

Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 15392 “Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis.” It includes abstracts on most talks presented during the seminar, a list of open problems that were discussed and partially solved during the meeting as well as a bibliography on the seminar topic that we compiled during the seminar.

Seminar September 20–25, 2015 – <http://www.dagstuhl.de/15392>

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.3 Complexity Measures and Classes, F.2.1 Numerical Algorithms and Problems, F.4.1 Mathematical Logic

Keywords and phrases Computability and complexity in analysis, computations on real numbers, reducibilities, descriptive complexity, computational complexity, reverse and constructive mathematics

Digital Object Identifier 10.4230/DagRep.5.9.77

Edited in cooperation with Rupert Hölzl

1 Executive Summary

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Reducibilities such as many-one, Turing or polynomial-time reducibility have been an extraordinarily important tool in theoretical computer science from its very beginning. In recent years these reducibilities have been transferred to the continuous setting, where they allow to classify computational problems on real numbers and other (continuous) data types.

On the one hand, Klaus Weihrauch’s school of computable analysis and several further researchers have studied a concept of reducibility that can be seen as an analogue of many-one reducibility for functions on such data. The resulting structure is a lattice that yields a refinement of the Borel hierarchy and embeds the Medvedev lattice. Theorems of for-all-exists form can be easily classified in this structure.



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Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis,
Dagstuhl Reports, Vol. 5, Issue 9, pp. 77–104

Editors: Vasco Brattka, Akitoshi Kawamura, Alberto Marcone, and Arno Pauly



Dagstuhl Reports

REPORTS

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

On the other hand, Stephen Cook and Akitoshi Kawamura have independently introduced a polynomial-time analogue of Weihrauch's reducibility, which has been used to classify the computational complexity of problems on real numbers and other objects. The resulting theory can be seen as a uniform version of the complexity theory on real numbers as developed by Ker-I Ko and Harvey Friedman.

The classification results obtained with Weihrauch reducibility are in striking correspondence to results in reverse mathematics. This field was initiated by Harvey Friedman and Stephen Simpson and its goal is to study which comprehension axioms are needed in order to prove certain theorems in second-order arithmetic. The results obtained so far indicate that Weihrauch reducibility leads to a finer uniform structure that is yet in basic agreement with the non-uniform results of reverse mathematics, despite some subtle differences.

Likewise one could expect relations between weak complexity theoretic versions of arithmetic as studied by Fernando Ferreira et al., on the one hand, and the polynomial-analogue of Weihrauch reducibility studied by Cook, Kawamura et al., on the other hand.

While the close relations between all these approaches are obvious, the exact situation has not yet been fully understood. One goal of our seminar was to bring researchers from the respective communities together in order to discuss the relations between these research topics and to create a common forum for future interactions.

We believe that this seminar has worked extraordinarily well. We had an inspiring meeting with many excellent presentations of hot new results and innovative work in progress, centred around the core topic of our seminar. In an Open Problem Session many challenging current research questions have been addressed and several of them have been solved either during the seminar or soon afterwards, which underlines the unusually productive atmosphere of this meeting.

A bibliography that we have compiled during the seminar witnesses the substantial amount of research that has already been completed on this hot new research topic up to today.

This report includes abstracts of many talks that were presented during the seminar, it includes a list of some of the open problems that were discussed, as well as the bibliography.

Altogether, this report reflects the extraordinary success of our seminar and we would like to use this opportunity to thank all participants for their valuable contributions and the Dagstuhl staff for their excellent support!

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3 Overview of Talks

3.1 Preliminary investigations into Eilenberg-Moore algebras arising in descriptive set theory

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Recently we proposed an abstract notion of a “jump-operator” to unify characterizations of limit-computability and other topological and recursion-theoretic complexity classes given by V. Brattka, M. Ziegler, and others. These operators determine functors on the category of (Baire-) represented spaces, are closely related to (strong) Weihrauch reducibility, and can be used to represent the major complexity hierarchies in descriptive set theory. In particular, sets of a given level of the Borel hierarchy correspond to realizable maps into particular “jumps” of the Sierpinski-space.

In a different context, P. Taylor has been developing a re-axiomatization of topology inspired by M. Stone’s celebrated duality theorem between topology and algebra. Within this paradigm, P. Taylor showed that many important concepts from topology can be described using the exponential object of maps into an object playing the role of the Sierpinski-space. In particular, fundamental aspects of Stone duality can be expressed in terms of Eilenberg-Moore algebras of a monad defined using the Sierpinski-space object. The resulting theory is quite general, and much can be expressed with very little assumptions on the Sierpinski-space object.

In this talk, we present preliminary investigations into interpreting some parts of P. Taylor’s theory using “jumps” of the Sierpinski-space as the basic Sierpinski-space object, and look at some examples of the resulting Eilenberg-Moore algebras. As a case study, we make some connections with the Jayne-Rogers theorem by applying recent results on that theorem by A. Pauly and myself.

This work was supported by JSPS Core-to-Core Program, A. Advanced Research Networks and by JSPS KAKENHI Grant Number 15K15940.

3.2 The mathematics and metamathematics of weak analysis

Fernando Ferreira (University of Lisboa, PT)

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In this survey talk, we start by remarking that it is well-known that the provably total functions of the base theory RCA_0 of reverse mathematics are the primitive recursive functions. We show how to set up a similar theory (called BTFA, an acronym for ‘base theory for feasible analysis’) whose provably total functions are (in an appropriate sense) the polytime computable functions. As with RCA_0 , one can add to this theory weak König’s lemma without proving new Π_2^0 -consequences. We draw attention to the pivotal rôle of the bounded collection scheme in defining BTFA and in the proof of the above conservation result, and also to some differences with the usual setting of reverse mathematics (weak König’s lemma can be formulated in BTFA not only for set trees but, more generally, for trees defined by bounded formulas).

We describe how to introduce the real numbers in the theory BTFA. Continuous functions can also be introduced, following the usual blueprint of reverse mathematics. The intermediate value theorem can be proved and, in particular, the real numbers form a real closed ordered field (but are more than just that). We discuss the rôle of (several forms of) weak König's lemma in the setting of BTFA in relation to the Heine-Borel theorem, the uniform continuity theorem and the attainment of maximum for continuous real functions defined on a closed bounded interval.

We also briefly describe two other theories of weak analysis: one related to Vaillant's class $\#P$ of counting functions and the other related to polyspace computability. We show how to introduce Riemann integration in the former theory and argue that, in a sense (namely, for continuous functions defined *à la* Simpson with a modulus of uniform continuity) this is the weaker theory in which integration can be done.

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3.3 The Weihrauch degrees of conditional distributions

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Joint work of Ackerman, Nathanael L.; Freer, Cameron; Roy, Daniel

Main reference N. L. Ackerman, C. Freer, D. Roy, “On computability and disintegration,” arXiv:1509.02992v1 [math.LO], 2015.

URL <http://arxiv.org/abs/1509.02992v1>

We show that the disintegration operator on a complete separable metric space along a projection map, restricted to measures having a unique continuous disintegration, is strongly Weihrauch equivalent to the limit operator Lim . When a measure does not have a unique continuous disintegration, we may still obtain a disintegration when some basis of continuity sets has the Vitali covering property with respect to the measure; the disintegration, however, may depend on the choice of sets. We show that, when the basis is computable, the resulting disintegration is strongly Weihrauch reducible to Lim , and further exhibit a single distribution realizing this upper bound.

3.4 Probabilistic computability and the Vitali Covering Theorem

Guido Gherardi (Universität der Bundeswehr – München, DE)

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Joint work of Brattka, Vasco; Gherardi, Guido; Hölzl, Rupert

Our recent work [3] has developed our investigation on probabilistic and Las Vegas computability for sequences of infinite length, already introduced and studied in [1] and [2].

Las Vegas computable (multi-valued) functions are those (multi-valued) functions on represented spaces that can be computed with positive success probability by *non deterministic TTE Turing machines*. Such devices constitute a more powerful variation of TTE Turing machines: they are allowed to integrate the information contained in the input by accessing auxiliary information contained in a *randomly selected* binary string (“oracle”). If such randomly accessed information is useful to solve the task, then a correct output is produced. Otherwise, after finitely many steps, the machine recognizes the failure and outputs in fact a failure message. If a (multi-valued) function $f : \subseteq X \rightrightarrows Y$ over represented spaces can be computed by a non deterministic TTE Turing machine under the condition that for every possible input the set of successful oracles has positive measure in the Cantor space, then f is said to be *Las Vegas computable*.

We also consider functions that can be simulated on non deterministic Turing machines that replace the oracle space $2^{\mathbb{N}}$ by $\mathbb{N} \times 2^{\mathbb{N}}$. In reality, such functions can also be computed by non deterministic Turing machines that maintain $2^{\mathbb{N}}$ as oracle space but that are allowed to do finitely many corrections on the output tape. For this reason we call such functions *Las Vegas computable with finitely many mind changes*. This new class of functions extends the previous one, and both classes are contained in the wider class of *probabilistic functions*, that is, the class of those functions computed by selecting the Baire space $\mathbb{N}^{\mathbb{N}}$ as oracle space and without demanding for failure messages in case of unsuccess.

As a very significant study case we have investigated the classical *Vitali covering theorem*: every sequence \mathcal{I} of open intervals that *Vitali covers* a Lebesgue measurable subset $A \subseteq [0, 1]$ (i.e., \mathcal{I} is such that every point of A is contained in arbitrarily small elements of \mathcal{I}) includes a countable sequence \mathcal{J} *eliminating* A (i.e., the elements of \mathcal{J} are pairwise disjoint and cover A up to measure 0). Several classically equivalent versions of the statement are of course possible. We have analyzed three natural versions for $A := [0, 1]$ that we are going to formulate after having introduced the following terminology. For every sequence of open intervals \mathcal{I} , a point x is *captured* by \mathcal{I} if it is contained in elements of \mathcal{I} of arbitrarily small diameter. Moreover \mathcal{I} is called *saturated* if every point covered by some element in \mathcal{I} is even captured by \mathcal{I} (therefore, every Vitaly cover is a saturated sequence). We have then the following versions of the theorem:

1. For every Vitali cover \mathcal{I} of $[0, 1]$ there exists a countable subsequence \mathcal{J} of \mathcal{I} that eliminates $[0, 1]$.
2. For every saturated sequence \mathcal{I} of open intervals that does not admit a countable subsequence eliminating $[0, 1]$ there exists a point $x \in [0, 1]$ that is not covered by \mathcal{I} .
3. For every sequence \mathcal{I} of open intervals that does not admit a countable subsequence eliminating $[0, 1]$ there exists a point $x \in [0, 1]$ that is not captured by \mathcal{I} .

These three classically equivalent versions define three different operators defined on Int , the set of all sequences of open rational intervals in \mathbb{R} :

1. $\text{VCT}_0 : \subseteq \text{Int} \rightrightarrows \text{Int}$ with

$$\text{VCT}_0(\mathcal{I}) := \{\mathcal{J} : \mathcal{J} \text{ is a countable subsequence of } \mathcal{I} \text{ eliminating } [0, 1]\}$$

for \mathcal{I} a Vitali cover of $[0, 1]$;

2. $\text{VCT}_1 : \subseteq \text{Int} \Rightarrow [0, 1]$ with

$$\text{VCT}_1(\mathcal{I}) := \{x \in [0, 1] : x \text{ is not covered by } \mathcal{I}\}$$

for \mathcal{I} saturated with no countable subsequence eliminating $[0, 1]$;

3. $\text{VCT}_2 : \subseteq \text{Int} \Rightarrow [0, 1]$ with

$$\text{VCT}_2(\mathcal{I}) := \{x \in [0, 1] : x \text{ is not captured by } \mathcal{I}\}$$

for \mathcal{I} with no countable subsequence eliminating $[0, 1]$.

It turns out that these operators are computationally very significant, in particular to characterize the notion of Las Vegas computability. In fact the following theorems hold:

► **Theorem 1.** VCT_0 is computable.

► **Theorem 2.** VCT_1 is Weihrauch complete with respect to the class of Las Vegas computable functions.

► **Theorem 3.** VCT_2 is Weihrauch complete with respect to the class of Las Vegas computable functions with finitely many mind changes.

Theorem 2 is proved by showing that $\text{VCT}_1 \equiv_W \text{PC}_{[0,1]}$, where $\text{PC}_{[0,1]}$ is the operator selecting points from closed subsets of $[0, 1]$ of positive Lebesgue measure. It was indeed proved in [2] that this operator is Weihrauch complete with respect to the class of Las Vegas computable functions. Analogously, Theorem 3 is proved by showing that $\text{VCT}_2 \equiv_W \text{PC}_{\mathbb{R}}$, where $\text{PC}_{\mathbb{R}}$ is the extension of the previous positive choice operator over $[0, 1]$ to the whole real line (one direction of the equivalence has been proved by Arno Pauly).

We point out that the Vitali Covering Theorem has been proved to be equivalent to the principle WWKL_0 in Reverse Mathematics ([4]). In fact, in computable analysis $\text{WWKL} \equiv_W \text{PC}_{[0,1]}$ holds, where WWKL is the natural operational interpretation of the proof theoretic principle WWKL_0 : every infinite binary tree of positive measure contains an infinite path. Nevertheless the situation in our framework is, as we have seen, more finely structured and at the same time particularly interesting, since the same theorem can be used to characterize three important different computational classes.

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3.5 Topological Complexity and Topological Weihrauch Degrees

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We describe the relation between various ways for measuring the topological complexity of computation problems: either by counting the number of comparison nodes that a

computation tree for the problem needs to have, or by the level of discontinuity of the problem or by the topological Weihrauch degree of the problem. The hierarchies defined via continuous Weihrauch reductions refine the hierarchy defined by the level. Examples from algebraic complexity theory, from information-based complexity and from algebraic topology are presented. Furthermore, we show that an initial segment of the topological Weihrauch degrees of computation problems given by relations with finite discrete range can be described by classes of labeled forests under suitable reducibility relations on the class of labeled forests.

3.6 Reverse Mathematics and Computability-Theoretic Reduction

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Reverse mathematics is a research program that aims to calibrate the strength of theorems of ordinary mathematics in the context of subsystems of second-order arithmetic. Typically, one performs this calibration over the weak base theory RCA_0 , which roughly corresponds to the level of computable mathematics. This practice has been quite successful in many respects, but its very success has led to a desire for more fine-grained tools than implication over RCA_0 . This talk will introduce a few notions of computability-theoretic reduction between principles of a certain form, one of which is equivalent to Weihrauch reducibility.

3.7 Formalized reducibility

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Joint work of Hirst, Jeffrey L.; Mummert, Carl; Gura, Kirill

Main reference K. Gura, J. L. Hirst, C. Mummert, “On the existence of a connected component of a graph,” *Computability*, 4(2):103–117, 2015.

URL <http://dx.doi.org/10.3233/COM-150039>

Some forms of reducibility can be formalized in higher order reverse mathematics, as axiomatized by Professor Kohlenbach [1]. Proving strong Weihrauch reductions in the higher order reverse mathematics setting yields both the usual reduction results and associated sequential reverse mathematics results as easy corollaries.

Several natural questions arise from considering these formal proofs. For what portions of type-2 constructible analysis would this sort of formalization be fruitful? What is the comparative logical strength of the various functional existence axioms generated in this way? What foundational insights can be gained here? What about other reducibilities?

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3.8 Universality, optimality, and randomness deficiency

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Joint work of Hölzl, Rupert; Paul Shafer

Main reference R. Hölzl, P. Shafer, “Universality, optimality, and randomness deficiency,” *Annals of Pure and Applied Logic*, 166(10):1049–1069, 2015.

URL <http://dx.doi.org/10.1016/j.apal.2015.05.006>

A Martin-Löf test \mathcal{U} is universal if it captures all non-Martin-Löf random sequences, and it is optimal if for every Martin-Löf test \mathcal{V} there is a constant c such that for all n the set \mathcal{V}_{n+c} is contained \mathcal{U}_n .

We study the computational differences between universal and optimal Martin-Löf tests as well as the effects that these differences have on both the notion of layerwise computability and the Weihrauch degree of LAY, the function that produces a bound for a given Martin-Löf random sequence’s randomness deficiency. We prove several robustness results concerning the Weihrauch degree of LAY. Along similar lines we also study the principle RD, a variant of LAY outputting the precise randomness deficiency of sequences instead of only an upper bound as LAY.

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3.9 Constructive reverse mathematics: an introduction

Hajime Ishihara (*JAIST – Ishikawa, JP*)

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Main reference H. Ishihara, “Constructive reverse mathematics: compactness properties,” in L. Crosilla, P. Schuster (eds.), “From Sets and Types to Analysis and Topology,” *Oxford Logic Guides*, Vol. 48, pp. 245–267, Oxford University Press, 2005.

URL global.oup.com/uk/isbn/0-19-856651-4

A mathematical theory consists of axioms describing mathematical objects in the theory, and logic being used to derive theorems from the axioms.

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet), and classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

Intuitionistic mathematics has axioms: the weak continuity for numbers (WCN) and the fan theorem (FAN), and constructive recursive mathematics has axioms: extended Church’s thesis (ECT) and Markov’s principle (MP). A common consequence of intuitionistic mathematics and constructive recursive mathematics is the Kreisel-Lacombe-Shoenfield-Tsejtin theorem (KLST) which is inconsistent with classical mathematics:

Every mapping from a complete separable metric space into a metric space is continuous.

The Friedman-Simpson-program (classical reverse mathematics) [2] is a formal mathematics using classical logic with a very weak set existence axiom. Its main question is “Which set existence axioms are needed to prove the theorems of ordinary mathematics?”, and many

classical theorems have been classified by set existence axioms of various strengths. Since classical reverse mathematics is formalized with classical logic, we cannot classify theorems in intuitionistic mathematics nor in constructive recursive mathematics which are inconsistent with classical mathematics such as KLST.

The purpose of constructive reverse mathematics [1] is to classify various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations.

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3.10 Decomposing Borel functions and generalized Turing degree theory

Takayuki Kihara (*University of California – Berkeley, US*)

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The Jayne-Rogers Theorem states that a function from an absolutely Souslin-F set into a separable metrizable space is first-level Borel measurable (that is, the preimage of each F_σ set under the function is again F_σ) if and only if it is decomposable into countably many continuous functions with Δ_2^0 domains. Recently, Gregoriades, K., and Ng [1, 2] used the Louveau separation theorem and the Shore-Slaman join theorem to show that if the preimage of a Σ_α^0 set under a function from an analytic space into a Polish space is again $\Sigma_{\beta+1}^0$ then the function is decomposable into countably many functions each of which is $\Sigma_{\gamma+1}^0$ -measurable for some γ with $\gamma + \alpha \leq \beta$. As shown by K. and Pauly [3], by combining other computability-theoretic methods, this theorem can be used to construct a family of continuum many infinite dimensional Cantor manifolds with property C in the sense of Haver/Addis-Gresham whose Borel structures at an arbitrary finite rank are mutually non-isomorphic.


Now we discuss possible extensions of this decomposition theorem of Borel functions. Is there a generalization of the theorem in higher measurability levels such as Nikodym's hierarchy of Selivanovskii's C-sets, Kolmogorov's R-sets and beyond? Is there a generalization in a wider category of topological spaces? We mainly focus on the latter problem, and give a few results on separation axioms and quasi-minimal enumeration degrees.

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3.11 Convergence Theorems in Mathematics: Reverse Mathematics and Weihrauch degrees versus Proof Mining

Ulrich Kohlenbach (TU Darmstadt, DE)

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We discuss the issue of how to formulate the computational content of convergence statements and compare the information provided by reverse mathematics, Weihrauch degrees and proof mining.

‘Proof Mining’ emerged as a systematic program during the last two decades as a new applied form of proof theory and has successfully been applied to a number of areas of core mathematics (see [3] for a book treatment of this paradigm covering the development up to 2008). This program has its roots in Georg Kreisel’s pioneering ideas of ‘unwinding of proofs’ going back to the 1950’s who asked for a ‘shift of emphasis’ in proof theory away from issues of mere consistency of mathematical theories (‘Hilbert’s program’) to the question ‘What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?’ Proof Mining is concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of infinitary noneffective principles. The main logical tools for this are so-called proof interpretations. Logical metatheorems based on such interpretations have been applied with particular success in the context of nonlinear analysis including fixed point theory (e.g. [8]), ergodic theory (e.g. [4, 11]), continuous optimization (e.g. [9, 5] and abstract Cauchy problems ([6]). The combinatorial content can manifest itself both in explicit effective bounds as well as in the form of uniformity results.

In this talk we focus on convergence theorems. In many cases one can show that a computable rate of convergence cannot exist (see e.g. [12]). In terms of (intuitionistic) reverse mathematics this usually corresponds to the fact that the Cauchy statement for the sequence (x_n) at hand implies the law-of-excluded-middle-principle for Σ_1^0 -formulas (Σ_1^0 -LEM which is also called LPO, see [16, 10]) and that the existence of a limit requires arithmetical comprehension ACA. In terms of Weihrauch degrees one often has, corresponding to this, that $\lim \equiv_W \lim(x_n)$ (see [12]). We show that Proof Mining provides more detailed information on noneffective convergence statements by extracting explicit and highly uniform subrecursive bounds on the so-called metastable (in the sense of Tao [14, 15]) reformulation of the Cauchy property. These bounds also allow for a detailed analysis of the convergence statements in terms of the algorithmic learnability of a rate of convergence which under certain conditions may result in oscillation bounds (see [10, 2]). In some cases this can be converted into full rates of convergence. We exemplify this with strong convergence results that are based on Fejér monotonicity of sequences defined by suitable iterations of nonlinear functions ([9]). We give applications of this in the context of the proximal point algorithm in Hilbert spaces ([9]) and to recent results ([1]) of convex feasibility problems in $\text{CAT}(\kappa)$ -spaces ([5]). We also discuss a recent asymptotic regularity result of a general alternated iteration procedure in $\text{CAT}(0)$ -spaces which applies to the resolvents of lower semi-continuous convex functions ([1]). From the *prima facie* highly noneffective convergence proof in [1] a simple exponential rate of convergence could be extracted using the logical machinery ([13]). In all these cases already the proof of the Cauchy property *prima facie* made use of ACA which, however, gets eliminated in the course of the extraction procedure.

We also briefly mention an explicit bound extracted recently in the context of nonlinear semigroups from a proof based on the weak (‘binary’) König’s lemma WKL ([7]).

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3.12 On the Uniform Computational Content of the Baire Category Theorem

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Joint work of Brattka, Vasco; Hendtlass Matthew; Kreuzer, Alexander P.

Main reference V. Brattka, M. Hendtlass, A. P. Kreuzer, “On the Uniform Computational Content of the Baire Category Theorem,” arXiv:1510.01913v1 [math.LO], 2015.

URL <http://arxiv.org/abs/1510.01913v1>

We study the uniform computational content of different versions of the Baire Category Theorem in the Weihrauch lattice. The Baire Category Theorem can be seen as a pigeonhole principle that states that a complete (i.e., “large”) metric space cannot be decomposed into countably many nowhere dense (i.e., “small”) pieces. The Baire Category Theorem is an illuminating example of a theorem that can be used to demonstrate that one classical theorem can have several different computational interpretations. For one, we distinguish two different logical versions of the theorem, where one can be seen as the contrapositive form of the other one. The first version aims to find an uncovered point in the space, given a sequence of nowhere dense closed sets. The second version aims to find the index of a closed set that is somewhere dense, given a sequence of closed sets that cover the space. Even though the two statements behind these versions are equivalent to each other in classical logic, they are not equivalent in intuitionistic logic and likewise they exhibit different computational behavior in the Weihrauch lattice. Besides this logical distinction, we also consider different ways how the sequence of closed sets is “given”. Essentially, we can distinguish between positive and negative information on closed sets. We discuss all the four resulting versions of the Baire Category Theorem. Somewhat surprisingly it turns out that the difference in providing the input information can also be expressed with the jump operation. Finally, we also relate the Baire Category Theorem to notions of genericity and computably comeager sets.

3.13 From Well-Quasi-Orders to Noetherian Spaces: Reverse Mathematics results and Weihrauch lattice questions

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Joint work of Frittaion, Emanuele; Hendtlass, Matthew; Marcone, Alberto; Shafer, Paul; Van der Meeren, Jeroen

Main reference E. Frittaion, M. Hendtlass, A. Marcone, P. Shafer, J. Van der Meeren, “Reverse mathematics, well-quasi-orders, and Noetherian spaces,” arXiv:1504.07452v2 [math.LO], 2015.

URL <http://arxiv.org/abs/1504.07452v2>

We study some theorems by Goubault-Larrecq from the viewpoint of reverse mathematics. These theorems deal with the relationship between well-quasi-orders and Noetherian spaces. The main result is the following:

► **Theorem 1** (RCA_0). *The following are equivalent:*

1. ACA_0 ;
2. if \mathcal{Q} is wqo then $\mathcal{A}(\mathcal{P}_f^b(\mathcal{Q}))$ is Noetherian;
3. if \mathcal{Q} is wqo then $\mathcal{U}(\mathcal{P}_f^b(\mathcal{Q}))$ is Noetherian;
4. if \mathcal{Q} is wqo then $\mathcal{U}(\mathcal{P}_f^\#(\mathcal{Q}))$ is Noetherian;
5. if \mathcal{Q} is wqo then $\mathcal{U}(\mathcal{P}^b(\mathcal{Q}))$ is Noetherian;
6. if \mathcal{Q} is wqo then $\mathcal{U}(\mathcal{P}^\#(\mathcal{Q}))$ is Noetherian.

These statements are of the form:

$$\forall X (\forall Z \Phi(X, Z) \implies \forall Y \Psi(X, Y))$$

with Φ and Ψ arithmetical (because both “ \mathcal{Q} is wqo” and “ $\mathcal{U}(\mathcal{Q})$ is Noetherian” are Π_1^1). Thus, even if they are Π_2^1 , they do not fit nicely in the problem/solution pattern usually used to translate Π_2^1 statements into multi-valued functions analyzed in the Weihrauch lattice setting.

We suggest to rewrite

$$\forall X (\forall Z \Phi(X, Z) \implies \forall Y \Psi(X, Y))$$

as

$$\forall X \forall Y (\neg \Psi(X, Y) \implies \exists Z \neg \Phi(X, Z)).$$

Now a problem is a pair consisting of a quasi-order \mathcal{Q} and a witness to the fact that $\mathcal{U}(\mathcal{P}^\#(\mathcal{Q}))$ is not Noetherian. Its solutions are the sequences witnessing that \mathcal{Q} is not wqo.


In fact the proofs of both directions of the reverse mathematics results actually work with

if $\mathcal{U}(\mathcal{P}^\#(\mathcal{Q}))$ is not Noetherian then \mathcal{Q} is not wqo

so the above translation in problem/solution form is quite faithful.

3.14 Separation of randomness notions in Weihrauch degrees

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Joint work of Hölzl, Rupert; Miyabe, Kenshi

We consider randomness notions in the Weihrauch degrees. Let WR, SR, CR, MLR, W2R, DiffR, and 2R be the classes of Kurtz random sets, Schnorr random sets, computably random sets, ML-random sets, weakly 2-random sets, difference random sets, and 2-random sets, respectively. These notions naturally induce operations in the Weihrauch degrees, that we denote by the same notations. In particular, MLR has been studied in the literature. Here, we only consider the usual Turing relativization.

We have the following reductions:

$$\text{WR} <_W \text{SR} \leq_W \text{CR} <_W \text{MLR} <_W \text{W2R} <_W \text{2R}.$$

The strictness of the inequalities above can be proved by looking at hyperimmune degrees, high degrees, and minimal degrees.

In contrast, we need to make use of uniformity to prove the separation between MLR and DiffR.

Whether the Weihrauch degrees of SR and CR can be separated remains an open question.

3.15 On the existence of a connected component of a graph

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Joint work of Gura, Kirill; Hirst, Jeffrey L.; Mummert, Carl

Main reference K. Gura, J. L. Hirst, C. Mummert, “On the existence of a connected component of a graph,”
Computability, 4(2):103–17, 2015.

URL <http://dx.doi.org/10.3233/COM-150039>

We study the reverse mathematics and computable analysis properties of countable graph theory. We focus on two problems. The first is to construct a single connected component of a countable graph, while the second is to decompose a countable graph into connected components. We show that each of these problems is strongly Weihrauch equivalent to its parallelized form and to the parallelized form of LPO. We also study problems relating to countable graphs in which each connected component is finite, and countable graphs with a finite number of connected components.

3.16 Closed choice and ATR

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The concept of “iterating taking a limit over some countable ordinal” can be formalized as a Weihrauch degree, and this Weihrauch degree is shown to be equivalent to $\text{UC}_{\mathbb{N}^{\mathbb{N}}}$ (unique choice on Baire space). $\text{UC}_{\mathbb{N}^{\mathbb{N}}}$ is strictly below $\text{C}_{\mathbb{N}^{\mathbb{N}}}$ (choice on Baire space), which is another candidate for a Weihrauch degree corresponding to ATR_0 .

Looking at determinacy principles, in reverse math Δ_1^0 -determinacy on Cantor space, Δ_0^0 -determinacy on Baire space and Σ_1^0 -determinacy on Baire space are all equivalent to ATR_0 . For Weihrauch degrees, the two former are equivalent to $\text{UC}_{\mathbb{N}^{\mathbb{N}}}$, whereas the latter is at least as hard as $\text{C}_{\mathbb{N}^{\mathbb{N}}}$.

3.17 On Weihrauch Degrees of k -Partitions of the Baire Space

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In [7] K. Weihrauch introduced some notions of reducibility for functions on topological spaces turned out useful for understanding the non-computability and non-continuity of decision problems in computable analysis and constructive mathematics. In particular, the following three notions of reducibility between functions $f, g : X \rightarrow Y$ on topological spaces were introduced: $f \leq_0 g$ (resp. $f \leq_1 g$, resp. $f \leq_2 g$) iff $f = g \circ H$ for some continuous function $H : X \rightarrow X$ (resp. $f = F \circ g \circ H$ for some continuous functions $H : X \rightarrow X$, resp. $F : Y \rightarrow Y$, $f(x) = F(x, g(H(x)))$) for some continuous functions $H : X \rightarrow X$ and $F : X \times Y \rightarrow Y$). In this way we obtain preorders $(Y^X; \leq_i)$, $i \leq 2$, on the set Y^X of all functions from X to Y .

The notions are nontrivial even for the case of discrete spaces $Y = k = \{0, \dots, k-1\}$ with $k < \omega$ points (we call functions $A : X \rightarrow k$ *k-partitions* of X because they are in a natural bijective correspondence with the partitions (A_0, \dots, A_{k-1}) of X where $A_j = f^{-1}(j)$). E.g., for $k = 2$ the relation \leq_0 coincides with the classical Wadge reducibility [3].

In [1, 2] P. Hertling gave useful “combinatorial” characterizations of initial segments of the degree structures under Weihrauch reducibilities on k -partitions of the Baire space $\mathcal{N} = \omega^\omega$ whose components are finite boolean combinations of open sets. The Baire space is important in this context because it is commonly used in computable analysis [8] for representing many other spaces of interest. In particular, the structure $(k^\mathcal{N}; \leq_0)$ induces (via total admissible representations) the fine hierarchies of k -partitions of quasi-Polish spaces discussed in [6].

In this work we attempt to extend the characterizations from [1, 2] to as large segments of Weihrauch degrees of k -partitions as possible. In particular, for any countable ordinal $\alpha \geq 2$ we try to characterize the quotient-posets of the preorders $((\Delta_\alpha^0)_k; \leq_i)$ for any $i \leq 2$, where $(\Delta_\alpha^0)_k$ is the class of k -partitions of \mathcal{N} with components in Δ_α^0 . It is easy to see that $(\Delta_\alpha^0)_k$ is an initial segment of $(k^\mathcal{N}; \leq_i)$ for each $i \leq 2$, i.e. $A \leq_i B \in (\Delta_\alpha^0)_k$ implies $A \in (\Delta_\alpha^0)_k$.

By a *forest* we mean a poset without infinite chains in which every upper cone $\{y \mid x \leq y\}$ is a chain. A *k-forest* is a triple $(F; \leq, c)$ consisting of a forest $(F; \leq)$ and a labeling $c : F \rightarrow k$. Let $\tilde{\mathcal{F}}_k$ denote the class of countable k -forests without infinite chains. Note that we use tilde in our notation in order to distinguish $\tilde{\mathcal{F}}_k$ from the class \mathcal{F}_k of finite k -forests considered in a series of previous publications.

A *0-morphism* (resp. *1-morphism*, resp. *2-morphism*) $f : (P; \leq, c) \rightarrow (P'; \leq', c')$ between k -forests is a monotone function $f : (P; \leq) \rightarrow (P'; \leq')$ satisfying $c = c' \circ f$ (resp. satisfying $\forall p, q \in P (c(p) \neq c(q) \rightarrow c'(f(p)) \neq c'(f(q)))$, resp. satisfying $\forall p, q \in P (p \leq q \wedge c(p) \neq c(q) \rightarrow c'(f(p)) \neq c'(f(q)))$).

For $i \leq 2$, the i -preorder on $\tilde{\mathcal{F}}_k$ is defined as follows: $(P, \leq, c) \leq_i (P', \leq', c')$, if there is an i -morphism from (P, \leq, c) to (P', \leq', c') . Obviously, \leq_0 implies \leq_1 and \leq_1 implies \leq_2 .

A basic result of this work is the following:

► **Theorem 1.** *For all $2 \leq k < \omega$ and $0 \leq i \leq 2$, the quotient-posets of $(\tilde{\mathcal{F}}_k; \leq_i)$ and $((\Delta_2^0)_k; \leq_i)$ are isomorphic.*

This results extends the mentioned characterizations from [1, 2]. For \leq_0 the result was in fact established in [4, 5]. Although for $i = 1, 2$ the isomorphism is induced by the same function from $\tilde{\mathcal{F}}_k$ to $(\Delta_\alpha^0)_k$ as in [4, 5], the proof requires some additional considerations. We believe that the result may be extended to a characterization of $((\Delta_\alpha^0)_k; \leq_i)$ for any $\alpha \geq 3$, although the proof becomes technically much more involved. So far we succeeded with the proof only for $i = 0$.


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3.18 A simple conservation proof for ADS


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It is known that the first-order part of infinite Ramsey’s theorem can be approximated by a version of Paris/Harrington Principle. In this talk, I will give a simple proof of a partial conservation result for ADS based on this idea.

3.19 Evaluating separations in the Weihrauch lattice

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This research aims to develop a method of evaluating separation results in the Weihrauch lattice. Two multi-valued functions, or their degrees, are said to be *separated* if and only if they are not Weihrauch equivalent, i.e. at least one direction of the mutual reducibilities fails. A number of separation results are already established by existing researches, though, the *strengths* of those separations have never been discussed.

Our proposal is to use a suitable ideal notion for evaluating the strengths of separations. Say a non-empty downward closed subset of the pointed Weihrauch lattice is a *stable ideal* if it is closed under the compositional product. Given a stable ideal \mathcal{I} and two degrees \mathcal{F} and \mathcal{G} , if we define $\mathcal{F} \leq_{\mathcal{I}} \mathcal{G}$ by the existence of an $I \in \mathcal{I}$ for which $\mathcal{F} \leq_W I \cdot \mathcal{G} \cdot I$, the relation $\leq_{\mathcal{I}}$ turns out to be a preorder. We shall then ask an appropriate \mathcal{I} , for separated degrees, such that they are still separated with respect to $\leq_{\mathcal{I}}$. The strength of the separation will approximately be evaluated by such an \mathcal{I} . In what follows we list some concrete examples of stable ideals.

The most primary example is given by *continuous degrees*. A continuous degree is the degree of a multi-valued function having a continuous choice function. The class of continuous degrees is indeed a stable ideal, and the induced partial order is characterized by the continuous Weihrauch reducibility.

Next let us introduce an ideal which captures the *rigidness* of the hierarchy of LLPO_n ’s [1]. A multi-valued function F is said to be *properly discontinuous* if its restriction to $\{\lim \alpha_i, \alpha_i \mid i \in \omega\}$ does not have a continuous choice function for some converging sequence $\{\alpha_i\}_{i \in \omega}$ on $\text{dom}(F)$ whose limit is still in $\text{dom}(F)$, and to be *improperly discontinuous* otherwise. All LLPO_n ’s and LPO are properly discontinuous. Accordingly a degree is *proper* and *improper* in cases that its arbitrary representative is properly and improperly discontinuous, respectively. The class \mathcal{L} of improper pointed degrees turns out to be a stable ideal while that of proper degrees is a filter of the Weihrauch lattice. If we say, given a reduction $\mathcal{F} \leq_W \mathcal{G}$, that \mathcal{G} is *rigidly separated* from \mathcal{F} in case $\mathcal{G} \leq_W \mathcal{G}/\mathcal{F}$, where $(-/-)$ is the

residual implication of the compositional product, then rigid separations never happen in above LPO since $\text{LPO} \leq \mathcal{F}, \mathcal{G}$ implies $\mathcal{G}/\mathcal{F} \in \mathcal{L}$; while the following hierarchy can be shown via rigid separations.

$$\cdots <_{\mathcal{L}} \text{LLPO}_{n+1} <_{\mathcal{L}} \text{LLPO}_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} \text{LLPO}_2 <_{\mathcal{L}} \text{LPO}$$

In particular LPO is the top with respect to $\leq_{\mathcal{L}}$. Hence the above *earliest* hierarchy found by K. Weihrauch in 1990s has a remarkable rigidity, which, for example, the classifications of closed choices (see [2]) do not have. We also remark to the fact that every improper degree is ω -indiscriminative. Hence for instance ω -indiscriminateness of the *cohesiveness* COH is automatically derived from its implicational presentation $\text{COH} \equiv_W \text{WKL}' / \lim$ [1].

For further variations of stable ideals, let us consider on analogs of the big five systems in reverse mathematics [4]. As usual, we interpret a *conditional* Π_2^1 -formula, i.e. a formula of the form $\varphi \equiv \forall X.(\varphi_0(X) \rightarrow \exists Y.\varphi_1(X, Y))$ where φ_0 and φ_1 are arithmetical, as the multi-valued function $\alpha \mapsto \{\beta \mid (\omega, \text{Pow}(\omega)) \models \varphi_0(\alpha) \wedge \varphi_1(\alpha, \beta)\}$. If we say a multi-valued function F is a *uniformity* when $F[\alpha]$ contains an α -computable point for every $\alpha \in \text{dom}(F)$, and say a degree is a uniformity degree when its arbitrary representative is a uniformity, then all conditional Π_2^1 -theorems of RCA are interpreted as uniformities. Moreover the reducibility $\leq_{\mathcal{U}}$, where \mathcal{U} is the stable ideal of pointed uniformity degrees, is characterized by computable reducibility proposed in [3]. It would not go too far to say that this ideal \mathcal{U} is the most natural one among all thinkable analogs of RCA. Also letting $\mathcal{U}(\text{WKL})$ and $\mathcal{U}(\lim)$ be the smallest stable ideals containing $\mathcal{U} \cup \{\text{WKL}\}$ and $\mathcal{U} \cup \{\lim\}$, respectively, we obtain the similar *soundness* works for WKL and ACA. In particular $\mathcal{U}(\lim)$ can be obtained as the downward closure of $\{\lim^i \cdot \mathcal{F} \mid i \in \omega, \mathcal{F} \in \mathcal{U}\}$. As expected, the three ideals \mathcal{U} , $\mathcal{U}(\text{WKL})$ and $\mathcal{U}(\lim)$ behaves as good *steps* for evaluating separation results.

It is very likely that natural analogs of ATR and Π_1^1 -CA can also be found, according to the characterizations of their ω -models. However, on another front, most pre-known separations in the Weihrauch lattice do not have the strengths of above ATR, and the above three ideals suffice.

As future directions, we suggest the following three. Firstly it would be significant to have a general result which enable us to convert a separation to a stronger separation; namely a pair of a stable ideal \mathcal{I} , probably one of the above listed examples, and a mapping $(F, G) \mapsto (F_{\mathcal{I}}, G_{\mathcal{I}})$ such that $F \not\equiv_W G$ implies $F_{\mathcal{I}} \not\equiv_{\mathcal{I}} G_{\mathcal{I}}$. Secondly the *correctness* of the analogs for the big five systems should be concerned; in particular it is natural to ask if the provability of a conditional Π_2^1 -sentence in $\text{RCA}_{(0)}$ is equivalent to the condition that its interpretation is a uniformity in every model of $\text{RCA}_{(0)}$. The similar questions should be asked for $\text{WKL}_{(0)}$, $\text{ACA}_{(0)}$ and possibly also for the rest two. Finally a deep result on the relationship of the induced reducibility $\leq_{\mathcal{U}}$ and the *computable entailment* (see [3]) is strongly desirable. Such a result would show a formal connection of reverse mathematics and the classification of Weihrauch degrees.

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3.20 Hyper-degrees of 2nd-order polynomial-time reductions

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Ko, Friedman, and Kawamura et al. have shown common operators in analysis to map polynomial-time computable arguments to \mathcal{NP} -hard ones, thus gauging their non-uniform complexity. 2nd-order polynomial-time reductions compare the uniform computational complexity of operators in analysis, that is, functionals on Baire space $\mathbb{N}^{\mathbb{N}}$, respectively; cmp. [2]. They (have to) grant more runtime on ‘long’ arguments, expressed by 2nd-order polynomials, that is, terms $P(n, \mu)$ over $+$, \cdot , 1 and 1st and 2nd-order variables $n \in \mathbb{N}$ and $\mu \in \mathbb{N}^{\mathbb{N}}$, respectively; cmp. Mehlhorn (1976) – but are criticized for permitting, on arguments of exponential length, runtimes bounded by any constant-height exponential tower. We suggest a refined analysis in terms of the *hyperdegree* of the 2nd-order polynomial bounds according to the following

► **Lemma 1.** *Let P denote a 2nd-order polynomial.*

- (a) *For every integer d , $P(n, n \mapsto n^d)$ is an ordinary integer polynomial.*
- (b) *The (thus well-defined) mapping $\mathbb{N} \ni d \mapsto \text{DEG}(P)(d) := \deg(P(n, n \mapsto n^d))$ is in turn an integer polynomial in d .*
- (c) *For every fixed ordinary polynomial μ it holds $\deg(P(n, \mu)) = \text{DEG}(P)(\deg \mu)$.*
- (d) *It holds $\text{DEG}(P(Q, \cdot)) = \text{DEG}(P) \cdot \text{DEG}(Q)$.*
- (e) *It holds $\text{DEG}(P(\cdot, Q)) = \text{DEG}(P)(\text{DEG}(Q))$.*
- (f) *The integer $\deg(\text{DEG}(P))$ coincides with the depth of P as defined in [1].*

In particular 2nd-order polynomials of hyperdegree one are closed under composition. This suggests to try to refine recent reductions [3, 6, 5].

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4 Open Problems

- **Question (Ackerman, Freer, Pauly, Roy).** “Does currying give rise to an interesting operator?”

Suppose $f: X \times Y \rightarrow Z$ is such that for all x , the map $f(x, \cdot): Y \rightarrow Z$ is continuous.

One could instead consider the “curried” version $F: X \rightarrow \mathcal{C}(Y, Z)$, defined by $x \mapsto f(x, \cdot)$.

Clearly $f \leq_W F$. If $Y = \mathbb{N}$ then $F \leq_W \hat{f}$. In general, can $f \mapsto F$ be seen as an interesting operator? What can one say about this map in general?

Background: For continuous f , the currying map is computable; see [8, Prop. 3, part 2]. In probability theory and elsewhere, often both the curried and uncurried functions are of interest; see, e.g., the discussion of disintegration and conditional distributions in [1, Def. 2.1].

- **Question (Brattka).** Does strong Weihrauch reducibility induce a lattice structure? It is clear that the sum operation induces an infimum and that the coproduct, which induces the supremum with respect to ordinary Weihrauch reducibility, does not yield a supremum for strong Weihrauch reducibility. But currently it is not known whether there is some other way to obtain a supremum for strong Weihrauch reducibility.
- **Question (Brattka).** How do different combinations of the stable version of Ramsey's Theorem for pairs and the cohesiveness problem compare to Ramsey's Theorem for pairs? We have

$$\text{SRT}_2^2 \sqcup \text{COH} \leq_W \text{SRT}_2^2 \times \text{COH} \leq_W \text{SRT}_2^2 * \text{COH}$$

and

$$\text{SRT}_2^2 \sqcup \text{COH} \leq_W \text{RT}_2^2 \leq_W \text{SRT}_2^2 * \text{COH}.$$

What else can be said?

In an upcoming paper initiated during the seminar, Damir Dzhafarov, Denis Hirschfeldt and Ludovic Patey establish several negative results about some of the remaining reductions.

- **Problem (Fernandes, Ferreira and Ferreira [2])** Define a notion of integration within BTFA that works well for a sufficiently robust class of continuous functions (e.g., a class that contains many analytical functions).
- **Conjecture (Fernandes, Ferreira and Ferreira [2])** Show that over BTFA (and within the framework of [3]), Weierstrass approximation theorem is equivalent to the totality of the exponential function.
- **Question (Fouché).** Complexity of Fourier dimension: Write $M_+[0, 1]$ for the Radon probability measures on the unit interval. An s -Fourier measure on the unit interval is a Radon measure μ such that its Fourier transform satisfies

$$|\hat{\mu}(\xi)|^2 \leq \frac{1}{(1 + |\xi|)^s},$$

for all real ξ . Define

$$\text{Fourm} := \mathcal{A}[0, 1] \times [0, 1] \rightrightarrows M_+[0, 1]$$

by

$$\mu \in \text{Fourm}(A, s) \leftrightarrow \mu \text{ is an } s\text{-Fourier measure and } \text{supp} \mu \subseteq A.$$

Determine the Weihrauch degree of Fourm.

- **Conjecture (Hölzl and Shafer [4]).** There exist universal tests \mathcal{U} and \mathcal{V} such that

$$\text{RD}_{\mathcal{U}} \not\equiv_{\text{SW}} \text{RD}_{\mathcal{V}}.$$

- **Question (Le Roux & Pauly [5])** Is there some $k \in \mathbb{N}$ such that $\text{XC}_{[0,1]} \star \text{XC}_{[0,1]} \leq_W \text{XC}_{[0,1]}^k$?

Here $\text{XC}_{[0,1]}$ is the restriction of closed choice on the unit interval to convex sets, i.e. intervals. \star denotes the sequential composition of Weihrauch degrees.

After the seminar, this question has been answered in the negative by Takayuki Kihara.

- **Question (Marcone)** “What do the Weihrauch hierarchies look like once we go to very high levels of reverse mathematics strength?”

So far the Weihrauch hierarchies have been used to obtain a finer picture of the relationships between statements that are provable in ACA_0 . Yet the reverse mathematics picture of the relationships between mathematical statements goes well beyond ACA_0 . Can we use the Weihrauch hierarchies to obtain information about the relationships between statements that are equivalent to ATR_0 or to $\Pi_1^1\text{-CA}_0$? One could start by looking at different forms of the perfect tree theorem.

- **Question (Pauly [7])** Is there some $k \in \mathbb{N}$ such that $\text{AoUC}_{[0,1]} \star \text{AoUC}_{[0,1]} \leq_W \text{AoUC}_{[0,1]}^k$?

Here $\text{AoUC}_{[0,1]}$ is the restriction of closed choice on the unit interval to sets that are either the entire unit interval or singletons. \star denotes the sequential composition of Weihrauch degrees.

After the seminar, this question has been answered in the negative by Takayuki Kihara.

- **Question (Yokoyama)** For given a coloring $P : [\mathbb{N}]^2 \rightarrow 2$, a *grouping for P* is an infinite family of ω -large finite sets $\{F_0 < F_1 < \dots\}$ such that

$$\forall i_1 < \dots < i_n \exists c < k \forall x_1 \in F_{i_1}, \dots, \forall x_n \in F_{i_n} P(x_1, \dots, x_n) = c$$

Now GP (grouping principle for ω -largeness) asserts that for any coloring $P : [\mathbb{N}]^2 \rightarrow 2$, there exists an infinite grouping for P . Then, what is the reverse mathematical strength of GP? Trivially, it is a consequence of RT_2^2 .

Ludovic Patey answered this question as follows. He showed that GP does not imply ADS, and it implies RRT_2^2 . Also, he showed that there exists an ω -model of $\text{RCA}_0 + \text{SGP}$ with only low sets, where SGP is a grouping principle for stable colorings. The second result give a good information to calibrate the proof theoretic strength of RT_2^2 . See [6].

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