Report from Dagstuhl Seminar 16031

Well Quasi-Orders in Computer Science

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 16031 "Well Quasi-Orders in Computer Science", the first seminar devoted to the multiple and deep interactions between the theory of Well quasi-orders (known as the Wqo-Theory) and several fields of Computer Science (Verification and Termination of Infinite-State Systems, Automata and Formal Languages, Term Rewriting and Proof Theory, topological complexity of computational problems on continuous functions). Wqo-Theory is a highly developed part of Combinatorics with ever-growing number of applications in Mathematics and Computer Science, and Well quasi-orders are going to become an important unifying concept of Theoretical Computer Science. In this seminar, we brought together several communities from Computer Science and Mathematics in order to facilitate the knowledge transfer between Mathematicians and Computer Scientists as well as between established and younger researchers and thus to push forward the interaction between Wqo-Theory and Computer Science.

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1 Executive Summary

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Computer Science, being a huge and complex conglomerate of theoretical disciplines, technological advances and social methodologies, strongly needs unifying concepts and techniques. In particular, relevant mathematical concepts and theories are required. The notion of well quasi-order (or almost-full relation, if transitivity is not required – a notion preferred by some authors) was discovered independently by several mathematicians in the 1950-s and quickly evolved to a deep theory with many applications and remarkable results. Soon afterwards, well and better quasi-orders started to appear more and more frequently in different parts of theoretical computer science such as automata theory, term rewriting, verification of infinite-state systems, computations with infinite data, and others. Accordingly, an increasing number of researchers from different fields of computer science use notions and methods of Wqo-Theory. Therefore, it seemed to be the right time to have a broad discussion on how to speedup this process and to better understand the role of well quasi-orders in theoretical computer science.

Topics of the seminar

During this seminar we concentrated on the following four topics:
1. Logic and proofs
2. Automata and formal languages
3. Topological issues
4. Verification and termination problems

Logic and proofs

Well quasi-orders, originally introduced in algebra, soon played an important role in proof theory: Higman's Lemma and Kruskal's Theorem are examples of theorems that are not provable in Peano Arithmetic. Determining the proof-theoretic strength of these (types of) theorems, as well as classifying them in terms of Reverse Mathematics, constituted an important endeavor. The concept of a WQO naturally extends to the more complex concept of a better quasi-order (BQO) which deals with infinite structures. Again, the proof theoretic strength of theorems on BQOs has been/must be investigated, and the theorems themselves can be used for more sophisticated termination problems. One of the open challenges is the strength of Fraïssé's order type conjecture. Non-constructive proofs of this type of theorems (on WQOs) include proofs using the so-called minimal-bad-sequence argument. Investigating their strengths and also their computational content, via Friedman's A-translation or Gödel's Dialectica Interpretation, has led to interesting results. To optimize these techniques so that realistic programs can be extracted from these classical proofs, using bar recursion, update recursion, selection functions, etc., is ongoing work.

Automata and formal languages

Well quasi-orders have many-fold connections to automata theory and formal language theory. In particular, there are nice characterizations of regular and context-free languages in terms of well quasi-orders, some lower levels of the concatenation hierarchies admit characterizations in terms of the subword relation and its relatives. Such characterizations sometimes help in getting new results, say on decidability of some levels of the concatenation hierarchy (Glasser, Schmitz, Selivanov). The same applies to ω-languages, though in this case the relationships are less investigated.

On the topological level, it is known that Wadge reducibility (or reducibility by functions on ω-words computable by finite automata) are well quasi-orders on the class of ω-regular finite partitions of the Cantor space. Using some variants of the Kruskal theorem on quasi-orderings of labeled trees, Selivanov was able to completely characterize the corresponding partial order, obtaining thus a complete extension of the Wagner hierarchy from sets of finite partitions.

The mentioned relationships between Wqo-theory and formal languages are currently not well systematized, and many natural questions remain open. Further insights in this topic is essential for the development of this field.

Topological issues

An important task in computing with infinite data is to distinguish between computable and non-computable functions and, in the latter case, to measure the degree of non-computability. Usually, functions are non-computable since they are not even continuous, hence a somewhat easier and more principal task is in fact to understand the degree of discontinuity of functions. This is achieved by defining appropriate hierarchies and reducibility relations.

In classical descriptive set theory, along with the well-known hierarchies, Wadge introduced and studied an important reducibility relation on subsets of the Baire space. As shown by van Engelen et al, von Stein, Weihrauch and Hertling, this reducibility of subsets of topological spaces can be generalized in various ways to a reducibility of functions on a topological space. In this way, the degrees of discontinuity of several important computational problems were classified. The transfer from sets to functions requires some notions and results of Wqo-theory in order to define and study hierarchies and reducibilities arising in this way.

Verification and termination problems

WQOs made their debut in computer science when Don Knuth suggested that Kruskal's Theorem might find an application in proving termination of programs. This was achieved a few years later by Nachum Dershowitz and the advent of recursive path orderings. Today, it is probably the area of software verification that provides the largest number of applications of WQOs in computer science. The decidability of coverability for well-structured transition systems (WSTS) crucially relies on the very properties of well quasi-orders. WSTS include Petri nets and their extensions, and more generally affine nets. They also include lossy channel systems, weak memory models, various process algebras, data nets, certain abstractions of timed Petri nets, and certain parametrized transition systems. The verification of new classes of transition systems prompts for new classes of WQOs. In addition to this, understanding the computational complexity of the resulting verification algorithms requires a finer analysis of minimal-bad-sequence arguments and their relation to hierarchies of recursive functions (Hardy, fast growing, etc.)

Report

One of the central purposes of the proposed seminar was to bring together researchers from Wqo-theory and those from the related areas of computer science who actively apply notions and techniques of Wqo-theory. We wanted thus to encourage more interaction between the different communities, leading finally to a significant development of the mentioned fields. Overall, the seminar was very stimulating. The initial concerns that the four topics of the seminar might remain separate was quickly brushed off, as verification talks relied on concepts from logic (e.g., maximal order types), topological issues resonated with verification (e.g., Noetherian spaces), and all participated actively in vivid sessions of setting up and discussing open questions.

To facilitate the interaction, each of the four topics of the seminar started with one or two introductory talks. In the topic **Logic and Proofs**, Diana Schmidt gave the first talk of the seminar “Ordinal notations, the maximal ordertypes of Kruskal’s Theorem and a tale of two cultures”, where she summarized work that started 40 years ago, but is still of interest to date. Andreas Weiermann complemented the introduction by addressing open problems concerning the so-called phase transitions (a topic which was then further elaborated by Lev Gordeev) in proof theory and the calculation of maximal ordertypes. A number of contributions focused on the formalization of Kruskal type theorems possibly including the extraction of computational content, leading to various practical exchange sessions in the evenings (see also next paragraph). The topic was continued on Tuesday by Alberto Marcone who gave a survey on WQOs and BQOs in Reverse Mathematics, followed various talks and discussions on open questions around better quasi-orderings. Another highlight of the second day were two contributions on the Graph Minor Theorem: Chun-Hung Liu reported on his recent proof of a conjecture by Robertson involving topological minors, and Michael Rathjen looked at the Graph Minor Theorem to determine its proof theoretic strength. Of the further contributions we want to specifically mention Julia Knight who made the connection between WQOs and Hahn-Fields.

The topic **Verification** was addressed by Philippe Schnoebelen in “Well quasi-orderings and program verification” on Monday afternoon, with a gentle introduction to the field of well-structured transition systems (WSTS). Although this was an introductory talk, he took the opportunity to introduce the class of priority channel systems as an example, whose decidability results rely on a form of well quasi-ordering with gap embedding. Such orderings require very high maximal order types, and led to the open question of giving a formula, or even lower and upper bounds, for those maximal order types. This has direct consequences on the complexity of verification. Alain Finkel proceeded to give another talk on the verification of WSTS, “Decidability results on infinitely branching WSTS”, which completed the introduction given by Philippe Schnoebelen, and quickly proceeded to explore the challenges of verifying WSTS that are infinitely branching. He explained that deciding termination, coverability, and boundedness could be done through computations that involve a so-called completion of the WSTS, generalizing the well-known Karp-Miller construction for Petri nets – some of these problems turn out to be decidable, some others not. Crucially, in the decidable cases, even if the WSTS is infinitely branching, its completion is always finitely branching. Maurice Pouzet made the observation that the key result used there, that every downwards closed set of a WQO is a union of finitely many ideals, is due to Kabil and himself in 1992, and that this was still true for the more general class of FAC (finite antichain) orders. FAC orders were the subject of the next presentation, by Mirna Džamonja, entitled “On the width of FAC orders”. She started from the fact that, while height, length and width of well quasi-orders are important notions, width generalizes to all FAC orders, and that we can compute the width of FAC orders defined from FAC constituents. One consequence is that for every ordinal α, there is a WQO of width exactly α. The main open question is to be able to compute the width of a finite product of FAC orders. The width of the product of two ordinals is known, but is given by a complex formula. Mirna Džamonja gave some new results on the question, in particular for some three-way products.

The final two Monday contributions were concerned with very different questions, namely mechanized proofs of Higman’s Lemma, one of the core results in well quasi-order theory. Christian Sternagel described a proof of Higman’s Lemma by so-called open induction, a concept akin to Scott induction in domain theory, as one of the participants noticed. The proof is a considerably simplified version of a proof of the same kind given by Alfons

Geser. Open induction was again the subject of Thomas Powell's talk "Open induction and the Dialectica interpretation" on Friday morning. Helmut Schwichtenberg concluded the afternoon of Monday, just before a rump session dedicated to open questions. He gave a talk "on the computational content of Higman's Lemma", stressing that different constructive proofs of the same theorem yield programs – by the Curry-Howard correspondence – that behave differently in practice. This resonated with the last morning talk of the same day, "An axiom-free Coq proof of Kruskal's Theorem", by Dominique Larchey-Wendling, on a recent constructive proof of the much more complex Kruskal theorem, inspired from and generalizing a proof due to Wim Veldman.

Other verification-oriented talks were given by Mizuhito Ogawa on Thursday morning, "Notes on regularity and WQOs, and well-structured pushdown systems", which gave new decidability results for coverability on extended forms of transition systems, through the use of so-called P-automata. In the evening, Roland Meyer explained how one can encode certain depth-bounded, breadth-bounded, and name-bounded processes of the pi-calculus into well-structured transition systems, and obtain decidability results through acceleration techniques, akin (again) to the Karp-Miller technique already mentioned above. Sylvain Schmitz provided us with a glimpse of the new complexity classes that had to be invented in recent years to characterize the complexity of the standard decidable problems for classical WSTS. These are classes of very high complexity: Ackermannian, hyper-Ackermannian, and others. This was a talk that unified the concerns of verification with the logical view, based on maximal ordertypes, and introduced by Andreas Weiermann and others. This provided a natural link with the last talk in the verification strand, "Trace universality for VASS", given by Simon Halfon, who explained what the problem was, that it relied on the fact that all bad sequences of finite subsets of d-tuples of natural numbers are finite, that it is Ackermann-complete for $d = 1$, and he then proceeded to explain what was known in the cases $d \geq 2$, on which he is working.

The topic **Topological Issues** was introduced by Victor Selivanov in "Well quasi-orders and Descriptive Set Theory" on Tuesday afternoon, and by Jean Goubault-Larrecq in "A Gentle Introduction to Noetherian Spaces" on Wednesday morning. Selivanov gave a short survey of the relationships between Wqo-Theory and Descriptive Set Theory, as well as a discussion of several interesting open questions in this field. Even the definition and basics of BQOs are related to Descriptive Set Theory, as was demonstrated in the talks by Alberto Marcone and Yann Pequignot on Tuesday. Other connections were given by Raphael Carroy on Tuesday (discussing some WQOs on continuous functions), by Oleg Kudinov on Wednesday (discussing joint results with Selivanov on definability issues for some popular WQOs) and by Peter Hertling on Friday (giving an overview of results about Weihrauch reducibilities). Goubault-Larrecq introduced an important topological extension of the notion of WQO which motivates several interesting open questions. Some of these questions were addressed in the subsequent Wednesday talks by Matthew de Brecht and Arno Pauly. Both mentioned directions in the topic **Topological Issues** look very prospective and deserve additional attention.

The topic **Automata and Formal Languages** was surveyed by Mizuhito Ogawa on Thursday morning who summarized several interesting facts relating WQOs to popular classes of languages and ω-languages on words and trees (cf. also the topic **Verification**). Automata and Formal Languages were also addressed on Tuesday by Selivanov who related WQOs to his extension of the Wagner hierarchy from the case of sets to the case of k-partitions. Another related talk was given on Thursday by Willem Fouché who discussed some applications of Ramsey theory to unavoidable regularities in words. Although **Automata and Formal**

Languages turned out slightly underrepresented at this seminar, the relationship between Wqo-Theory and Formal Languages seems quite important and deserves further investigation which we hope to include in a future seminar.

Overall, our seminar attracted 44 participants (10 from Germany, 22 from other European countries, 12 from Canada, Japan, Russia, South Africa, and USA) who contributed 33 talks. In addition, we included several problem sessions where we summarized all problems mentioned in the seminar. As a result of these sessions we give a list of open problems at the end of this report. Looking at the feedback the seminar was very well received amongst the participants. Positively mentioned was that the seminar involved "people from different backgrounds" who "can still share interest", or in other words "hearing people from different research areas discuss similar questions", and that "one week is too short :-)". Thoroughly enjoyed was also our two hour long walk in the snow on Wednesday afternoon. The great success of the seminar is not only due to the participants, but also to the staff in Saarbrücken and Dagstuhl, who did a splendid job in facilitating the seminar and making our stay a very pleasant one. Special thanks go to Susanne Bach-Bernhard for all the interaction related to the organization of the seminar and to Jutka Gasiorowski for her support in producing this report.

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3 Overview of Talks

3.1 A question about bad arrays

Andreas R. Blass (University of Michigan – Ann Arbor, US)

If a quasi-ordering $\leq$ of a set Q is not a better quasi-order, then this fact is witnessed by a continuous bad array, which means a continuous function F to Q (with the discrete topology) from the Baire space B of infinite subsets of the set $\mathbb{N}$ of natural numbers (topologized as a subspace of the product $2^{\mathbb{N}}$ of discrete two-point spaces) such that no set X in B has $F(X) \leq F(X - \{min(X)\})$. Identifying B with the set of paths through the tree T of finite increasing sequences of natural numbers, we have that each continuous map F from B to Q is given by a function into Q from a barrier in the tree T. The complexity of F can be measured by the ordinal height of the part of T lying between this barrier and the root. The smaller this height, the farther F is from being better quasi-ordered. In particular, the height is never 0, and it can take the value 1 if and only if Q is not even well quasi-ordered.

For any finite value h of the height, Ramsey's theorem allows one to arrange for F to have a very uniform structure, so that the order relation between $F(X)$ and $F(Y)$ depends only on the relative ordering, in $\mathbb{N}$, of the first h elements of X and the first h elements of Y. When $h = 2$, this uniformity makes the quasi-ordering $\leq$ unique on the range of F; the only such quasi-order is the Rado example.

I asked whether there is a similar uniqueness for larger finite values of h. Maurice Pouzet provided a negative answer. So I refined the question: How many such uniform examples are there, as a function of the height h? This seems to be an open problem.

3.2 Ordering functions

Raphael Carroy (University of Torino, IT)

Main reference R. Carroy, "A quasi-order on continuous functions", Journal of Symbolic Logic, Vol. 78(2), pp. 633–648, 2013.)
URL http://dx.doi.org/10.2178/jsl.7802150

I examine a notion of reduction between functions: a function f is *continuously reducible* to a function g whenever there are two continuous functions σ and τ such that $f = \tau \circ g \circ \sigma$. This is the topological equivalent of the *strong Weihrauch reducibility*.

After briefly discussing the relevance of other quasi-orders existing in the literature, I begin to analyze continuous functions between zero-dimensional Polish spaces with respect to continuous reducibility. I prove that the identity is complete among continuous functions, and reduces to any Borel function with uncountable image.

I also prove that this well orders the family of continuous functions with compact domains. Concerning more general families of functions, I generalize the Cantor-Bendixson analysis of closed sets to continuous functions, showing that it stratifies those with countable image into countably many layers, and describing the general structure of these layers.

Finally, I apply this analysis to obtain that embeddability between closed subsets of the Baire space is a better quasi-order.

3.3 Noetherian spaces and quantifier elimination

Matthew de Brecht (NICT – Osaka, JP)

Noetherian spaces are topological spaces which can be viewed as natural generalizations of well quasi-orders. They are defined as spaces whose open set lattice satisfies the ascending chain condition, or equivalently, as spaces in which every open set is compact. They are not Hausdorff in general.

We prove a simple quantifier elimination result for countably based sober Noetherian spaces. In particular, we show that if X and Y are countably based sober Noetherian spaces, and P is a boolean combination of open subsets of the product space $X \times Y$, then the projection of P onto Y is a boolean combination of open sets. This result is essentially a weak version of a well known theorem in algebraic geometry due to Chevalley concerning images of morphisms between schemes.

The result we present is purely topological, and can be viewed as an exercise to better understand Noetherian spaces. Our proof methods will use common tools from descriptive set theory, such as the Baire category theorem and the Hausdorff-Kuratowski theorem.

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3.4 Some uses of WQOs and BQOs in modal logic

Dick de Jongh (University of Amsterdam, NL)

In modal logic sometimes the class of models (or better: frames) can be seen as WQOs or BQOs. On that basis such logics can be shown to be finitely axiomatizable. I will sketch the case of the extensions of S4.3 and of $S5^2$. The latter is work of my student and now colleague Nick Bezhanishvili with Ian Hodkinson.

References

1 Nick Bezhanishvili, Ian Hodkinson. *All normal extension of S5-squared are finitely axiomatizable*. Studia Logica, vol. 78, 443-457, 2004.

3.5 Basics on (infinitely branching) WSTS

Alain Finkel (ENS – Cachan, FR)

Joint work of Michael Blondin, Jean Goubault-Larrecq, Pierre McKenzie, Alain Finkel
Main reference A. Finkel, J. Goubault-Larrecq, "Forward Analysis for WSTS, Part II: Complete WSTS", Logical Methods in Computer Science, Vol. 8(3), pp. 1–35, 2012.
URL http://arxiv.org/abs/1208.4549
Main reference M. Blondin, A. Finkel, P. McKenzie, "Handling Infinitely Branching WSTS", in Proc. of 41st Int'l Colloquium on Automata, Languages, and Programming (ICALP'14), LNCS, Vol. 8573, pp. 13–25, Springer, 2014.
URL http://dx.doi.org/10.1007/978-3-662-43951-7_2

WSTS (introduced in ICALP'87) is a model that allows verification of safety properties of infinite-state systems. We will recall the definition and the essential results of WSTS. Most decidability results concerning well-structured transition systems apply to the *finitely branching* variant. Yet some models (inserting automata, ω-Petri nets, ...) are naturally infinitely branching. Here we develop tools to handle infinitely branching WSTS by exploiting the crucial property that in the (ideal) completion of a well quasi-ordered set, downward closed sets are *finite* unions of ideals. Then, using these tools, we derive decidability results and we delineate the undecidability frontier in the case of the termination, the coverability and the boundedness.

3.6 On the width of FAC orders

Mirna Džamonja (University of East Anglia, Norwich, GB)

Joint work of Sylvain Schmitz, Philippe Schnoebelen, Mirna Džamonja

We investigate the ordinal invariants height, length and width of well quasi-orders, with the particular emphasis on width, which is an invariant also interesting in the case of the larger class of orders with finite antichain condition (FAC). We show that the width in the class of FAC orders is completely determined by the width in the class of WQOs, in the sense that if we know how to calculate the width of any WQO then we have a procedure to calculate the width of any given FAC order. We give formulas for the behavior of the width function under various classically defined operations with partial orders and obtain as a consequence a theorem that shows that for any ordinal α there is a WQO poset whose width is α. We make some progress towards a Minimax Theorem for the width function, complementing what was known about the height and the length. In addition to the Minimax Theorem, one of the main remaining questions is to give a complete formula for the width of the Cartesian products of WQOs. Even the width of the product of two ordinals is only known through a complex recursive formula. Although we have not given a complete answer to this question we have advanced the state of knowledge by considering some more complex special cases and in particular by calculating the width of certain products containing three factors.

3.7 BQOs and where they come from

Thomas E. Forster (University of Cambridge, UK)

A quasi-order $\langle X, \leq_X \rangle$ can be lifted naturally to a quasi-order on $V_\infty(X)$, the cumulative hierarchy based on X as a set of atoms. $\leq_X$ is BQO iff this lifted quasi-order is well founded. It turns out that this condition is equivalent to the condition that the lift to $H_{\aleph+1}(X)$ (the hereditary countable sets over X as a set of atoms) is well founded (but this uses DC). The proof is in [1]. Two questions: (i) can we dispense with DC? (ii) $H_{\aleph_1}(X)$ is a rather set-theoretic construction. Might we be able to use instead something like the free countable completion of $\langle X, \leq_X \rangle$?

References

1 "Better-quasi-orderings and Coinduction", *Theoretical Computer Science*, **309**, Issues 1-3, 2 December 2003, pp. 111–123.

3.8 The algorithmic complexity of recognizing unavoidable regularities of words.

Willem L. Fouché (UNISA – Pretoria, ZA)

Main reference W. L. Fouché, "Unavoidable regularities and factor permutations of words", in Proc. of the Royal Society of Edinburgh: Section A Mathematics, Vol. 125(3):519–524, 1995.
URL http://dx.doi.org/10.1017/S0308210500032650

Many theorems in combinatorics can be interpreted as statements expressing unavoidable regularities in words. Examples are Van der Waerden's theorem or Graham-Rothschild theorems in Ramsey theory. It is a highly non-trivial task to understand the complexity of the bounds in terms of time and space of where and how these phenomena become manifest. It sometimes took decades before it was established that some of these phenomena belong to primitive recursive arithmetic [Shelah]. On the other hand, many such results have been shown not to be provable in Peano arithmetic [Paris-Harrington].

These results are frequently finitisations of topological results, results which in themselves do have constructive content [Coquand].

In this talk we shall explore the algorithmic content of the following statement, first discovered by topological means together with invoking notions involving well quasi-orderings which was then shown to be presentable by an Ackermann type recursion, leaving open the problem whether we can do better than that.

The result in question is as follows:

For every n, r we can find some $N = N(n, r)$ such that any word W on r symbols of length N will contain, for any permutation p of $\{1, \ldots, n\}$ a subword (factor) of the form $w_1 \ldots w_n X w_{p(1)} \ldots w_{p(n)}$, where $w_1, \ldots, w_n, X$ are all subwords of W.

We shall discuss the recursive complexity of $N(n, r)$.

3.9 On Harvey Friedman's Finite Phase Transitions

Lev Gordeev (Universität Tübingen, DE)

The *proof theoretic integer of a given theory* S (abbreviated $PTI(S)$) is the least integer n such that every arithmetical Σ_1 sentence that has a proof in S with at most $10,000$ symbols, has witnesses less than n. A good source of examples is in the area surrounding Kruskal's theorem. Consider Friedman's finite form (unstructured).

▶ **Theorem** (H .F.). *For all non-negative* k *there exist* $n > 0$ *such that the following holds. For all finite rooted trees* $T_1, ..., T_n$ *, where* $|T_i| < i + k - 1$ *, there exist* $i < j$ *such that* T_i *is homomorphically embeddable into* T_j*.*

▶ **Definition** (H. F.). *Let* $F(k)$ *be the least* n *such that Theorem holds.*

▶ **Question** (H. F.). *How fast grows function* F *and where are its jumps, i.e. phase transitions?*

▶ **Theorem** (L. G.).
1. $F(0) = 2,\ F(1) = 3,\ F(2) = 6,\ F(3) = 125.$ *However*
2. $F(4) > H_{\epsilon_0}(10^{10^{100}})$ *[H being the Hardy function].*
3. $PTI(PA) < F(4) < PTI(PA + Consis(PA)).$ *Moreover*
4. PA *proves that* $F(4)$ *does exist, but the length of any proof thereof must exceed* $10,000$ *symbols.*

3.10 A gentle introduction to Noetherian spaces

Jean Goubault-Larrecq (ENS – Cachan, FR)

To prepare for the next two talks, I'll explain what Noetherian spaces are, and how they generalize WQOs. The exposition will stress concepts that play similar roles on the WQO side and on the topological side: upward closed subsets become opens, monotone subsequences become self-convergent subnets, and so on.

For example, a WQO is a quasi-ordered set where every monotonic sequence of upward-closed subsets stabilizes; a Noetherian space is a topological space where every monotonic sequence of opens stabilizes. A WQO is a quasi-ordered set where every upward-closed subset is the upward closure of finitely many points; a Noetherian space is a topological space where every open is compact. A WQO is a quasi-ordered set where every sequence has a monotonic subsequence; a Noetherian space is a topological space where every net has a self-convergent subnet (a net is self-convergent if and only if it converges to every of its points. I will mention the Alexandroff topology below, and in such a topology, the self-convergent nets are exactly those that are eventually monotone; in particular, every monotone sequence gives you an example of a self-convergent net).

The precise connection is as follows: every topological space can be considered as a quasi-ordered set, with the so-called *specialization quasi-ordering*, defined by $x \leq y$ iff every open neighborhood of x contains y; conversely, the *Alexandroff topology* of a quasi-ordered set is the collection of all its upward-closed subsets. Starting from a quasi-order $\leq$, building its Alexandroff topology, then the specialization quasi-ordering of the latter, we retrieve $\leq$. Doing a similar round-trip from topological spaces to topological spaces does not give you back the original topology in general, unless it happened to be an Alexandroff topology: there are many topologies that have the same specialization quasi-ordering, and this is the source of some additional freedom that Noetherian spaces provide us, compared to WQOs. For example, the powerset of a Noetherian space, with the lower Vietoris topology, is again Noetherian. The analogous result for WQOs (that the powerset of a WQO under domination would be WQO) has been known to be false since Rado.

I will also mention that most of the constructions that are known to preserve WQO-ness also preserve Noetherianity: the space of all finite words on a Noetherian alphabet is Noetherian (an extension of Higman's Lemma), the space of all finite trees with vertices labeled by a Noetherian alphabet is Noetherian (generalizing Kruskal's Theorem), notably.

Finally, I'll mention two open problems:

- Following on Diana Schmidt, Andreas Weiermann, and others, is there a notion equivalent to that of maximal order types for Noetherian spaces? I contend that the ordinal height of the lattice of closed subsets should be the right notion, but we now need a theory of those: what is it for products of spaces, for sums, for spaces of words, etc.?
- What would be the natural analogue of BQOs in that topological theory of Noetherian spaces? One possible answer would be Noetherian spaces themselves, since they are already closed under the powerset construction. I don't think this is satisfactory, in particular in the view of Yann Péquignot's talk (with Raphaël Carroy) that the ideal completion remainder of a WQO is BQO iff the WQO is already BQO itself.

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1 Jean Goubault-Larrecq. *Non-Hausdorff topology and domain theory*. Cambridge University Press, New Mathematical Monographs 22, 2013. See in particular Section 9.7.

3.11 Trace Universality for VASS

Simon Halfon (ENS – Cachan, FR)

In [1], Esparza et al. have shown that the problem of trace universality for Petri Nets is decidable. The algorithm relies on the finiteness of bad sequences in the WQO $\mathbb{P}_f(\mathbb{N}^d)$. The complexity of the problem is addressed in the case $d = 1$ in [2]. They have shown that the problem is Ackermann-complete, using the tools introduced in [3] for the upper bound. In this talk I will present what is known on the complexity of this problem.

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3.12 On Initial Segments of Topological Weihrauch Degrees

Peter Hertling (Universität der Bundeswehr – München, DE)

Topological Weihrauch reducibility gives a very fine way of measuring the topological complexity of computation problems. We present old and new results stating that initial segments of topological Weihrauch degrees of certain classes of computation problems can be characterized in a combinatorial way by reducibilities between forests.

3.13 Well quasi-orderings and Hahn fields

Julia Knight (University of Notre Dame, US)

Joint work of Julia Knight, Karen Lange

Mourgues and Ressayre [2] showed that every real closed field has an integer part, where this is a discrete ordered subring appropriate for the range of a floor function. The proof gives an explicit procedure for embedding the given real closed field in a Hahn field. We wanted to measure the complexity of this procedure. For this, we needed to bound the lengths of roots of polynomials over the Hahn field, in terms of the lengths of the coefficients. In [3], we gave a conjecture, which we proved in [4]. We used results of de Jongh-Parikh [1] on well quasi-orderings.

References

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3.14 Definability in some well partial orders

Oleg Kudinov (Sobolev Institute of Mathematics,Novosibirsk, RU)

Joint work of Oleg Kudinov, Victor Selivanov

Well quasi-orders appear in many fields of Mathematics and Computer Science, where usually they play a role of classification tool. To our knowledge, the structure of concrete important WQOs (especially, definability aspect) was not investigated in details so far. In this talk we present some results on definability in some concrete WQOs, in particular, in the subword order on words and in the homomorphic quasi-order on labeled forests. In the case when the rank of a WQO is ω we are able in several cases to characterize completely the first-order definability. For WQOs of higher rank we discuss partial results and some open problems.

3.15 An axiom free Coq proof of Kruskal's tree theorem

Dominique Larchey-Wendling (LORIA – Nancy, FR)

URL http://www.loria.fr/~larchey/Kruskal

We present a Coq (http://coq.inria.fr) implementation of a purely inductive proof of Kruskal's tree theorem:

> If R is a well quasi-order on the type X then `homeo_embed`(R) is a WQO on the type of finite trees decorated by values in X.

Contrary to classical proofs, there are a few instances of intuitionistic proofs for the Kruskal tree theorem. Some of these proofs requires the further assumption that the ground relation R is decidable (e.g. Monika Seisenberger's proof [2] or Jean Goubault-Larrecq's proof [1]). Wim Veldman's proof [3] is the only published proof that does not require that assumption of decidability, but it requires *Brouwer's thesis*. Moreover, none of these proofs had been mechanized before.

We implement a typed variant of Wim Veldman's intuitionistic proof and we show that the use of the axiom called "Brouwer's thesis" is not necessary in that setting which makes our proof an axiom free one (w.r.t. the CIC on which Coq is based).

We use Thierry Coquand *et al.* [4] inductive definition of *Almost-Full* (AF) relations as an alternative to Wim Veldman's. We present the architecture of Wim Veldman's proof and its fundamental constituents: Ramsey's theorem, the Fan theorem, combinatorial principles and Evaluation maps. We show how to replace Wim Veldman *stump* based induction by lexicographic products of relations well founded upto a projection.

The source code for this project can be accessed at the following web page: http://www.loria.fr/~larchey/Kruskal.

References

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3.16 Robertson's conjecture on well quasi-ordering and topological minors

Chun-Hung Liu (Princeton University, US)

Joint work of Robin Thomas, Chun-Hung Liu

One of the most prominent results in graph theory is Robertson and Seymour's Graph Minor Theorem: finite graphs are well quasi-ordered by the minor relation [1]. They also proved that finite graphs are well quasi-ordered by the weak immersion relation [2], confirming a conjecture of Nash-Williams.

Topological minor relation is a graph containment relation that is closely related to the minor and the immersion relations. Kruskal's Tree Theorem and Robertson and Seymour's Weak Immersion Theorem imply that finite trees and finite subcubic graphs, respectively, are well quasi-ordered by the topological minor relation. However, unlike the minor and the weak immersion relation, the topological minor relation does not well quasi-order finite graphs in general.

In the late 1980's, Robertson conjectured that the known obstruction is the only obstruction. More precisely, he conjecture that for every positive integer k, finite graphs that do not contain a topological minor isomorphic the graph obtained from the path of length k by duplicating each edge are well quasi-ordered by the topological minor relation. Joining with Robin Thomas, we prove this conjecture. The proof will be sketched in the talk.

An application of this result is that every topological-minor-closed property on certain classes of graphs can be characterized by finitely many graphs. It leads to the existences of cubic time algorithms to test those properties. But more applications of this theorem remain requiring further investigations.

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3.17 Wqo and Bqo Theory in Reverse Mathematics

Alberto Marcone (University of Udine, IT)

This is a survey talk about WQO and BQO theory in reverse mathematics.

3.18 Dimensions of Mobility

Roland Meyer(University of Kaiserslautern, DE)

We study natural semantic fragments of the pi-calculus: depth-bounded processes (there is a bound on the longest communication path), breadth-bounded processes (there is a bound on the number of parallel processes sharing a name), and name-bounded processes (there is a bound on the number of shared names). We give a complete characterization of the decidability frontier for checking if a pi-calculus process in one subclass belongs to another. Our main construction is a general acceleration scheme for pi-calculus processes. Based on this acceleration, we define a Karp and Miller (KM) tree construction for the depth-bounded pi-calculus. The KM tree can be used to decide if a depth-bounded process is name-bounded, if a depth-bounded process is breadth-bounded by a constant k, and if a name-bounded process is additionally breadth-bounded. Moreover, we give a procedure that decides whether an arbitrary process is bounded in depth by a given k.

We complement our positive results with undecidability results for the remaining cases. While depth- and name-boundedness are known to be Σ_1-complete, we show that breadth-boundedness is Σ_2-complete, and checking if a process has a breadth bound at most k is Π_1-complete, even when the input process is promised to be breadth-bounded.

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3.19 Notes on regularity and well quasi-ordering

Mizuhito Ogawa (JAIST – Ishikawa, JP)

In [2], Ehrenfeucht et al. showed that a set L of finite words (over finite alphabet) is regular if and only if L is $\leq$-closed under some monotone well quasi-order $\leq$ over finite words. This note briefly surveys extensions to finite trees and ω-words [4]. They are obtained by similar proofs by modifying the standard congruence in Myhill-Nerode theorem to those in [3, 1]. The extensions are,

1. a tree language L is regular if and only if L is $\leq$-closed under some monotone well quasi-order $\leq$ over finite trees.
2. an ω-language L is regular if and only if L is $\preceq$-closed under a *periodic* extension $\preceq$ of some monotone WQO over finite words, and
3. an ω-language L is regular if and only if L is $\preceq$-closed under a WQO $\preceq$ over ω-words that is a *continuous* extension of some monotone WQO over finite words.

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3.20 Well-structured pushdown systems

Mizuhito Ogawa (JAIST – Ishikawa, JP)

Joint work of Xiaojuan Cai, Mizuhito Ogawa

Well-structured transition systems (WSTS) have been widely investigated [2]. We introduce an extension of WSTS with the stack, called *well-structured pushdown systems* (WSPDS), which is a pushdown system with well quasi-ordered states and stack alphabet [1, 3, 4]. The decidability of their properties, such as coverability, boundedness, and termination are discussed. The boundedness and the termination are decidable under the strong monotonicity and the monotonicity, respectively. The decidability of the coverability has been shown under certain conditions, e.g.,

- When the states are 1-dimensional vectors and the stack alphabet is finite [4].
- When the states are finite and the stack alphabet is well quasi-ordered [1].

The latter is based on P-automata techniques, and the convergence of a P-automaton with the minimization rules implies the decidability of the coverability.

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3.21 Noetherian Spaces in TTE

Arno Pauly (University of Brussels, BE)

Many topological notions have associated computable notions (eg [2]). Here, we investigate the computable counterpart of Noetherian space. Unfortunately, it turns out that no non-empty space is computably Noetherian in the straight-forward sense.

Based on the idea of relativizing topological notions w.r.t. some endofunctor introduced in [1], and then investigate the ∇-computably Noetherian spaces. These turn out to be well-behaved, and constitute a prime candidate for the correct notion of being Noetherian within computable analysis.

References

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3.22 From well to better: the space of ideals

Yann Pequignot (Universität Wien, AT) and Raphael Carroy (University of Torino, IT)

Main reference R. Carroy and Y. Pequignot, "From well to better, the space of ideals", in Fundamenta Mathematicae, Vol. 227, pp. 247–270, 2014.
URL http://dx.doi.org/10.4064/fm227-3-2

On the one hand, the ideals of a well quasi-order (WQO) naturally form a compact topological space into which the WQO embeds. On the other hand, Nash-Williams' barriers are given a uniform structure by embedding them into the Cantor space. We prove that every map from a barrier into a WQO restricts on a barrier to a uniformly continuous map, and therefore extends to a continuous map from a countable closed subset of the Cantor space into the space of ideals of the WQO. We then prove that, by shrinking further, any such continuous map admits a canonical form with regard to the points whose image is not isolated. As a consequence, we obtain a simple proof of a result on better quasi-orders (BQO); namely, a WQO whose set of non-principal ideals is a BQO is actually a BQO.

3.23 Problems on well quasi-orders and hereditary classes

Maurice Pouzet (University Claude Bernard – Lyon, FR)

I will present some problems on well quasi-ordering and their interactions with hereditary classes of relational structures. Among the dozen of problems presented, only two are recent and due respectively to Atmitas and Lozin (2015) and to Abraham, Bonnet and Kubis (2008). The others go back to the seventies and are about the ordinal length of hereditary classes of graphs; the relationship between WQO and BQO for hereditary classes; the effect of labeling members of a hereditary class by a WQO poset; the preservation of the WQO character by adding a linear order; the extension of Laver's theorem to hereditary classes; the effect of the WQO character of a hereditary class of finite structure on the asymptotic growth of the enumerative function of that class; and some other problems.

3.24 A constructive interpretation of open induction

Thomas Powell (Universität Innsbruck, AT)

Nash-Williams' well known minimal-bad-sequence argument can be elegantly reformulated as an instance of open induction over the lexicographic ordering on infinite sequences. In this talk I focus on how Gödel's Dialectica interpretation can be used to give a constructive interpretation to general induction principles, and in particular discuss the problem of giving a direct realizer to the Dialectica interpretation of open induction which can be used to extract natural programs from Nash-Williams proofs of Higman's lemma and Kruskal's theorem. I conclude by presenting several related open problems.

3.25 Bounds for the strength of the graph minor theorem

Michael Rathjen (University of Leeds, GB)

Joint work of M. Krombholz, M. Rathjen

The graph minor theorem, GM, is arguably the most important theorem of graph theory. The strength of GM exceeds that of the standard classification systems of Reverse Mathematics known as the "big five". The plan is to survey the current knowledge about the strength of GM, presenting lower and upper bounds.

3.26 Ordinal notations, the maximal order types of Kruskal's Tree Theorem, and a tale of two cultures

Diana Schmidt (Heilbronn, DE)

Main reference Diana Schmidt, "Well-Partial Orderings and their Maximal Order Types," Habilitationsschrift, Mathematics Faculty, Heidelberg University, 1979.

1. Why ordinal notations are useful, and what they are: Ordinal notations are terms built by applying functions from the ordinals to the ordinals, starting with 0. They are used to represent large ordinals. Such ordinal notations correspond in a natural way to labeled trees such as those in Kruskal‘s Tree Theorem.
2. What my 1979 Habilitationsschrift theorem means (intuitively) for ordinal notations: it computes the maximal order types of the ordinal notation systems which correspond to the tree well quasi orderings in Kruskal's tree theorem.
3. How I came to prove that theorem, and who else contributed to the proof (Schütte, Gandy, de Jongh, Parikh).
4. A tale of two cultures: also in 1979, Nachum Dershowitz submitted his paper "Orderings for term rewriting systems", which depends essentially on Kruskal's tree theorem. There was no internet; he was a computer scientist, I a mathematician. It was not til 15 years later that Andreas Weiermann unearthed the Habilitationsschrift and bridged the culture gap.

References

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3.27 Complexity Classes Beyond Elementary

Sylvain Schmitz (ENS – Cachan, FR)

Main reference S. Schmitz, "Complexity hierarchies beyond Elementary", ACM Transactions on Computation Theory, Vol. 8(1:3), 2016.
URL http://dx.doi.org/10.1145/2858784

Well quasi-orders provide termination or finiteness arguments for many algorithms, and miniaturized versions can furthermore be employed to prove complexity upper bounds for those algorithms. We have however an issue with these bounds: they go way beyond the familiar complexity classes used in complexity theory. I shall discuss a definition of complexity classes suitable for the task. In particular, unlike the subrecursive classes they are based on, those classes support the usual notions of reduction and completeness.

3.28 Well-quasi-orderings for program verification and computational complexity

Philippe Schnoebelen (ENS – Cachan, FR)

Joint work of Sylvain Schmitz, Christoph Haase, Philippe Schnoebelen
Main reference S. Schmitz and Ph. Schnoebelen, "The power of well-structured systems", in Proc. of the 24th Int'l Conf. on Concurrency Theory (CONCUR'13), LNCS, Vol. 8052, pp. 5–24, Springer, 2013.
URL http://dx.doi.org/10.1007/978-3-642-40184-8_2

Well-structured systems (WSTS) are a generic family of computational models where transitions are monotonic w.r.t. an effective well quasi-ordering of the states. This allows generic decidability proofs and verification algorithms for the verification of behavioral properties (like safety, liveness, . . .) [1, 2] Recent work by the authors aim at extracting computational complexity bounds from decidability proofs that rely on well quasi-orderings [3, 4].

References

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3.29 Linearization as Conservation

Peter M. Schuster (University of Verona, IT), Davide Rinaldi, and Daniel Wessel (University of Trento, IT)

A variant of Szpilrajn's Order Extension Principle (OEP) says that every partial order can be extended to a linear order. While OEP as it stands is a form of the Axiom of Choice, Negri–von Plato–Coquand 2004 have proved a proof-theoretic, purely syntactical counterpart: for quasi-orders, linearity is conservative when it comes to prove Horn sequents. This has now turned out a special case of the universal Krull-Lindenbaum conservation theorem we have gained from a criterion for conservation given by Scott 1974.

3.30 Higman's Lemma and its Computational Content

Helmut Schwichtenberg (LMU, München)

Joint work of Monika Seisenberger (Swansea University), Franziskus Wiesnet (LMU München), Helmut Schwichtenberg

Higman's Lemma is a fascinating result in infinite combinatorics, with manyfold applications in logic and computer science, that has been proven using different methods several times. The aim of the talk is to look at Higman's Lemma from a computational point of view. We give a proof of Higman's Lemma that uses the same combinatorial idea as Nash-Williams' indirect proof using the so-called minimal-bad-sequence argument, but which is constructive. For the case of a two letter alphabet such a proof was given by Coquand. Using more flexible structures, we present a proof that works for an arbitrary well quasi-ordered alphabet. We report on a formalization of this proof in the proof assistant Minlog, and discuss machine extracted terms (in an extension of Gödel's system T) expressing its computational content.

3.31 Well quasi-orders and descriptive set theory: some results and questions

Victor Selivanov (A. P. Ershov Institute – Novosibirsk, RU)

The existing hierarchies of sets have very easy structure (their levels are almost well ordered under inclusion) and they are sufficient for expressing apparently all interesting topological properties of sets. In contrast, existing classifications of functions and equivalence relations seem to be insufficient to express many specific properties of these objects. The situation is relatively clear for functions with finite range and for equivalence relations with finitely many classes but is much less clear for more complex objects.

In this talk, we survey some earlier results and discuss some more recent results and open questions in the specified direction, considering classifications from descriptive set theory and automata theory. We try to demonstrate that this topic is closely related to WQO- and BQO-theory. We give some relevant references for the interested reader.

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3.32 A Mechanized Proof of Higman's Lemma by Open Induction

Christian Sternagel (Universität Innsbruck, AT)

I present a recent Isabelle/HOL formalization of a short proof of Higman's lemma using open induction. The proof is based on Geser's technical report "A Proof of Higman's Lemma by Open Induction (1996)" but considerably simplified and amending an intermediate lemma.

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3.33 Some Challenges Related to Wqo-Theory

Andreas Weiermann (University of Ghent, BE)

An important result of De Jongh and Parikh states that every well partial order can be extended to a well order of maximal possible order type. We discuss the role of this order type in bounding complexities arising from applications of well quasi-orders to termination problems. We also discuss Friedman style miniaturizations of well quasi-orders and indicate possible phase transitions. One specific example concerns Kruskal's theorem for which the critical constant is 0.639577689994720133112899870565731384115276481914419... (these decimal numbers have been calculated with great accuracy by Moritz Firsching).

Remaining challenges will be to determine critical constants related to other WQO-principles (joint work with Lev Gordeev) and the calculation of maximal order types for more general tree classes (joint work with Michael Rathjen and Jeroen Van der Meeren).

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4 Discussion and Open Problems

The following list of open problems reflects the discussion at the workshop, and can be used for further reference.

1. (Philippe Schnoebelen) What extra parameter could make the maximal order type functional again? That is, writing $o(X)$ for the maximal order type of a WQO X, it holds that $o(X+Y)$, $o(X \times Y)$ and $o(X^*)$ are completely determined by $o(X)$ and $o(Y)$ (for instance), but $o(\mathbb{P}_f(X))$ does not depend just on $o(X)$. ($\mathbb{P}_f(X)$ is the finite powerset of X, quasi-ordered by domination: $A \leq^\flat B$ iff for every $a \in A$, there is $b \in B$ such that $a \leq b$.) We know that $1 + o(X) \leq o(\mathbb{P}_f(X)) \leq 2^{o(X)}$, and the bounds are reached for some X. Considering dependencies on width and height are still not enough.
   Answers/suggestions:
   a. (Thomas Forster) The rank of the tree of bad quadratic arrays.
   b. (Lev Beklemishev) Look at the simpler case of subsets of two elements.
   c. (Julia Knight) Conjecture: $o([X]^2)$ might be $o([o(X)]^2)$ – for finite non-linear X, it is false.
2. (Andreas Weiermann) Let α be an ordinal and $(X, \leq)$ be a well quasi-order. Let $S_\alpha(X)$ be the set of all sequences $f\colon \beta \to X$ with $\beta < \alpha$ such that f has a finite range. For $f, g \in S_\alpha(X)$ with $f\colon \beta \to X$ and $g\colon \gamma \to X$ let $f \leq g$ if there exists a strictly monotonic function $h\colon \beta \to \gamma$ such that $f(\delta) \leq g(h\delta)$ for all $\delta < \beta$. Nash-Williams proved that $S_\alpha(X)$ is a well quasi-order. Is it possible to give good bounds for the maximal order type for $S_\alpha(X)$ in terms of α and the maximal order type of X? This problem has been considered in Diana Schmidt's Habilitationsschrift.

3. (Andreas Weiermann) What is the maximal order type of the set of finite trees, with labels $\{0, 1, \cdots, n\}$, and a gap condition? The case with 2 labels is known, see Jeroen van der Meeren's PhD thesis.
4. (Alberto Marcone) Laver's theorem states that the collection of countable scattered linear orders is WQO (even BQO) under embedding. A theorem by Hausdorff states that those orders can all be obtained by a certain enrichment process, indexed by ordinals. The Hausdorff rank $rh_H(L)$ of such an order L is the least ordinal where we obtain it by this construction. We already know the maximal order type of the subcollection of countable scattered linear orders with finite Hausdorff rank. As an attempt to approach Laver's theorem from below, rather that from above, and its proof-theoretic strength, what is the maximal order type of the collection of scattered linear orders of Hausdorff rank $< \omega^2$? It is a long-standing open problem, maybe too dangerous to give to a PhD student (Andreas Weiermann).
5. (Jean Goubault-Larrecq) The proper analogue of the notion of maximal order type for Noetherian spaces seems to be the ordinal height of their poset of closed subsets. Can we develop their De Jongh-Parikh theory? E.g., what is it for products, for sums, for spaces of words, etc.?
6. (Maurice Pouzet) Let P be a WQO. Is it true that $rank_{CB}(Idl(P)) = rank_{CB}(Idl(o(P)))$, where $rank_{CB}$ denotes Cantor Bendixson rank, $Idl(P)$ is the set of ideals of P (up-directed, non-empty, downward closed), and $o(P)$, as an ordinal, is considered as a WQO itself? The point is that we have a formula for the right-hand side.
7. (Lev Beklemishev) Is there any relationship between Noetherian spaces and scattered spaces?
8. (Yann Pequignot) Informal conjecture by Nash-Williams: is every "naturally occurring" WQO actually a BQO? Have you encountered any counterexample in your research? See also next problem, and problem 14.
   Variant (a): is every "naturally occurring" WQO an ω^2-WQO? (And all other variants of the same type.)
9. (Thomas Forster) It is still unknown whether the minor relation on finite graphs is a BQO. Is it?
   Variant (a): Are subcases of that relation, which were already known to be WQO before the Robertson-Seymour result, already BQO? Graphs of bounded tree-width are known to be BQO.
10. (Sylvain Schmitz) Can we develop a reverse mathematics programme for WSTS? Is the proof-theoretic ordinal of the statement "this property is decidable for that model of WSTS" always the maximal order type of the underlying WQO?
11. (Raphael Carroy) Are continuous functions a WQO under $\leq_1$, where $f \leq_1 g$ iff there are continuous functions F, G such that $f = F \circ g \circ G$?
12. (Jean Goubault-Larrecq) While Noetherian spaces seem to be the proper topological analogue of the order-theoretic notion of WQO, what would be the analogue for BQOs?
13. (Thomas Forster) Is there another definition of BQO that would help us in any of the BQO-related questions, letting us have slicker proofs?
14. (Maurice Pouzet) If you have a hereditary class of finite graphs (w.r.t. embeddability) which is WQO, is that class BQO?
15. (Philippe Schnoebelen, Sylvain Schmitz)
   (This is a "reference request" type of question, it was prompted by Dick de Jongh's talk Friday morning.)

A quasi-ordering $(A, \leq)$ leads to a natural notion of embedding on $Mat[A]$, the set of (finite) rectangular matrices $M, N, \ldots$ with elements from A. One lets $M \leq_{Mat} N$ when there is a submatrix N' of N (i.e., a matrix derived from N by removing some lines and columns) s.t. M and N' have same dimensions and $M[i,j] \leq N'[i,j]$ for all i, j.
Asking for which qos $(A, \leq)$ one has $(Mat[A], \leq_{Mat})$ WQO is an exercise or homework problem that we sometimes give to our students after teaching them Higman's Lemma. We won't spoil the fun by answering here. The question is: do you know of some work where this question is mentioned/answered? What would be the best reference?

16. (Julia Knight) An integer part for a real closed ordered field R is a discrete ordered subring I such that for all $r \in R$, there exists $i \in I$ with $i \leq r < i + 1$. Mourgues and Ressayre proved that every real closed ordered field has an integer part. If R is countable, with universe ω, then the procedure of Mourgues and Ressayre yields an integer part that is $\Delta^0_{\omega^\omega}(R)$. Is there one that is $\Delta^0_2(R)$? See the next question, suggested by Beklemishev.
17. (Lev Beklemishev) Analyze from the point of view of reverse mathematics the theorem of Mourgues and Ressayre saying that every real closed ordered field has an integer part.
18. (Julia Knight) A divisible ordered Abelian group G is *Archimedean* if for all $a, b \in G^{>0}$, there exist natural numbers m, n such that $ma > b$ and $nb > a$. Let G be an Archimedean divisible ordered Abelian group. Suppose $S \subseteq G^{\geq 0}$, and let $[S]$ be the semi-group generated by elements of S. Let α be a multiplicatively indecomposable ordinal. If S has order type at most α, then so does $[S]$. But what can we say if G is *not* Archimedean?
19. (Julia Knight) Let G be an Archimedean divisible ordered Abelian group and let K be a field that is algebraically closed, or real closed. Suppose $p(x)$ is a polynomial over $K((G))$ with $Supp(p) \subseteq G^{\geq 0}$, and let r be a root with $w(r) > 0$. Let α be multiplicatively indecomposable. If $Supp(p)$ has order type at most α, then r has length at most α. What can we say if G is *not* Archimedean?
20. (Lev Beklemishev) In (D. Gabelaia, A. Kurucz, F. Wolter, M. Zakharyaschev. Non-primitive recursive decidability of products of modal logics with expanding domains. *Annals of Pure and Applied Logic* 142 (1), 245–268) a natural notion of *expanding product* of Kripke frames was considered. Let (W, R) be a Kripke frame, and let F be a function assigning to each $x \in W$ a Kripke frame $F(x) = (W_x, R_x)$. We assume that whenever $x, y \in W$ and xRy, the frame $F(x)$ is a subframe of $F(y)$ in the sense that $W_x \subseteq W_y$ and $R_x = R_y \cap (W_x^2)$. An *e-frame (expanding frame)* associated with (W, F) is the set

$$\bigsqcup_{x \in W} F(x) = \{(x, u) : x \in W, y \in W_x\}$$

equipped with two binary relations R_1, R_2 such that

$$(x, u)R_1(y, v) \iff (xRy \wedge u = v),$$

$$(x, u)R_2(y, v) \iff (x = y \wedge uR_x v).$$

Expanding frames are models of propositional bimodal logic. By using Kruskal's Theorem, the authors of the paper cited above show that the bimodal logic determined by the class of all e-products of finite irreflexive trees is decidable. However, they also prove an Ackermannian lower bound by reducing to it the decision problem for lossy channel systems. Similar results are obtained for several other classes of frames.
What are sharp upper and lower bounds on the complexity of the decision problem for the bimodal logic of the class of e-products of finite irreflexive trees? Similar questions are also open for several other natural classes of frames studied in the paper cited above, in particular for linear frames.

Variant (a): find an axiomatization of the bimodal logic determined by the class of all e-products of finite irreflexive trees.
See also problem 26.

21. (Victor Selivanov) Define and investigate new natural topologically relevant WQOs (reducibilities) on Borel measurable functions $f : X \to Y$ between topological spaces (the particular case when both X, Y coincide with the Baire space is already important). Previous results in this direction (due to Wadge, Carlson-Laver, van Engelen-Miller-Steel, Weihrauch, Hertling, Selivanov, Carroy and others, some references may be found in my conference presentation) show that this research programme might be of great interest for descriptive set theory but the reducibilities considered so far do not seem sufficient for a deep understanding of Borel measurable functions.
22. (Victor Selivanov) For a qo Q, let $\mathcal{T}_Q$ (resp. $\widetilde{\mathcal{T}}_Q$) be the set of finite (resp. of at most countable well founded) Q-labeled trees (T, c_T) equipped with the homomorphism qo $\leq_h$ defined as follows: $(S, c_S) \leq_h (T, c_T)$ iff there is a monotone (not necessarily injective) function $\varphi : S \to T$ such that $\forall x \in S(c_S(x) \leq c_T(\varphi(x)))$. Several versions of these constructions that were introduced and studied in my publications (see e.g. LNCS 6735 (2011), p. 260-269, APAL 163 (2012), p. 1075-1107, Arxiv (2014): 1406.3942v1) turn out to be relevant to classifying some topological objects. From well known facts of WQO-theory it follows that $\mathcal{T}_Q$ (resp. $\widetilde{\mathcal{T}}_Q$) is WQO (resp. BQO) provided that Q is WQO (resp. BQO). The question is to understand the relationships between Q and $\mathcal{T}_Q$ (resp. $\widetilde{\mathcal{T}}_Q$), in particular to compute the ranks (heights), the maximal order types and other natural invariants (like the automorphism group of the corresponding quotient-orders) of $\mathcal{T}_Q, \widetilde{\mathcal{T}}_Q$ for natural Q. The question is interesting and non-trivial also for iterations of these constructions (and their modifications), e.g. for $\mathcal{T}_{\mathcal{T}_k}$ and $\widetilde{\mathcal{T}}_{\widetilde{\mathcal{T}}_k}$ where k is the antichain with $k < \omega$ elements.
23. (Victor Selivanov) Continue the systematic investigation of (un)decidability and definability issues of natural WPOs on words, trees, forests, graphs and other structures relevant to WQO theory and Computer Science. Some interesting work in this direction is already done by Comon, Kuske, Selivanov, Kudinov, Schnoebelen, and many others.
One challenging concrete problem is: what is a precise estimate of the m-degree of first-order theory of the quotient-order of $\widetilde{\mathcal{T}}_k$ (for $k \geq 3$) from the previous question?
Variant (a): can you characterize the first-order definable relations in the quotient-order of $\mathcal{T}_{\mathcal{T}_k}$ (also for $k \geq 3$)?
In solving such questions the tools developed in (Kudinov-Selivanov, LNCS 5635 (2009), p. 290-299) seem especially relevant since they probably generalize to many natural well founded partial orders.
24. (Victor Selivanov) Are the Weihrauch reducibilities $\leq_1, \leq_2$ WQO on the Borel-measurable functions from the Baire space $\mathcal{N}$ to the discrete space with countably many points? Let us recall that $f \leq_1 g$ iff there are continuous functions F, G such that $f = F \circ g \circ G$, while $f \leq_2 g$ iff there are continuous functions F, G such that for every x, $f(x) = F(x, g(G(x)))$.
Variant (a): Is the continuous reducibility WQO on the Borel equivalence relations with countably many equivalence classes?
25. (Victor Selivanov, Oleg Kudinov) Let $(F_P, \leq_h)$ denote the factorization (i.e., the quotient order) of the set of all finite forests with vertices labeled by elements from WQO $(P, \leq)$, and $\leq_h$ is the homomorphic quasi-ordering of problem 22 (it is WQO again; please do not confuse it with the homeomorphic embedding as mentioned in Kruskal's Theorem). The detailed properties of such WQOs are not established so far even for finite P. So, the question is to characterize them in terms of finite P: 1) height(F_P); 2) $o(F_P)$; 3) $Th(F_P)$. For the last point the conjecture is that this theory is decidable iff width$(P) < 3$.

26. (Sylvain Schmitz) Same questions as in problem 20 for one-variable FOLTL with counting over expanding domains on finite linear orders (C. Hampson and A. Kurucz. Undecidable propositional bimodal logics and one-variable first-order linear temporal logics with counting. *ACM Transactions on Computational Logic* 16(3:27), 2015).

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