin2 and Max Permutation CSPs parameterized above tight lower bounds, were overviewed. A proof that Max-r-Sat parameterized above average is fixed-parameter tractable was presented.

For more information, see, e.g., [1, 2, 3].

References

1 N. Alon, G. Gutin, E.J. Kim, S. Szeider, and A. Yeo, Solving MAX-k-SAT Above a Tight Lower Bound. *Algorithmica*, 61 (2011), 638–655. Preliminary version in *ACM-SIAM Symposium on Discrete Algorithms (SODA) 2010*, pp. 511–517.

2 R. Crowston, M. Fellows, G. Gutin, M. Jones, F. Rosamond, S. Thomassé and A. Yeo, Simultaneously Satisfying Linear Equations Over $\mathbb{F}_2$: MaxLin2 and Max-r-Lin2 Parameterized Above Average. In *FSTTCS 2011*, LIPICS Vol. 13, 229–240.

3 R. Crowston, G. Gutin, M. Jones, V. Raman and S. Saurabh, Parameterized Complexity of MaxSat Above Average. *Theor. Comput. Sci.*, to appear. Pleliminary version in *LATIN 2012*, Lect. Notes Comput. Sci. 7256 (2012), 184–194.

3.10 On the NP-hardness of Max-Not-2

Johan Håstad (KTH Stockholm, SE)

We prove that, for any $\epsilon > 0$, it is NP-hard to, given a satisfiable instance of Max-NTW (Not-2), find an assignment that satisfies a fraction $\frac{5}{8} + \epsilon$ of the constraints. This, up to the existence of ϵ, matches the approximation ratio obtained by the trivial algorithm that just pick an assignment at random and thus the result proves that Max-NTW is approximation resistant on satisfiable instances. This result makes our understanding of arity three Max-CSPs with regards to approximation resistance complete.

3.11 Approximation Resistance on Satisfiable Instances for Predicates with Few Accepting Assignments

Sangxia Huang (KTH – Stockholm, SE)

In this talk we consider approximability of Max-CSP(P), where P is a Boolean predicate of arity k, and the goal is to find an assignment satisfying as many clauses as possible. A predicate is said to be approximation resistant if it is hard to achieve better approximation ratio than random assignment. A related problem is approximation resistance on satisfiable instances, where we are given satisfiable instances. The situation could be quite different in these cases, for instance PARITY (LinearEquation on k variables) is approximation resistant but on satisfiable instances we can solve optimally using Gaussian Elimination.

While there has been lots of progress in understanding approximation resistance of predicates and many tight upper- and lower-bounds are known, approximation resistance on satisfiable instances is still largely a mystery. In this talk, we will survey some known results on this problem, the role of PARITY, and present a construction which proves that some predicate on k variables with $2^{\tilde{O}(k^{1/3})}$ accepting assignments is approximation resistant on satisfiable instances, improving the previous bound of $2^{O(k^{1/2})}$ by (Håstad and Khot, 2001).

3.12 VCSPs on Three Elements

Anna Huber (University of Durham, GB)

Joint work of Huber, Anna; Krokhin, Andrei; Powell, Robert
Main reference A. Huber, A. Krokhin, R. Powell, "Skew Bisubmodularity and Valued CSPs," in Proc. of the 24th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA), pp. 1296–1305, 2013.
URL http://knowledgecenter.siam.org/0236-000082

An instance of the Finite-Valued Constraint Satisfaction Problem (VCSP) is given by a finite set of variables, a finite domain of values, and a sum of rational-valued functions, each function depending on a subset of the variables. The goal is to find an assignment of values to the variables that minimises the sum.

This talk investigates VCSPs in the case when the variables can take three values and provides a tight description of the tractable cases.

3.13 Functional Clones

Mark Jerrum (Queen Mary University of London, GB)

Joint work of Bulatov, Andrei A.; Dyer, Martin; Goldberg, Leslie Ann; Jerrum, Mark; McQuillan, Colin
Main reference A.A. Bulatov, M. Dyer, L.A. Goldberg, M. Jerrum, C. McQuillan, "The expressibility of functions on the Boolean domain, with applications to Counting CSPs," arXiv:1108.5288v4 [cs.CC].
URL http://arxiv.org/abs/1108.5288

In the classical setting, where CSPs model decision problems, a certain notion of expressibility, namely pp-definablity, plays an important role, as do relational clones, i.e., sets of relations closed under pp-definability. The corresponding concepts for valued CSPs (VCSPs), which model optimisation problems, have recently received a good deal of attention, and have yielded significant insight. What is the correct notion of expressibility, i.e., analogue of pp-definability, for (weighted) counting CSPs (#CSPs)? Equivalently, how should we define "functional clone"? The answer is not completely straightforward, as some delicate choices have to be made.

After proposing an answer to the above question, I will go on to investigate the structure of the lattice of functional clones in the Boolean, conservative case. In a second talk, Leslie Goldberg will continue this theme by investigating the consequence of these results for the complexity of #CSPs in the Boolean, conservative case.

3.14 Complexity of SAT Problems, Clone Theory and the Exponential Time Hypothesis

Peter Jonsson (Linköping University, SE)

Joint work of Jonsson, Peter; Lagerkvist, Victor; Nordh, Gustav; Zanuttini, Bruno
Main reference P. Jonsson, V. Lagerkvist, G. Nordh, B. Zanuttini, "Complexity of SAT Problems, Clone Theory and the Exponential Time Hypothesis," in Proc. of the 24th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA), pp. 1264–1277, 2013.
URL http://knowledgecenter.siam.org/0236-000094

The construction of exact exponential-time algorithms for NP-complete problems has for some time been a very active research area. Unfortunately, there is a lack of general methods for studying and comparing the time complexity of algorithms for such problems. We propose a method based on the lattice of partial clones and demonstrate it on the SAT problem. By using this method, we identify a relation R_e such that SAT(R_e) is, in a certain sense, the computationally easiest NP-complete SAT problem. We additionally demonstrate that SAT(R_e)-2 (i.e. SAT(R) restricted to instances where no variable appears in more than two clauses) is NP-complete, too. We then relate SAT(R_e)-2 to the exponential-time hypothesis (ETH) and show that ETH holds if and only if SAT(R)-2 is not sub-exponential. This constitutes a strong connection between ETH and the SAT problem under both severe relational and severe structural restrictions. In the process, we also prove a stronger version of Impagliazzo et. al's sparsification lemma for k-SAT; namely that all finite, NP-complete

Boolean languages can be sparsified into each other. This should be compared with Santhanam and Srinivasan's recent negative result which states that the same does not hold for all infinite Boolean languages.

3.15 Linear Programming and VCSPs Revisited

Vladimir Kolmogorov (IST Austria – Klosterneuburg, AT)

Main reference V. Kolmogorov, "The power of linear programming for valued CSPs: a constructive characterization," arXiv:1207.7213v4 [cs.CC]
URL http://arxiv.org/abs/1207.7213

I study which classes of finite-valued VCSPs can be solved exactly by the Basic Linear Programming relaxation (BLP). Thapper and Zivny proved that BLP solves a language iff it admits a symmetric fractional polymorphism of every arity. I show that it's sufficient to have a *binary* symmetric fractional polymorphism.

Combined with the recent dichotomy result of Thapper and Zivny, this implies that a finite-valued language can either be solved by BLP or it is NP-hard.

Link to the paper: http://pub.ist.ac.at/~vnk/papers/BLP.html

3.16 Some of the CSP's Solvable in Linear Datalog

Marcin Kozik (Jagiellonian University – Kraków, PL)

Joint work of Kozik, Marcin; Barto, Libor; Willard, Ross

Larose and Tesson conjectured that a complement of a CSP with the template in a variety omitting types 1,2 and 5 is definable in linear datalog. The result has been proved for templates with majority polymorphism (Dalmau, Krokhin) and near-unanimity polymorphism (Barto, Kozik, Willard). I will discuss the last proof and some further developments.

3.17 A Tutorial on Algebra and CSP, Part 1

Andrei Krokhin (University of Durham, GB)

I will explain why algebra works in classifying the complexity of CSPs and how algebraic classification results underpin complexity classification results. The second part of tutorial, given by Ross Willard, will be devoted to showing how algebra is used to design and analyse algorithms for CSPs.

3.18 Holant Problems: CSPs Where Each Variable Appears Exactly Twice

Pin-Yan Lu (Microsoft Research Asia, CN)

Holant Problem is a general framework to capture local constraints. CSP can be viewed as a special family of Holant problems where equality constraints of all arities are assumed to be available. This framework allows for the expression of (perfect) matching problems, a class of substantive combinatorial problems that have proved pivotal in complexity theory. Dichotomy is still open in this framework even for the Boolean domain case both for decision version and counting version. In this talk, I will review some of the results in this framework with an emphasis on some new phenomena comparing to the CSP framework. I will talk about decision version (NP), counting version (#P) and parity version ($\oplus$ P) with a focus on the counting version.

3.19 Local Search is Better than Random Assignment for Bounded Occurrence Ordering k-CSPs

Konstantin Makarychev (Microsoft Research – Redmond, US)

We prove that the Bounded Occurrence Ordering k-CSP Problem is not approximation resistant. We give a very simple local search algorithm that always performs better than the random assignment algorithm. Specifically, the expected value of the solution returned by the algorithm is at least Alg > Avg + $a(B,k)$ (Opt - Avg), where "Opt" is the value of the optimal solution; "Avg" is the expected value of the random solution; and $a(B,k) = \Omega_k(B^{-(k+O(1))}$ is a parameter depending only on "k" (the arity of the CSP) and "B" (the maximum number of times each variable is used in constraints).

The question whether bounded occurrence ordering k-CSPs are approximation resistant was raised by Guruswami and Zhou (APPROX 2012) who recently showed that bounded occurrence 3-CSPs and "monotone" k-CSPs admit a non-trivial approximation.

3.20 Approximation Algorithm for Non-Boolean MAX k-CSP

Yury Makarychev (Toyota Technological Institute – Chicago, US)

Joint work of Makarychev, Konstantin; Makarychev, Yury
Main reference K. Makarychev, Y. Makarychev, "Approximation Algorithm for Non-boolean MAX k-CSP. Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques," in Proc. of 15th Int'l APPROX Workshop and 16th Int'l RANDOM Workshop, LNCS, Vol. 7408, pp. 254–265, Springer, 2012.
URL http://dx.doi.org/10.1007/978-3-642-32512-0_22

We present a randomized polynomial-time approximation algorithm for Max k-CSP$_d$. In Max k-CSP$_d$, we are given a set of predicates of arity k over an alphabet of size d. Our goal is to find an assignment that maximizes the number of satisfied constraints. Our algorithm has approximation factor $\Omega(kd/d^k)$ (when $k \geq \Omega(\log d)$). The best previously known algorithm has approximation factor $\Omega(k \log d/d^k)$.

Our bound is asymptotically optimal. We also give an approximation algorithm for the boolean Max k-CSP_2 problem with a slightly improved approximation guarantee.

3.21 Constraint Satisfaction with Counting Quantifiers

Barnaby Martin (Middlesex University, GB)

We consider CSPs and QCSPs augmented with counting quantifiers.

3.22 CSPs and Fixed-Parameter Tractability

Daniel Marx (MTA – Budapest, HU)

We survey fixed-parameter tractability results appearing in the context of constraint satisfaction. The focus of the talks in on explaining the different type of questions that can be asked and the briefly summarizing the known results without going into the technical details.

3.23 The Dichotomy Conjecture: Sketching the Algebraic Landscape Near the Boundary

Ralph McKenzie (Vanderbilt University – Nashville, US)

I shall look at the chief classes of finite algebras that have assumed importance in the algebraic attempts to resolve the dichotomy conjecture, and survey many of the results that have emerged, sketching their significance for constraint satisfaction, and as contributions to pure algebra.

3.24 Topological Birkhoff

Michael Pinsker (University Paris-Diderot, FR)

Joint work of Pinsker, Michael; Bodirsky, Manuel

I will present a new method for hardness proofs of CSPs with (infinite) omega-categorical templates: we prove that if S is an omega-categorical countable structure whose polymorphism clone allows a continuous homomorphism to the trivial clone of projections, then CSP(S) is NP-hard.

The proof of this statement is based on a generalization of the finite version of Birkhoff's HSP theorem to omega-categorical structures, and I will outline the connection with this theorem. Moreover, I will describe the larger program in which we wish to reduce CSPs of infinite templates to CSPs of finite templates, and of which the above-mentioned result is an important step.

3.25 Efficient Algorithms via Polymorphisms in the Value Oracle Model

Prasad Raghavendra (University of California – Berkeley, US)

In this work, we design efficient algorithms via fractional polymorphisms for problems that are not specified as constraint satisfaction problems. Specifically, we show the following result:

1) Suppose we are given access to a function F in the value oracle model, along with an operation/polymorphism on the domain that never increases the value of F. Submodular minimization is a well-known example of this nature, with the operation being 2-bit AND and 2-bit OR.

We show that under restrictions on the operation/polymorphism, the function F can be minimized in pseudopolynomial time. This shows that the tractability of submodular minimization in the value-oracle model is a special case of a general phenomenon.

3.26 Tree Projections and Structural Decomposition Methods: The Power of Local Consistency and Larger Islands of Tractability

Francesco Scarcello (University of Calabria, IT)

Joint work of Greco, Gianluigi; Scarcello, Francesco
Main reference G. Greco, F. Scarcello, "Tree Projections and Structural Decomposition Methods: The Power of Local Consistency and Larger Islands of Tractability," arXiv:1205.3321v2 [cs.DB], 2012. (Preliminary version in the Proceedings of PODS'10.)
URL http://arxiv.org/abs/1205.3321

Evaluating conjunctive queries and solving constraint satisfaction problems are fundamental problems in database theory and artificial intelligence, respectively. These problems are NP-hard, so that several research efforts have been made in the literature for identifying tractable classes, known as islands of tractability, as well as for devising clever heuristics for solving efficiently real-world instances. Many heuristic approaches are based on enforcing on the given instance a property called local consistency, where (in database terms) each tuple in every query atom matches at least one tuple in every other query atom. Interestingly, it turns out that, for many well-known classes of instances, such as the acyclic ones, enforcing local consistency is even sufficient to solve the given instance correctly. However, the precise power of such a procedure was unclear, but for some very restricted cases. We provide the answers to the long-standing questions about the precise power of algorithms based on enforcing local consistency. The classes of instances where enforcing local consistency turns out to be a correct CSP-solving procedure are however not efficiently recognizable. In fact, the paper finally focuses on certain subclasses defined in terms of the novel notion of greedy tree projections. These latter classes are shown to be efficiently recognizable and strictly larger than most islands of tractability known so far, both in the general case of tree projections and for specific structural decomposition methods.

3.27 Structural Parameterizations of Language Restricted Constraint Satisfaction Problems

Stefan Szeider (Vienna University of Technology, AT)

Joint work of Szeider, Stefan; Bova, Simone

We study the fixed-parameter tractability of the constraint satisfaction problem. We restrict constraint relations to languages in a family of NP-hard languages, classified by a purely combinatorial criterion that generalizes Boolean matrices with fixed row and column sum. For various natural and established structural parameterizations of the instances, we characterize the fixed-parameter tractable constraint languages in the family.

3.28 The Complexity of Finite-Valued CSPs

Johan Thapper (Ecole Polytechnique – Palaiseau, FR)

Joint work of Thapper, Johan; Zivny, Stanislav
Main reference J. Thapper, S. Zivny, "The complexity of finite-valued CSPs," arXiv:1210.2987v2 [cs.CC].
URL http://arxiv.org/abs/1210.2987

I will give a complete complexity classification for finite-valued CSPs. The final result states that every core language Γ either admits a binary idempotent and symmetric fractional polymorphism, in which case the basic linear programming relaxation solves VCSP(Γ) exactly, or Γ satisfies a simple hardness condition that allows for a polynomial-time reduction from Max-Cut.

3.29 A Tutorial on Algebra and CSP, Part 2

Ross Willard (University of Waterloo, CA)

In this talk I explained (roughly) the two currently most-general polynomial-time CSP algorithms ("local consistency" and "few subpowers"), indicating in particular the role of algebra and polymorphisms.

3.30 Algebraic Characterizations of Testable CSPs

Yuichi Yoshida (NII – Tokyo, JP)

Joint work of Yoshida, Yuichi; Arnab Bhattacharyya
Main reference A. Bhattacharyya, Y. Yoshida, "Testing Assignments of Boolean CSPs," ECCC TR12–103, 2012.
URL http://eccc.hpi-web.de/report/2012/103/

Given an instance I of a CSP, a tester for I distinguishes assignments satisfying I from those which are far from any assignment satisfying I. The efficiency of a tester is measured by its query complexity, the number of variable assignments queried by the algorithm. In this talk, we show a characterization of the hardness of testing Boolean CSPs in terms of the associated algebra. In terms of computational complexity, we show that if a non-trivial Boolean CSP is sublinear- query testable (resp., not sublinear-query testable), then the CSP is in NL (resp., P-complete, parityL-complete or NP-complete) and that if a sublinear-query testable Boolean CSP is constant-query testable (resp., not constant-query testable), then counting the number of solutions of the CSP is in P (resp., #P- complete). Additionally, we conjecture that a CSP instance is testable in sublinear time if its Gaifman graph has bounded treewidth. We confirm the conjecture when a near- unanimity operation is a polymorphism of the CSP. We also mention a similar characterization for list H-homomorphism problem.

3.31 Linear Programming and VCSPs

Stanislav Zivny (University of Warwick, GB)

Joint work of Thapper, Johan; Zivny, Stanislav
Main reference J. Thapper, S. Zivny, "The Power of Linear Programming for Valued CSPs," in Proc. of the 53rd Annual IEEE Symp. on Foundations of Computer Science (FOCS'12), pp. 669–678, IEEE, 2012.
URL http://dx.doi.org/10.1109/FOCS.2012.25

This talk presents a recent result on the power of LP for valued CSPs: a valued constraint language L is solvable by the basic linear programming relaxation (BLP) iff L admits symmetric fractional polymorphisms of all arities (and 2 more equivalent statements).

Our results establish tractability of several previously widely-open classes of VCSPs including (i) VCSPs that are submodular on *arbitrary* lattices, (ii) VCSPs that are bisubmodular (also known as k-submodular) on *arbitrary* domains, and (iii) VCSPs that are submodular on *arbitrary* trees.

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