Report from Dagstuhl Seminar 21472

Geometric Logic, Constructivisation, and Automated Theorem Proving

Edited by

Thierry Coquand[1], Hajime Ishihara[2], Sara Negri[3], and Peter M. Schuster[4]

1 **University of Gothenburg, SE,** thierry.coquand@cse.gu.se
2 **JAIST – Ishikawa, JP,** ishihara@jaist.ac.jp
3 **University of Genova, IT,** sara.negri@unige.it
4 **University of Verona, IT,** petermichael.schuster@univr.it

Abstract

At least from a practical and contemporary angle, the time-honoured question about the extent of intuitionistic mathematics rather is to which extent any given proof is effective, which proofs of which theorems can be rendered effective, and whether and how numerical information such as bounds and algorithms can be extracted from proofs. All this is ideally done by manipulating proofs mechanically or by adequate metatheorems, which includes proof translations, automated theorem proving, program extraction from proofs, proof analysis and proof mining. The question should thus be put as: What is the computational content of proofs?

Guided by this central question, the present Dagstuhl seminar puts a special focus on coherent and geometric theories and their generalisations. These are not only widespread in mathematics and non-classical logics such as temporal and modal logics, but also a priori amenable for constructivisation, e.g., by Barr's Theorem, and last but not least particularly suited as a basis for automated theorem proving. Specific topics include categorical semantics for geometric theories, complexity issues of and algorithms for geometrisation of theories including speed-up questions, the use of geometric theories in constructive mathematics including finding algorithms, proof-theoretic presentation of sheaf models and higher toposes, and coherent logic for automatically readable proofs.

Seminar November 21–26, 2021 – http://www.dagstuhl.de/21472

2012 ACM Subject Classification Theory of computation → Constructive mathematics; Theory of computation → Proof theory; Theory of computation → Automated reasoning

Keywords and phrases automated theorem proving, categorical semantics, constructivisation, geometric logic, proof theory

Digital Object Identifier 10.4230/DagRep.11.10.151

Edited in cooperation with Fellin, Giulio

1 Executive Summary

Thierry Coquand
Hajime Ishihara
Sara Negri
Peter M. Schuster

A central question has remained from the foundational crisis of mathematics about a century ago: What is the extent of intuitionistic mathematics? From a practical angle, the question is to which extent any given proof is effective, which proofs of which theorems can be rendered effective, and whether and how numerical information such as bounds and algorithms can be extracted from proofs. Ideally, all this is treated by manipulating proofs mechanically and/or by adequate proof-theoretic metatheorems (proof translations, automated theorem proving, program extraction from proofs, proof analysis, proof mining, etc.). In this vein, the central question should rather be put as follows: What is the computational content of proofs?

Guided by this form of the central question, the Dagstuhl Seminar 21472 put a special focus on coherent and geometric theories and their generalisations. These indeed are fairly widespread in mathematics and non-classical logics such as temporal and modal logics, a priori amenable for constructivisation in the vein of Barr's Theorem, and particularly suited as a basis for automated theorem proving. Specific topics included categorical semantics for geometric theories, complexity issues of and algorithms for geometrisation of theories with the related speed-up questions, the use of geometric theories in constructive mathematics up to finding algorithms, proof-theoretic presentation of sheaf models and higher toposes, and coherent logic for automated proving.

The Dagstuhl Seminar 21472 attracted researchers and practitioners from all over the world, including participants from various research areas in order to broaden the scope of the seminar and to create connections between communities. The seminar participants presented and discussed their research by means of programmed and ad-hoc talks, and a tutorial on Agda the well developed proof assistant based on dependent type theory – was held over several time slots. Numerous new research directions were developed in small working groups: for example, new perspectives on classifying toposes in algebraic geometry, applications of dynamical methods to quadratic forms, and Zorn induction to capture transfinite methods computationally.

The tireless efforts by Dagstuhl staff notwithstanding, it would not be fair to say that this seminar did not suffer from the pandemic-related travel restrictions by which many invitees were confined to remote participation, which of course made hard if not impossible that they took part at the invaluable informal exchange on-site characteristic for events held at Dagstuhl. Under the given circumstances, however, the seminar was still judged a success by all the participants. Following an unconditional request by many, the organisers intend to propose a follow-up Dagstuhl seminar on a related topic in the near future – if possible, all on-site.

2 Table of Contents

3 Overview of Talks

3.1 Progress and challenges in program extraction

Ulrich Berger (Swansea University, GB)

Program extraction from proofs is a technique to exploit the computational content of constructive proofs to extract programs that are provably correct. The technique builds on realizability as introduced by Kleene and Kreisel in the 1940s and 1950s.

We report about recent progress in program extraction, on the one hand regarding the inclusion of limited forms of nonconstructve principles such as the axiom of choice and the law of excluded middle, on the other hand regarding capturing computations that go beyond the usual functional paradigm, namely nondeterminism and concurrency. It turns out that limited form of the law of excluded middle is required to extract concurrent programs. We report on case studies regarding exact real number computation, infinite Gray code and Gaussian elimination for matrices with exact real number entries. This is joint work Hideki Tsuiki, Dieter Spreen, and Monika Seisenberger.

The main current challenge is the general axiom of choice. Raoult gave a reformulation of the axiom of choice as an induction principle (Open Induction). However, this does not seem to be amenable for program extraction (only restricted forms of Open Induction permit program extraction). In joint work with Peter Schuster, we are currently exploring different formulations of the axiom of choice, in its form as Zorn's Lemma, as 'induction-like principles (Zorn Induction), that might permit program extraction.

3.2 Loop-checking and the uniform word problem for join-semilattices with an inflationary endomorphism

Marc Bezem (University of Bergen, NO)

Main reference Marc Bezem, Thierry Coquand: "Loop-checking and the uniform word problem for join-semilattices with an inflationary endomorphism", Theoretical Computer Science, 2022.
URL http://dx.doi.org/10.1016/j.tcs.2022.01.017

We solve in polynomial time two decision problems that occur in type checking when typings depend on universe level constraints.

3.3 Bridging the foundational gap: updating algebraic geometry in face of current challenges regarding formalizability, constructivity and predicativity

Ingo Blechschmidt (Universität Augsburg, DE)

The Lean community recently reached a major milestone in formalizing the definition of schemes, the objects of study in algebraic geometry. However, their development spans more than 10,000 lines of code. A fundamental notion such as that of schemes should not be such demanding to formalize.

We argue that this defect is due to the reliance on transfinite methods in the classical presentation of the foundations of algebraic geometry, which the Lean community decided to follow. Just as they are inappropriate from a constructive and predicative point of view, they don't provide a good basis for formalization. In fact, those three concerns are closely related, perhaps even sides of the same coin.

The talk explores the tension between the foundation of algebraic geometry and these modern challenges, and reports on work in progress recasting the foundation of algebraic geometry to face these challenges, including a constructive and predicative framework for setting up cohomology of quasicoherent sheaves.

References

1 M. Barakat and M. Lange-Hegermann. An axiomatic setup for algorithmic homological algebra and an alternative approach to localization. *J. Algebra Appl.*, 10(2):269–293, 2011.
2 M. Barr. Toposes without points. *J. Pure Appl. Algebra*, 5:265–280, 1974.
3 A. Blass. Injectivity, projectivity, and the axiom of choice. *Trans. Amer. Math. Soc.*, 255:31–59, 1979.
4 I. Blechschmidt. Flabby and injective objects in toposes, 2021.
5 M. Brandenburg. *Tensor categorical foundations of algebraic geometry.* PhD thesis, Universität Münster, 2014.
6 J. Cole. The bicategory of topoi and spectra. *Repr. Theory Appl. Categ.*, (25):1–16, 2016.
7 T. Coquand. Computational content of classical logic. In A. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, pages 33–78. Cambridge University Press, 1997.
8 T. Coquand, H. Lombardi, and P. Schuster. The projective spectrum as a distributive lattice. *Cah. Topol. Géom. Différ. Catég.*, 48(3):220–228, 2007.
9 T. Coquand, H. Lombardi, and P. Schuster. Spectral schemes as ringed lattices. *Ann. Math. Artif. Intell.*, 56:339–360, 2009.
10 A. Grothendieck. Introduction to functorial algebraic geometry, part 1: affine algebraic geometry (lecture notes by F. Gaeta), 1973.
11 M. Hakim. *Topos annelés et schémas relatifs*, volume 64 of *Ergeb. Math. Grenzgeb.* Springer, 1972.
12 G. Kempf. Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves. *Rocky Mountain J. Math.*, 10(3):637–646, 1980.
13 H. Lombardi and C. Quitté. *Commutative Algebra: Constructive Methods.* Springer, 2015.
14 M. Maietti. Joyal's arithmetic universes as list-arithmetic pretoposes. *Theory Appl. Categ.*, 23(3):39–83, 2010.
15 C. McLarty. What does it take to prove Fermat's last theorem? Grothendieck and the logic of number theory. *Bull. Symbolic Logic*, 16(3):359–377, 2010.

16 C. McLarty. The large structures of Grothendieck founded on finite-order arithmetic. *Rev. Symbolic Logic*, 13(2):296–325, 2020.
17 S. Posur. A constructive approach to Freyd categories. *Appl. Categ. Structures*, 29:171–211, 2021.
18 P. Schuster. Formal zariski topology: positivity and points. *Ann. Pure Appl. Logic*, 137(1):317–359, 2006.
19 M. Tierney. On the spectrum of a ringed topos. In A. Heller and M. Tierney, editors, *Algebra, Topology, and Category Theory. A Collection of Papers in Honor of Samuel Eilenberg*, pages 189–210. Academic Press, 1976.
20 S. Vickers. *Locales and Toposes as Spaces*, pages 429–496. Springer, 2007.

3.4 An automated method to reasoning about differentiable functions

Gabriele Buriola (University of Verona, IT)

Joint work of Gabriele Buriola, Domenico Cantone, Gianluca Cincotti, Eugenio G. Omodeo, Gaetano T. Spartà
Main reference Gabriele Buriola, Domenico Cantone, Gianluca Cincotti, Eugenio G. Omodeo, Gaetano T. Spartà: "A Decidable Theory of Differentiable Functions with Convexities and Concavities on Real Intervals", in Proc. of the 35th Italian Conference on Computational Logic – CILC 2020, Rende, Italy, October 13-15, 2020, CEUR Workshop Proceedings, Vol. 2710, pp. 231–247, CEUR-WS.org, 2020.
URL http://ceur-ws.org/Vol-2710/paper15.pdf

This contribution concerns an enrichment of pre-existing decision algorithms, which in their turn augmented a fragment of Tarski's elementary algebra with one-argument real functions endowed with continuous first derivative. In its present (still quantifier-free) version, our decidable language embodies addition of functions; the issue we address is the one of satisfiability. As regards real numbers, individual variables and constructs designating the basic arithmetic operations are available, along with comparison relators. As regards functions, we have another sort of variables, out of which compound terms are formed by means of constructs designating addition and – outermostly – differentiation. An array of predicates designate various relationships between functions, as well as function properties, that may hold over intervals of the real line; those are: function comparisons, strict and nonstrict monotonicity / convexity / concavity, comparisons between the derivative of a function and a real term. Our decision method consists in preprocessing the given formula into an equi-satisfiable quantifier-free formula of the elementary algebra of real numbers, whose satisfiability can then be checked by means of Tarski's decision method. No direct reference to functions will appear in the target formula, each function variable having been superseded by a collection of stub real variables; hence, in order to prove that the proposed translation is satisfiability-preserving, we must figure out a flexible-enough family of interpolating C^1 functions that can accommodate a model for the source formula whenever the target formula turns out to be satisfiable.

3.5 Deductive systems and Grothendieck topologies

Olivia Caramello (University of Insubria – Como, IT)

Main reference Olivia Caramello: "Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges' " Oxford University Press, 2017.
URL https://doi.org/10.1093/oso/9780198758914.001.0001

I will show that the classical proof system of geometric logic over a given geometric theory is equivalent to new proof systems based on the notion of Grothendieck topology. These equivalences result from a proof-theoretic interpretation of the duality between the quotients of a given geometric theory and the subtoposes of its classifying topos. Interestingly, these alternative proof systems turn out to be computationally better-behaved than the classical one for many purposes, as I will illustrate by discussing a few selected applications.

3.6 A General Glivenko–Gödel Theorem for Nuclei

Giulio Fellin (University of Verona, IT) and Peter M. Schuster (University of Verona, IT)

Main reference Giulio Fellin, Peter Schuster: "A General Glivenko-Gödel Theorem for Nuclei", in Proc. of the Proceedings 37th Conference on Mathematical Foundations of Programming Semantics, MFPS 2021, Hybrid: Salzburg, Austria and Online, 30th August – 2nd September, 2021, EPTCS, Vol. 351, pp. 51–66, 2021.
URL http://dx.doi.org/10.4204/EPTCS.351.4

Glivenko's theorem says that, in propositional logic, classical provability of a formula entails intuitionistic provability of double negation of that formula. We generalise Glivenko's theorem from double negation to an arbitrary nucleus, from provability in a calculus to an inductively generated abstract consequence relation, and from propositional logic to any set of objects whatsoever. The resulting conservation theorem comes with precise criteria for its validity, which allow us to instantly include Gödel's counterpart for first-order predicate logic of Glivenko's theorem. The open nucleus gives us a form of the deduction theorem for positive logic, and the closed nucleus prompts a variant of the reduction from intuitionistic to minimal logic going back to Johansson.

The present study was carried out within the projects "A New Dawn of Intuitionism: Mathematical and Philosophical Advances" (John Templeton Foundation, ID 60842) and "Reducing complexity in algebra, logic, combinatorics – REDCOM" ("Ricerca Scientifica di Eccellenza 2018", Fondazione Cariverona); and within GNSAGA of INdAM.

3.7 Proof mining a nonlinear ergodic theorem for Banach spaces

Anton Freund (TU Darmstadt, DE)

Joint work of Anton Freund, Ulrich Kohlenbach
Main reference Anton Freund, Ulrich Kohlenbach: "Bounds for a nonlinear ergodic theorem for Banach spaces. Ergodic Theory and Dynamical Systems", CoRR, Vol, abs/2108.08555, 2021
URL https://arxiv.org/abs/2108.08555

Proof mining uses tools from logic to extract quantitative (and sometimes new qualitative) results from seemingly noneffective proofs in core mathematics (see the textbook by Ulrich Kohlenbach [3]). This talk presents joint work of Kohlenbach and the speaker [1], which is concerned with nonexpansive maps on Banach spaces: by analysing a proof due to Kazuo Kobayasi and Isao Miyadera [2], we obtain a rate of metastability for the strong convergence of Cesàro means. In the talk, we focus on one particular step of the analysis, which deals with a seemingly noneffective use of a limit inferior. This focus allows us to explain fundamental ideas of proof mining by means of a concrete mathematical example.

Both Anton Freund and Ulrich Kohlenbach were supported by the "Deutsche Forschungsgemeinschaft" (DFG, German Research Foundation) – Projects 460597863, DFG KO 1737/6-1 and DFG KO 1737/6-2.

References

1. Anton Freund and Ulrich Kohlenbach, *Bounds for a nonlinear ergodic theorem for Banach spaces*, Ergodic Theory and Dynamical Systems, to appear. Preprint available as `arXiv:2108.08555`.
2. Kazuo Kobayasi and Isao Miyadera, *On the strong convergence of the Césaro means of contractions in Banach spaces*, Proc. Japan Acad. 56 (1980) 245-249.
3. Ulrich Kohlenbach, *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*, Springer Monographs in Mathematics, Springer, Berlin and Heidelberg, 2008.

3.8 Conservation theorems on semi-classical arithmetic

Makoto Fujiwara (Meiji University – Kawasaki, JP)

Joint work of Makoto Fujiwara, Taishi Kurahashi
Main reference Makoto Fujiwara, Taishi Kurahashi: "Conservation theorems on semi-classical arithmetic", CoRR, Vol. abs/2107.11356, 2021
URL https://arxiv.org/abs/2107.11356

It is well-known that classical arithmetic PA is Π_2-conservative over intuitionistic arithmetic HA. Using a generalized negative translation, we relativize this result with respect to theories of semi-classical arithmetic, which lie in-between PA and HA. In particular, it follows from our main result that PA is Π_{k+2}-conservative over HA + Σ_k-LEM where Σ_k-LEM is the low-of-excluded-middle scheme for formulas of Σ_k form.

3.9 Gluing classifying toposes along open subtoposes

Matthias Hutzler (Universität Augsburg, DE)

A geometric theory classified by some Grothendieck topos can be regarded as a syntactic presentation of the theory. In this talk, we consider the question how to construct such a syntactic presentation for a topos from syntactic presentations of a covering family of open subtoposes, and how to capture appropriate additional gluing data in a syntactic way.

Here, extensions (or expansions) of geometric theories, which can add new sorts, symbols and axioms, and which can be regarded as syntactic presentations of geometric morphisms, play an important role. As an instructive example, we construct a geometric theory classified by the big Zariski topos of the projective line, which is covered by two copies of the big Zariski topos of the affine line, both classifying local algebras with one distinguished element.

References

1 O. Caramello. *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'.* Oxford University Press, 2017.

2 M. Hakim. *Topos annelés et schémas relatifs*, volume 64 of *Ergeb. Math. Grenzgeb.* Springer, 1972.

3 M. Hutzler. Internal language and classified theories of toposes in algebraic geometry. Master's thesis, University of Augsburg, 2018.

4 M. Hutzler. *Syntactic presentations for glued toposes and for crystalline toposes.* PhD thesis, University of Augsburg, 2021.

5 D. Tsementzis. A syntactic characterization of Morita equivalence. *J. Symbolic Logic*, 82(4):1181–1198, 2017.

6 G. Wraith. Generic galois theory of local rings. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of sheaves*, volume 753 of *Lecture Notes in Math.*, pages 739–767. Springer, 1979.

3.10 Negative Results in Universal Proof Theory

Rosalie Iemhoff (Utrecht University, NL)

In this talk I explain how a property of logics, such as uniform interpolation, can be used to establish that a logic does not have proof systems of a certain kind, in this case sequent calculi with good structural properties and other desirable qualities. This connection between the properties of a logic and its proof systems is based on a proof method for uniform interpolation that applies to any intermediate, substructural, modal or intuitionistic modal logic that has a sequent calculus of that kind.

Some of the relevant references:

References

1 A. Akbar Tabatabai, R. Iemhoff, and R. Jalali. Uniform Lyndon Interpolation for Basic Non-normal Modal Logics. In: Silva A., Wassermann R., de Queiroz R. (eds) Logic, Language, Information, and Computation. WoLLIC 2021. Lecture Notes in Computer Science, vol 13038, Springer, 2021.

2 I. van der Giessen and R. Iemhoff. Sequent Calculi for Intuitionistic Gödel-Löb Logic, *Notre Dame Journal of Formal Logic* 62(2): 221–246 (May 2021). DOI: 10.1215/00294527-2021-0011

3 R. Iemhoff. Uniform interpolation and the existence of sequent calculi, *Annals of Pure and Applied Logic* 170 (11), 2019, p. 1–37.

3.11 Theorem Proving as Constraint Solving with Coherent Logic

Predrag Janicic (University of Belgrade, RS) and Julien Narboux (University of Strasbourg, FR)

Main reference Predrag Janičić, Julien Narboux, Theorem Proving as Constraint Solving with Coherent Logic, submitted, 2021

We think coherent logic is well suited framework for automatic generation of readable proofs. In contrast to common automated theorem proving approaches, in which the search space is a set of some formulae and what is sought is again a (goal) formula, we propose an approach based on searching for a proof (of a given length) as a whole. Namely, a proof of a formula in a fixed logical setting can be encoded as a sequence of natural numbers meeting some conditions and a suitable constraint solver can find such sequence. The sequence can then be decoded giving a proof in the original theory language. This approach leads to several unique features, for instance, it can provide shortest proofs. We use SAT and SMT solvers for solving sets of constraints. We implemented the proposed method and we present its features, perspectives and performance.

3.12 Proof mining in nonconvex optimization

Ulrich Kohlenbach (TU Darmstadt, DE)

Proof mining uses so-called proof interpretations, such as suitable forms of Gödel's functional interpretation, to extract explicit computational information from given prima facie noneffective proofs in mathematics. In recent years this has been successfully applied in convex optimization with the extraction of effective rates of asymptotic regularity for cyclic projection methods ([3]) and effective rates of metastability for Proximal Point Type algorithms such as PPA and HPPA which approximate zeros of maximally monotone operators ([2, 4, 5]). In order to be able to treat also nonconvex/nonconcave optimization problems one has to generalize the concept of monotone operator. Recently, Bauschke et al. [1] studied as such a generalization so-called comonotone operators. In the case studies [4, 5] of applying proof mining to PPA and HPPA it becomes apparent that the monotonicity of A is used only in a restricted form which makes it easily possible to adopt the extracted bounds as well as the underlying qualitative convergence theorems also to comonotone operators ([6]). This illustrates how proof mining also facilitates the generalization of proofs.

This research is supported by the 'Deutsche Forschungsgemeinschaft' (Project DFG KO 1737/6-2).

References

1 Bauschke, H.H., Moursi, W.A, Wang, X., Generalized monotone operators and their averaged resolvents. Math. Programming 189, pp. 55-74 (2021).
2 Dinis, B., Pinto, P., Quantitative Results on the Multi-Parameters Proximal Point Algorithm. J. Convex Anal. 28, pp. 729-750 (2021)
3 Kohlenbach, U., A polynomial rate of asymptotic regularity for compositions of projections in Hilbert space. Foundations of Computational Mathematics 19, pp. 83-99 (2019).

4 Kohlenbach, U., Quantitative analysis of a Halpern-type proximal point algorithm for accretive operators in Banach spaces. J. Nonlin. Convex Anal. 9, pp. 2125-2138 (2020).
5 Kohlenbach, U., Quantitative results on the proximal point algorithm in uniformly convex Banach spaces. J. Convex Anal. **28**, pp. 11-18 (2021).
6 Kohlenbach, U., On the Proximal Point Algorithm and its Halpern-type variant for generalized monotone operators in Hilbert space. To appear in: Optimization Letters.

3.13 Geometric theories versus Grothendieck toposes, questions w.r.t. a possible constructive elementary approach

Henri Lombardi (University of Franche-Comté – Besancon, FR)

We use the terminology and notations of dynamical theories. See [1, 2, 4, 8, 12, 13, 14].

Dynamical theories, introduced in [8], are a version without logic, purely computational, of geometric theories. See also the paper [1] describing some advantages of this approach, and pioneering articles [19, Sections 1.5 and 4.2], [18] and [11].

Dynamical algebraic structures are explicite in [12, 14] and implicite in [8], where they are described through their presentations. They are also implicite in [13] and, last but not least, in [9, D5], which was a main source: it is possible to compute inside the algebraic closure of a discrete field, even if it is impossible to construct the structure. So it suffices to consider the algebraic closure as a dynamical algebraic structure à la D5 rather than a usual algebraic structure: *lazy evaluation à la D5 gives a constructive semantic for the algebraic closure of a discrete field.*

Since geometric theories, which are concrete objects are closely related to Grothendieck toposes, our aim is to describe, using a constructive external mathematical world à la Bishop ([3]) all the work on toposes in terms of geometric theories and dynamical theories. See related work in [5, 6, 7, 15, 17] and a preliminary draft in [16].

References

1 Bezem, M. and Coquand, T. (2005). Automating coherent logic. In *Logic for programming, artificial intelligence, and reasoning. 12th international conference, LPAR 2005, Montego Bay, Jamaica, December 2–6, 2005. Proceedings*, pages 246–260. Berlin: Springer.
2 Bezem, M. and Coquand, T. (2019). Skolem's theorem in coherent logic. *Fundam. Inform.*
3 Bishop, E. (1967). *Foundations of constructive analysis.* McGraw-Hill, New York.
4 Coquand, T. (2005). A completeness proof for geometrical logic. In *Logic, methodology and philosophy of science. Proceedings of the 12th international congress, Oviedo, Spain, August 2003*, pages 79–89. London: King's College Publications.
5 Coquand, T. and Lombardi, H. (2006). A logical approach to abstract algebra. *Math. Structures Comput. Sci.*, 16(5):885–900.
6 Coquand, T. and Lombardi, H. (2016). Anneaux à diviseurs et anneaux de Krull (une approche constructive). *Comm. Algebra*, 44:515–567.
7 Coquand, T., Lombardi, H., and Quitté, C. (2022). Dimension de Heitmann des treillis distributifs et des anneaux commutatifs. In *Publications Mathématiques de l'Université de Franche-Comté Besançon. Algèbre et théorie des nombres. Années 2003–2006.* Besançon: Laboratoire de Mathématiques de Besançon, 2006, p. 57–100, version corrigée.
8 Coste, M., Lombardi, H., and Roy, M.-F. (2001). Dynamical method in algebra: effective Nullstellensätze. *Ann. Pure Appl. Logic*, 111(3):203–256.

9 Della Dora, J., Dicrescenzo, C., and Duval, D. (1985). About a new method for computing in algebraic number fields. In *EUROCAL '85. Lecture Notes in Computer Science no. 204, (Ed. Caviness B.F.)*, pages 289–290. Springer, Berlin.

10 Kemper, G. and Yengui, I. (2020). Valuative dimension and monomial orders. *J. Algebra*, 557:278–288.

11 Lifschitz, V. (1980). Semantical completeness theorems in logic and algebra. *Proc. Amer. Math. Soc.*, 79(1):89–96.

12 Lombardi, H. (1998). Relecture constructive de la théorie d'Artin-Schreier. *Ann. Pure Appl. Logic*, 91(1):59–92.

13 Lombardi, H. (2002). Dimension de Krull, Nullstellensätze et évaluation dynamique. *Math. Z.*, 242(1):23–46.

14 Lombardi, H. (2006). Structures algébriques dynamiques, espaces topologiques sans points et programme de Hilbert. *Ann. Pure Appl. Logic*, 137(1-3):256–290.

15 Lombardi, H. (2020). Spectral spaces versus distributive lattices: a dictionary. In *Advances in rings, modules and factorizations. Selected papers based on the presentations at the international conference on rings and factorizations, Graz, Austria, February 19–23, 2018*, pages 223–245. Cham: Springer.

16 Lombardi, H. (2021). Théories géométriques pour l'algèbre constructive. `http://hlombardi.free.fr/Theories-geometriques.pdf`.

17 Lombardi, H. and Quitté, C. (2015). *Commutative algebra: constructive methods. Finite projective modules.* Algebra and applications, 20. Springer, Dordrecht. Translated from the French (Calvage & Mounet, Paris, 2011, revised and extended by the authors) by Tania K. Roblot.

18 Matijasevič, J. V. (1975). A metamathematical approach to proving theorems in discrete mathematics. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 49:31–50, 177. Theoretical applications of the methods of mathematical logic, I.

19 Prawitz, D. (1971). Ideas and results in proof theory. In *Proceedings of the Second Scandinavian Logic Symposium (Univ. Oslo, Oslo, 1970)*, pages 235–307. Studies in Logic and the Foundations of Mathematics, Vol. 63. North-Holland, Amsterdam.

3.14 Verifiable Solving of Geometric Construction Problems in the Framework of Coherent Logic

Vesna Marinkovic (University of Belgrade, RS)

Geometry construction problems are one of the longest studied problems in mathematical education. Solving construction problem corresponds to proving constructively a theorem in a coherent logic form. Automated solving of construction problems has been studied, however rarely in rigorous logical terms.

In this talk I will present a formal logical framework describing a traditional four phases process of solving construction problems and a mechanism for automated generation of solutions, both formalized and human-readable. For this purpose a solver for construction problems ArgoTriCS and a prover for coherent logic ArgoCLP (both developed in our research group) are used. It turns out that coherent logic is a natural framework for carrying out two of four phases. In order to obtain proofs as close as possible to ones generated by humans, automatically generated proofs in coherent logic form are simplified using a simplification procedure integrated within ArgoCLP.

3.15 No speedup for geometric theories

Michael Rathjen

Main reference Michael Rathjen: "No speedup for geometric theories", CoRR, Vol. abs/2105.04661v1, 2021, also to appear in: M. Benini, O. Beyersdorff, M. Rathjen, P. Schuster (eds.), Mathematics for Computation, World Scientific Publishing.
URL https://arxiv.org/abs/2105.04661v1
Main reference Michael Rathjen: "Remarks on Barr's Theorem: Proofs in Geometric Theories", pp. 347–374, De Gruyter, 2016.
URL http://dx.doi.org/doi:10.1515/9781501502620-019

Geometric theories based on classical logic are conservative over their intuitionistic counterparts for geometric implications. In the talk I plan to look at two aspects of geometric theories.

The first will be concerned with the cost of transforming a classical proof in a geometric theory into an intuitionistic one. The latter result (sometimes referred to as Barr's theorem) is squarely a consequence of Gentzen's Hauptsatz. Prima facie though, cut elimination can result in superexponentially longer proofs, posing the question of whether this transformation can be achieved in feasibly many steps.

There is also an infinitary version of geometric theories formulated in the logics $L_{\infty\omega}$ with arbitrary infinite (of any set size) disjunctions and conjunctions. These logics are very expressive. I'd like to discuss the constructivity of the proof that classical $L_{\infty\omega}$-proofs of infinite geometrical implications can be turned into intuitionistic proofs. Can this proof be carried out in CZF? An even more basic question presents itself: What is the proper notion of infinite proof? The latter question is also relevant if one works in classical set theory without AC (recall Barwise's completeness theorem). Also the choice of the proof system is relevant, for instance, if one wants to show that if a proof of phi from T exists in a forcing extension then there is also one in the ground model (assuming phi and T are in the latter).

3.16 Constructiveness and lattices in Lorenzen's work

Stefan Neuwirth (University of Franche-Comté – Besancon, FR)

Joint work of Thierry Coquand, Henri Lombardi, Stefan Neuwirth
Main reference "The free pseudocomplemented semilattice generated by a preordered set", work in progress.
URL http://lmb.univ-fcomte.fr/IMG/pdf/the_free_pseudocomplemented_semilattice_generated_by_a_preordered_set.pdf

This is joint work with Thierry Coquand and Henri Lombardi.

Let $(M, \leq_M)$ be a preordered set.

Let us define the free meet-semilattice over M. Let us consider the set H of unordered lists of elements of M, denoted by $a = \alpha_1 \wedge \cdots \wedge \alpha_n$; we shall define a relation $\leq_H$ on H by the following deduction rules:

(1) if $\alpha \leq_M \beta$ then $\alpha \leq_H \beta$;
(2) if $a \leq_H c$ then $a \wedge b \leq_H c$;
(3) if $c \leq_H a$ and $c \leq_H b$ then $c \leq_H a \wedge b$.

It is easy to prove that the converse holds in (3) and that $\leq_H$ is transitive by showing the admissibility of the corresponding deduction rules. Furthermore, as (2) and (3) introduce relations only between elements one of which is a list of at least two elements of M, the converse holds as well in (1): this is a conservativity result.

A meet-semilattice with least element 0 is *pseudocomplemented* if for every b there is c such that $a \wedge b \leq 0$ if and only if $a \leq c$; the element c is denoted by $\bar{b}$. Let us define the free pseudocomplemented meet-semilattice over M. Let us consider the set H generated inductively from M as the set of unordered lists of elements of H or of formal pseudocomplements of elements of H; we shall define a relation $\leq_H$ on H by the deduction rules (1)–(5) with:

(4) if $a \wedge b \leq_H 0$ then $a \leq_H \bar{b}$;

(5) if $a \leq_H b$ then $a \wedge \bar{b} \leq_H c$.

It is easy to prove that the converse holds in (1), (3), (4); but it is quite difficult to prove that $\leq_H$ is transitive.

A meet-semilattice is *σ-complete* if for every sequence $(a_1, a_2, \dots)$ there is a meet $\bigwedge(a_1, a_2, \dots)$. Let us define the free σ-complete pseudocomplemented meet-semilattice over M. Let us consider the set H generated as before but with the additional inductive clause of containing the sequences $(a_1, a_2, \dots)$ of elements of H written as formal meets $\bigwedge(a_1, a_2, \dots)$; we shall define a relation $\leq_H$ on H by the deduction rules (1)–(8) with:

(6) if $a_k \wedge b \leq_H c$ then $\bigwedge(a_1, a_2, \dots) \wedge b \leq_H c$;

(7) if $c \leq_H a_1$, $c \leq_H a_2$, ..., then $c \leq_H \bigwedge(a_1, a_2, \dots)$;

(8) if $a \wedge a \wedge b \leq_H c$ then $a \wedge b \leq_H c$.

It is easy to prove that the converse holds in (1), (3), (4), (7); the proof of the transitivity of $\leq_H$ is much easier here because of the inclusion of the contraction rule (8) among the deduction rules.

The first two constructions appear in Paul Lorenzen's "Algebraische und logistische Untersuchungen über freie Verbände" (1951). The third one appears in his manuscript "Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis" (1944), in which he explains why this construction is the semilattice counterpart to the proof of consistency of elementary number theory: this theory may be viewed as contained in the free σ-complete pseudocomplemented meet-semilattice over the set M of numerical propositions preordered by material implication. "The fact that the logic calculuses are semilattices or lattices permits a simple logistic application of free lattices" (Lorenzen 1951).

This talk is also an invitation to reflect upon mathematical objects (like the semilattices here) as given dynamically by rules instead of being considered statically as completed totalities.

References

1. Thierry Coquand and Stefan Neuwirth. Lorenzen's proof of consistency for elementary number theory. *Hist. Philos. Logic*, 41(3), 281–290, 2020. arXiv:2006.08996.
2. Paul Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *J. Symb. Log.*, 16(2), 81–106, 1951. `http://www.jstor.org/stable/2266681`. Translation by S. Neuwirth: 'Algebraic and logistic investigations on free lattices', 2017, arXiv:1710.08138.
3. Paul Lorenzen. Ein halbordnungstheoretischer Widerspruchsfreiheitsbeweis [A proof of freedom from contradiction within the theory of partial order]. *Hist. Philos. Logic*, 41(3), 265–280, 2020. arXiv:2006.08996. Dual German-English text, edited and translated by Stefan Neuwirth.

3.17 The distributivity of the category of dependent objects over the Grothendieck category

Iosif Petrakis (LMU München, DE)

In [1] and [2] the type-theoretic axiom of choice, or the distributivity of the Π-type over the Σ-type, is translated into Bishop set theory (BST) as the distributivity of the Π-set over the Σ-set. We present this distributivity categorically, as the distributivity of the category of dependent objects over the Grothendieck category. Similarly to the fact that the category of dependent objects is defined through the Grothendieck category and the functor category, in BST the Π-set can be defined through the Σ-set and the function set.

References

1 I. Petrakis: Dependent sums and Dependent Products in Bishop's Set Theory, in P. Dybjer et. al. (Eds) TYPES 2018, LIPIcs, Vol. 130, Article No. 3, 2019.
2 I. Petrakis: *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU, Munich, 2020.

3.18 Supercompactly generated theories

Morgan Rogers (University of Insubria – Como, IT)

There are a few standard ways to identify theories classified by a given topos. I discussed how to do so starting from a theory of presheaf type, in the special case of a topos obtained from a principal site, which is to say a site whose covering families are generated by a class of individual covering morphisms, based on the fourth chapter of my forthcoming thesis. I only got as far as presenting the case of topologies on the simplex category, but I illustrated the background principles involved.

3.19 Proofs and computation with infinite data

Helmut Schwichtenberg (LMU München, DE)

Joint work of Helmut Schwichtenberg, Nils Köpp

It is natural to represent real numbers in $[-1, 1]$ by streams of signed digits $-1, 0, 1$. Algorithms operating on such streams can be extracted from formal proofs involving a unary coinductive predicate CoI on (standard) real numbers x: a realizer of $\mathrm{CoI}(x)$ is a stream representing x. We address the question how to obtain bounds for the lookahead of such algorithms: how far do we have to look into the input streams to compute the first n digits of the output stream? We present a proof-theoretic method how this can be done. The idea is to replace the coinductive predicate $\mathrm{CoI}(x)$ by an inductive predicate $\mathrm{I}(x, n)$ with the intended meaning that we know the first n digits of a stream representing x. Then from a formal proof of $\mathrm{I}(x, n+1) \to \mathrm{I}(y, n+1) \to I(1/2(x+y), n)$ we can extract an algorithm for the average function whose lookahead is $n+1$ for both arguments.

3.20 Coherent logic in representation and proving of informal proofs

Sana Stojanovic-Djurdjevic (University of Belgrade, RS)

Main reference Sana Stojanovic Durdevic: "From informal to formal proofs in Euclidean geometry", Ann. Math. Artif. Intell., Vol. 85(2-4), pp. 89–117, 2019.
URL http://dx.doi.org/10.1007/s10472-018-9597-7

There are several different approaches to verification of proofs from mathematical textbooks. I will discuss one idea for using coherent logic for representation of semi-formal textbook proofs. Also, coherent logic vernacular can be used for automatic generation of more detailed proof objects, and eventually generate formal proofs in language of different interactive theorem provers. This approach is tested on two sets of theorem proofs using classical axiomatic system for Euclidean geometry created by David Hilbert, and a modern axiomatic system E created by Jeremy Avigad, Edward Dean, and John Mumma.

References

1 Stojanovic-Djurdjevic, S., From Informal to Formal proofs in Euclidean Geometry, Annals of Mathematics and Artificial Intelligence, Volume 85, pp 89-117, 2019

3.21 Terminating sequent calculi for a class of intermediate logics

Matteo Tesi (Scuola Normale Superiore – Pisa, IT)

Syntactic decision procedures for propositional intuitionistic logic usually exploit a suitably formulated sequent calculus. There are various approaches known in the literature, the reader can see [3] for an extended survey. These systems fail to satisfy one of the following four *desiderata*: 1. a simple termination procedure which does not require a loop-checking, 2. the invertibility of every rule of the calculus which eliminates the need for backtraking, 3. the extraction of a finite countermodel out of a failed proof search and 4. modularity, i.e. the possibility to extend the general methodology to various extension of intuitionistic logic.

We offer a new method based on labelled sequent calculi [2] which meets the *desiderata* listed above. To start with, we propose a variant with respect to the usual Kripke semantics for intuitionistic logic. In particular, we introduce *strict* Kripke models, i.e. models based on finite transitive and irreflexive orders.

The standard truth condition for the implication is replaced by the following. $x \Vdash A \to B$ if and only if the two conditions:

1. If $x \Vdash A$, then $x \Vdash B$
2. For all y (if $x < y$ and $y \Vdash A$, then $y \Vdash B$).

hold. The two semantics are shown to be equivalent and thus intuitionistic propositional logic proves sound and complete with respect to the strict semantics. This is shown using the finite model property for intuitionistic propositional logic [1] and by providing an easy transformation of finite partial orders into finite strict orders and vice versa. We introduce the following abbreviation:

$$x \Vdash A > B \equiv \text{for all } y \text{ (if } x < y \text{ and } y \Vdash A\text{, then } y \Vdash B)$$

and we show that in every strict intuitionistic model condition 2. is equivalent to:

$2'$. For all y (if $x < y$ and $y \Vdash A$ and $y \Vdash A > B$, then $y \Vdash B$)

The new semantics is employed to obtain a labelled sequent calculus $\mathbf{G3I}_{<}$ in which the rules for the implication $\to$ are obtained through those for the new connective $>$. The following rules govern the implication connective:

$$\frac{x : A > B, \Gamma \Rightarrow \Delta, x : A \qquad x : B, x : A > B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta} \, L\to \qquad \frac{\Gamma \Rightarrow \Delta, x : A > B \qquad x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B} \, R\to$$

$$\frac{x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A \qquad y : B, x < y, x : A > B, \Gamma \Rightarrow \Delta}{x < y, x : A > B, \Gamma \Rightarrow \Delta} \, L> \qquad \frac{x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A > B} \, R>, y \text{ fresh}$$

The termination of the calculus $\mathbf{G3I}_{<}$ is proved by showing that every proof search ends and yields either a proof or a strict countermodel. This gives a completeness result and a decision procedure for intuitionistic logic. The termination depends on the formulation of the rule R> prevents the formation of loops.

Finally, the sequent calculus $\mathbf{G3I}_{<}$ can be extended with relational rules which preserve the properties of the base system. We focus on the extensions for intermediate logics characterized by classes of frames with a condition of the form $\forall \overline{x} \varphi$ where φ is a quantifier-free formula. The termination strategy encompasses all these systems and so we obtain terminating calculi for intermediate logics with a universal frame condition and the finite model property.

References

1 Chagrov, A., Zakharyaschev, M., *Modal Logic*, Oxford University Press, 1997.
2 Dyckhoff, R., Negri, S., *Proof analysis in intermediate logics*, Archive for Mathematical Logic 51, pp. 71-92, 2012.
3 Dyckhoff, R., *Intuitionistic decision procedures since Gentzen*, in Kahle R., Strahm T., Studer T. (eds) Advances in Proof Theory. Progress in Computer Science and Applied Logic, vol 28. Birkhäuser, Cham., 2016.
4 Negri, S., *Proof analysis in modal logic*, Journal of Philosophical Logic 34, 507, 2005.

3.22 Some remarks about Skolem-Noether Theorem

Thierry Coquand

We discuss a constructive proof of Skolem-Noether Theorem. In particular, the original proof of Skolem was an early example of the technique of Galois descent. This is part of a general constructive study of the theory of central simple algebra.

References

1 Coquand, T., Lombardi, H. & Neuwirth, S. Constructive basic theory of central simple algebras. (2021)

4 Working groups

4.1 Tutorial on Agda, the dependently typed proof assistant

Ingo Blechschmidt (Universität Augsburg, DE) and Matthias Hutzler (Universität Augsburg, DE)

We give an introduction to Agda, a dependently typed proof assistant, loosely following a tutorial by Martín Escardó given at Proof and Computation 2018 in Fischbachau.

References

1 Escardó, M. Introduction to Univalent Foundations of Mathematics with Agda. (2021), https://www.cs.bham.ac.uk/ mhe/HoTT-UF-in-Agda-Lecture-Notes/
2 Wadler, P., Kokke, W. & Siek, J. Programming Language Foundations in Agda. (2020), https://plfa.inf.ed.ac.uk/20.07/

4.2 Working group on classifying toposes in algebraic geometry

Ingo Blechschmidt (Universität Augsburg, DE), Ulrik Buchholtz (TU Darmstadt, DE), Matthias Hutzler (Universität Augsburg, DE), Henri Lombardi (University of Franche-Comté – Besancon, FR), and Stefan Neuwirth (University of Franche-Comté – Besancon, FR)

A logical way to present a Grothendieck topos is to give a geometric theory which is classified by the topos. This point of view originated from Monique Hakim's PhD thesis, in which she determined such syntactic presentations of two important toposes in algebraic geometry, the Zariski topos and the étale topos.

However, for many related toposes in algebraic geometry, similar syntactic presentations are still lacking. This state of affairs only started to change in recent years, when the theories corresponding to the fppf and the surjective topologies and when theories presenting the infinitesimal and the crystalline topos have been determined.

In this working group, we studied several of the remaining toposes, and made progress on several such, namely the cl, cdf, cdp and the f toposes.

References

1 M. Anel. Grothendieck topologies from unique factorisation systems. 2009.
2 I. Blechschmidt. *Using the internal language of toposes in algebraic geometry.* PhD thesis, University of Augsburg, 2017.
3 O. Gabber and S. Kelly. Points in algebraic geometry. 2014.
4 M. Hakim. *Topos annelés et schémas relatifs*, volume 64 of *Ergeb. Math. Grenzgeb.* Springer, 1972.
5 M. Hutzler. Internal language and classified theories of toposes in algebraic geometry. Master's thesis, University of Augsburg, 2018.
6 M. Hutzler. *Syntactic presentations for glued toposes and for crystalline toposes.* PhD thesis, University of Augsburg, 2021.
7 S. Schröer. Points in the fppf topology. 2014.

8 G. Wraith. Generic galois theory of local rings. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of sheaves*, volume 753 of *Lecture Notes in Math.*, pages 739–767. Springer, 1979.

4.3 Zorn Induction

Peter M. Schuster (University of Verona, IT) and Ulrich Berger (Swansea University, GB)

We put forward Zorn Induction as a competitor of Raoult's Open Induction [3] in the undertaking to rephrase as classically equivalent but computationally interesting induction principles the minimal (or maximal) element principle known as Zorn's Lemma [7]. As compared to Open Induction, Zorn Induction works with chains rather than directed subsets, and refers to a strict partial order. We expect Zorn Induction to be of use for computation just as is Open Induction [1, 2], also in abstract algebra [5, 4]. A challenge will be how to capture the nondeterministic consequent of Zorn Induction, as this does not fit directly the setting of least fixed points [6].

By a *(strict) partial order* we understand a pair $(X, <)$ where X is a set and $<$ is a transitive and irreflexive relation on X. An element $x \in X$ is a *lower bound* of a subset Y of X if $x < y$ for all $y \in Y$. By $\mathrm{lb}(Y)$ we denote the set of lower bounds of Y. Let $\mathcal{P}$ be a property of subsets of X. We say that a subset A of X is *$\mathcal{P}$-progressive* if, for all subsets Y of X having property $\mathcal{P}$, if A contains all lower bounds of Y, then A contains an element of Y, i.e.,

$$\forall Y \subseteq X(\mathcal{P}(Y) \wedge \mathrm{lb}(Y) \subseteq A \to Y \cap A \neq \emptyset)\,.$$

We now can formulate *Zorn Induction* as the principle

$$\forall A \subseteq X(A \text{ chain-progressive } \to X \subseteq A).$$

With classical logic, Zorn Induction is equivalent to Zorn's Lemma in the form

$$\forall B \subseteq X(B \text{ inductive and nonempty } \to \ B \text{ has a minimal element})$$

where B is *inductive* if every chain contained in B has a lower bound in B, and b is a *minimal element* in B if $b \in B$ and there is no $y \in B$ with $y < b$.

References

1 Berger, U. A computational interpretation of open induction. *Proceedings – Symposium On Logic In Computer Science.* **19** pp. 326 – 334 (2004,8)
2 Coquand, T. A Note on the Open Induction Principle. (1997) `www.cse.chalmers.se/~coquand/open.ps`
3 Raoult, J. Proving Open Properties by Induction. *Inf. Process. Lett.*. **29** pp. 19-23 (1988)
4 Rinaldi, D. & Schuster, P. A universal Krull–Lindenbaum theorem. *Journal Of Pure And Applied Algebra.* **220**, 3207-3232 (2016)
5 Schuster, P. Induction in Algebra: a First Case Study. *Logical Methods In Computer Science.* **9** (2013,9)
6 Tarski, A. A lattice-theoretical fixpoint theorem and its applications.. *Pacific Journal Of Mathematics.* **5**, 285 – 309 (1955)
7 Zorn, M. A remark on method in transfinite algebra. *Bulletin Of The American Mathematical Society.* **41**, 667 – 670 (1935)

Participants

- Karim Johannes Becher
University of Antwerp, BE
- Arnold Beckmann
Swansea University, GB
- Ulrich Berger
Swansea University, GB
- Ulrik Buchholtz
TU Darmstadt, DE
- Gabriele Buriola
University of Verona, IT
- Giulio Fellin
University of Verona, IT
- Anton Freund
TU Darmstadt, DE
- Matthias Hutzler
Universität Augsburg, DE
- Rosalie Iemhoff
Utrecht University, NL
- Ulrich Kohlenbach
TU Darmstadt, DE
- Henri Lombardi
University of Franche-Comté – Besancon, FR
- Julien Narboux
University of Strasbourg, FR
- Stefan Neuwirth
University of Franche-Comté – Besancon, FR
- Eugenio Orlandelli
University of Bologna, IT
- Iosif Petrakis
LMU München, DE
- Morgan Rogers
University of Insubria – Como, IT
- Peter M. Schuster
University of Verona, IT
- Matteo Tesi
Scuola Normale Superiore – Pisa, IT

Remote Participants

- Jan Belle
LMU München, DE
- Marc Bezem
University of Bergen, NO
- Pierre Boutry
INRIA – Sophia Antipolis, FR
- Olivia Caramello
University of Insubria – Como, IT
- Liron Cohen
Ben Gurion University – Beer Sheva, IL
- Thierry Coquand
University of Gothenburg, SE
- Laura Crosilla
University of Oslo, NO
- Tiziano Dalmonte
University of Turin, IT
- Makoto Fujiwara
Meiji University – Kawasaki, JP
- Hajime Ishihara
JAIST – Ishikawa, JP
- Predrag Janicic
University of Belgrade, RS
- Tatsuji Kawai
JAIST – Nomi, JP
- Vesna Marinkovic
University of Belgrade, RS
- Kenju Miyamato
LMU München, DE
- Sara Negri
University of Genova, IT
- Takako Nemoto
Hiroshima Institute of Technology, JP
- Satoru Niki
Ruhr-Universität Bochum, DE
- Edi Pavlovic
LMU München, DE
- Cosimo Perini Brogi
University of Genova, IT
- Thomas Powell
University of Bath, GB

- Michael Rathjen
University of Leeds, GB
- Helmut Schwichtenberg
LMU München, DE
- Monika Seisenberger
Swansea University, GB
- Sana Stojanovic-Djurdjevic
University of Belgrade, RS
- Daniel Wessel
LMU München, DE
- Chuangjie Xu
fortiss GmbH – München, DE