

From Proofs to Computation in Geometric Logic and Generalizations

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Abstract

What is the computational content of proofs? This is one of the main topics in mathematical logic, especially proof theory, that is of relevance for computer science. The well-known foundational solutions aim at rebuilding mathematics constructively almost from scratch, and include Bishop-style constructive mathematics and Martin-Löf's intuitionistic type theory, the latter most recently in the form of the so-called homotopy or univalent type theory put forward by Voevodsky.

From a more practical angle, however, the question rather is to which extent *any given* proof is effective, which proofs of which theorems can be rendered effective, and whether and how numerical information such as bounds and algorithms can be extracted from proofs. Ideally, all this is done by manipulating proofs mechanically and/or by adequate metatheorems (proof translations, automated theorem proving, program extraction from proofs, proof mining, etc.).

A crucial role for answering these questions is played by *coherent and geometric theories and their generalizations*: not only that they are fairly widespread in modern mathematics and non-classical logics (e.g., in abstract algebra, and in temporal and modal logics); those theories are also a priori amenable for constructivisation (see Barr's Theorem, especially its proof-theoretic variants, and the numerous Glivenko-style theorems); last but not least, effective theorem-proving for coherent theories can be automated with relative ease and clarity in relation to resolution.

Specific topics that substantially involve computer science research include categorical semantics for geometric theories up to the proof-theoretic presentation of sheaf models and higher toposes; extracting the computational content of proofs and dynamical methods in quadratic form theory; the interpretation of transfinite proof methods as latent computations; complexity issues of and algorithms for geometrization of theories; the use of geometric theories in constructive mathematics including finding algorithms, ideally with integrated developments; and coherent logic for obtaining automatically readable proofs.

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1 Executive Summary

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The Dagstuhl Seminar 24021 emerged as a response to the challenges faced by its predecessor, Dagstuhl Seminar 21472, which grappled with pandemic-induced travel restrictions that hindered in-person attendance. The tireless efforts of the Dagstuhl staff notwithstanding, the earlier seminar relied on remote participation, limiting the depth of engagement and interaction among attendees. Many participants advocated for a follow-up seminar on a related topic, which materialized in the form of the present gathering.

Freed from the constraints of travel restrictions, this seminar hosted a dynamic environment characterized by extensive interactions, both structured and informal. Evening sessions provided a platform for casual discussions, enabling participants to delve deeper into topics of interest and forge meaningful connections across different communities within the field. Most notably, the seminar structure allowed ample time for spontaneous working groups.

As compared to the Dagstuhl Seminars the organisers attended in the past, this seminar stands out for an intense interaction between the participants. In the few months after we have already observed push effects to current research in several directions, e.g. strong negation in constructive mathematics and proof systems, synthetic algebraic geometry especially in the context of homotopy type theory, and topos theory. According to our humble opinion these effects can also be traced back to our stressing of interactive communication formats during the week in Dagstuhl, as there are the lightning talks sessions (one-hour slots of short talks of 5 minutes each) and especially the working groups, unusual both in number and participation.

As a pity, one of the four organisers, Thierry Coquand, could not attend this Dagstuhl Seminar. He still played a decisive role in forming the programme, especially concerning the crucial topics of geometric logic, topos theory and synthetic algebraic geometry with homotopy type theory.

An experimental addition to this year's seminar was the integration of an informal Discord server, serving as a digital hub for participant engagement. This platform not only facilitated pre-seminar planning and coordination but also granted participants a stronger sense of ownership over the seminar proceedings. Through features such as photo-sharing of blackboards and solicitation of talk topics, many participants actively shaped the agenda and direction of discussions.

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3 Overview of Talks

3.1 Geometric classes for the lazy topos theorist

Peter Arndt (Heinrich-Heine-Universität Düsseldorf, DE)

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Any geometric morphism between Grothendieck toposes factors as a surjection followed by a dense inclusion, followed by a closed inclusion. If the given geometric morphism f goes from the classifying topos $Set[T]$ of a geometric theory T to the classifying topos $Set[S]$ of a geometric theory S , then the two new toposes showing up in the factorization can be described as follows: The first one is the classifying topos of S' , the geometric theory consisting of all sequents that are valid in $f^*(M_S)$, the S -model in $Set[T]$ classified by f . And the second one is the classifying topos of S'' , the geometric theory consisting of S and all geometric sequents of the form $\varphi \vdash \perp$ that are valid in $f^*(M_S)$.

Computing the intermediate toposes can be very hard for general geometric theories. However, the following three observations point to a way of making the factorization usable for real mathematical situations:

1. The factorization still works for κ -geometric morphisms, and the intermediate toposes are the classifying toposes for the corresponding κ -geometric theories (i.e. geometric theories, but with infinitary conjunctions of size $< \kappa$)
2. Any accessible category is the category of Set-models of a κ -geometric *presheaf theory*.
3. Any accessible functor between accessible categories arises from an essential κ -geometric morphism between classifying presheaf toposes.

Thus, if we are willing to introduce the extra cardinal κ , we can place ourselves into the world of presheaf κ -toposes and essential κ -geometric morphisms. Here everything becomes rather easily computable:

Theorem: For an essential κ -geometric morphism between presheaf κ -toposes, induced by a functor $f: C \rightarrow D$ between the small index categories, the intermediate toposes of the factorization are the presheaf toposes on the full subcategory of D whose objects are in the essential image of f , and the full subcategory of D whose objects admit a morphism into the essential image of f , respectively.

Sample applications:



A. The theory of 3-colorable graphs can be axiomatized by adding to the theory of graphs all those geometric negations that are valid for the triangle graph. This can be seen by factorizing the geometric morphism from Set to the classifying topos of graphs, that chooses the triangle graph. The domain of the closed inclusion part is the classifying topos of 3-colorable graphs, because a graph is 3-colorable if and only if it admits a morphism to the triangle graph.

B. There are geometric sequents allowing to distinguish free finitely generated groups with different numbers of generators. This can be seen by considering the two geometric morphisms from Set to the classifying topos of groups that pick free groups on different numbers n, m (both ≥ 2) of generators. The surjection-inclusion factorizations yield the classifying toposes of the geometric theories of these free groups. By the theorem, they are the presheaf toposes over the full subcategories whose object are the free group with m , resp. n , generators. Explicit computation shows that their idempotent completions are non-equivalent, hence the presheaf toposes are non-equivalent, hence the classified geometric theories are different.

C. There are no negated κ -geometric formulas that are satisfied by all Special groups (a notion from quadratic form theory) coming from von Neumann-regular rings, but not by general special groups. This can be seen by considering the morphism from the classifying κ -topos of von Neumann-regular rings to the classifying κ -topos of special groups ($\kappa \geq \aleph_1$), and using the result that every special group admits a morphism the special group of a von Neumann-regular ring.

3.2 The Countable Reals

Andrej Bauer (University of Ljubljana, SI)

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 Andrej Bauer

Joint work of Andrej Bauer, James E. Hanson

Main reference Andrej Bauer, James E. Hanson: “The Countable Reals”, CoRR, Vol. abs/2404.01256, 2024.

URL <https://arxiv.org/abs/2404.01256>

In 1874 Georg Cantor published a theorem stating that every sequence of reals is avoided by some real, thereby showing that the reals are not countable. Cantor’s proof uses classical logic. There are constructive proofs, although they all rely on the axiom of countable choice. Can the real numbers be shown uncountable without excluded middle and without the axiom of choice?

In this evening session, we have a tour of the recent result that, perhaps astoundingly, higher-order intuitionistic logic cannot show the reals to be uncountable. The result rests on the construction of a certain sequence of reals, shown to exist by Joseph S. Miller from University of Wisconsin–Madison, with a strong counter-diagonalization property, and a novel variant of realizability making use of oracles with a certain uniformity restriction.

3.3 Quantified Boolean Formulas: Proofs, Solving and Circuits

Olaf Beyersdorff (Friedrich-Schiller-Universität Jena, DE)

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 Olaf Beyersdorff

Main reference Olaf Beyersdorff, Benjamin Böhm: “Understanding the Relative Strength of QBF CDCL Solvers and QBF Resolution”, Log. Methods Comput. Sci., Vol. 19(2), 2023.

URL [http://dx.doi.org/10.46298/LMCS-19\(2:2\)2023](http://dx.doi.org/10.46298/LMCS-19(2:2)2023)

This talk gives an overview on connections between proof complexity and algorithms for propositional satisfiability (SAT) and quantified Boolean formulas (QBF). We particularly cover the correspondence between conflict-driven clause learning algorithms (CDCL) for SAT and QBF to resolution proof systems. These connections enable the analysis of SAT/QBF algorithms through proof complexity. While in SAT there is a tight relation between propositional resolution and (non-deterministic) CDCL, the picture is more complex as we can show the incomparability of Q-Resolution and (non-deterministic) QCDCL. Nevertheless, lower bounds on QBF resolution systems yield run-time lower bounds for different QCDCL variants. Such lower bounds can be obtained via strong relations between QBF proof systems and circuit classes.

References

- 1 Olaf Beyersdorff. Proof complexity of quantified Boolean logic – a survey. In Marco Benini, Olaf Beyersdorff, Michael Rathjen, and Peter Schuster, editors, *Mathematics for Computation (M4C)*, pages 353–391. World Scientific, 2022.
- 2 Olaf Beyersdorff, Joshua Blinkhorn, Meena Mahajan, and Tomás Peitl. Hardness characterisations and size-width lower bounds for QBF resolution. *ACM Transactions on Computational Logic*, 2022.
- 3 Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artif. Intell.*, 175(2):512–525, 2011.

3.4 Skolem’s Theorem in Coherent Logic

Marc Bezem (*University of Bergen, NO*)

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Joint work of Marc Bezem, Thierry Coquand

Main reference Marc Bezem, Thierry Coquand: “Skolem’s Theorem in Coherent Logic”, *Fundam. Informaticae*, Vol. 170(1-3), pp. 1–14, 2019.

URL <http://dx.doi.org/10.3233/FI-2019-1853>

After a brief introduction to skolemization and to coherent logic, we discuss three possible research directions, relevant for automated reasoning.

- Generalization of skolemization from (properly quantified) atoms to coherent sentences. Due to the implication in the latter, one could call this conditional (or hypothetical) skolemization.
- More efficient proof systems for coherent logic. Ground forward reasoning is conceptually very simple and therefore attractive, but it leads to exponentially longer proofs as compared to non-ground rules.
- Comparison of the length of skolemized and non-skolemized proofs. The latter are known to be (much) longer in the case of first-order logic, but this is unknown for the coherent fragment.

3.5 A primer to realizability theory

Ingo Blechschmidt (*Universität Augsburg, DE*)

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Main reference Andrej Bauer: “Realizability as the connection between computable and constructive mathematics”, 2005.

URL <https://math.andrej.com/asset/data/c2c.pdf>

Realizability theory provides a bridge between constructive and computable mathematics. In its basic form, the fundamental result is that there is a mechanical way to extract from any given constructive number-theoretic proof a computable witness, a so-called “realizer” for instance, a realizer for the infinitude of primes is a Turing machine computing larger and larger prime numbers.

From this basic picture, originally due to Kleene, gradually several enhancements emerged: to higher-order statements and proofs; to proofs set in flavors of classical set theory; and to different machine models.

This evening talk featured a leisurely introduction to realizability theory. For building intuition, we focused on the following examples, exploring for each its status in classical mathematics, constructive mathematics, ordinary realizability and realizability based on infinite time Turing machines:

1. Every number is prime or not prime.
2. Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.
3. Every map $\mathbb{N} \rightarrow \mathbb{N}$ is (Turing-)computable.
4. Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.
5. Markov's principle holds.
6. Countable choice holds.
7. Heyting arithmetic is categorical.
8. A statement holds iff it is realized.
9. There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.

Our decision to approach our exploration in a constructive metatheory allowed us to discuss inheritance of classical principles. For instance, Markov's principle is realized if and only if it holds in the metatheory, while countable choice is realized even if it fails in the metatheory.

3.6 Geometric Type Theory

Ulrik Buchholtz (University of Nottingham, GB)

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This was an exploratory talk, outlining the structure of an envisioned geometric type theory.

The idea is to have a type theory capable of formalizing geometric reasoning, proving results such as $((\phi \rightarrow \perp) \rightarrow \perp) = 1$ for the generic proposition ϕ , and similarly other non-geometric properties of generic objects; and also capable of constructing common geometric maps and bundles.

The intended model is based on (small) toposes (or $(\infty, 1)$ -toposes for a homotopical version), where

- contexts are such toposes \mathcal{X}

and with two different type judgments:

- $\mathcal{X} \vdash A \text{ étale}_i$ corresponding to sheaves over \mathcal{X} in \mathcal{U}_i (classified by the usual univalent universes \mathcal{U}_i)
- $\mathcal{X} \vdash Y \text{ space}_i$ corresponding to geometric morphisms $p : \mathcal{Y} \rightarrow \mathcal{X}$ in \mathcal{U}_i .

We mentioned that maybe we should allow Y itself to be in \mathcal{U}_j , necessitating two indices, $\text{space}_{i,j}$.

Correspondingly, there should be two different element judgments:

- $\mathcal{X} \vdash a : A$ corresponding to sections of sheaves,
- $\mathcal{X} \vdash y : Pt_i(Y)$ corresponding to geometric sections $y : \mathcal{X} \rightarrow \mathcal{Y}$ in \mathcal{U}_j .

We also need to consider two fragments:

- A geometric fragment ($\Sigma, \text{Id}, \text{finitary HITs}$) stable under all geometric morphisms, and
- A full fragment ($\Pi, \text{W}, \text{universes}$) stable only under étale morphisms.

An inspiration is Mike Shulman's observation that in the $(\infty, 1)$ -world atomic=logical=étale.

A question arose as to whether we prove that spaces are exponentiable, or whether we need a separate judgment for that. If we write $[Y, Z]$ for the exponential, then if A, B are étale, the Π -type $A \rightarrow B$ should be the discrete coreflection $(A \rightarrow B) \rightarrow [A, B]$.

For the generic proposition (and hence all propositions) ϕ , we should get $[[\phi, 0], 0] = \phi$.

3.7 Making predicative sense of impredicative foundations

Ulrik Buchholtz (*University of Nottingham, GB*)

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Main reference Solomon Feferman: “Does reductive proof theory have a viable rationale?” *Erkenntnis* 53: 63–96, 2000.

URL <https://doi.org/10.1023/A:1005622403850>

Although classical principles such as the law of excluded middle or the axiom of choice are not available in constructive mathematics, constructive mathematics can make sense of them: The double-negation translation interprets classical logic in intuitionistic logic, and Gödel’s L translation embeds ZFC in ZF. In this way, proofs in classical mathematics can be understood as blueprints for constructive proofs; classical proofs can be decrypted to unearth obscured constructive content. Several other techniques for making constructive sense of classical proofs exist as well. Adding the law of excluded middle or the axiom of choice typically does not increase the consistency strength at all.

In contrast, there is no similar translation from impredicative mathematics to predicative mathematics; indeed, impredicative foundations are typically much stronger than their predicative counterparts. Hence it seems that predicative foundations cannot make sense of impredicative systems.

The evening talk explored that, fantastically, in some cases it is still possible to extract predicative content from impredicative proofs. As direct translations are no longer possible, the talk offered a whirlwind tour of the required tool, ordinal analysis, including a discussion of the true limits of predicative reasoning.

3.8 Introduction to synthetic algebraic geometry

Felix Cherubini (*Chalmers University of Technology – Göteborg, SE*) and Matthias Hutzler (*University of Gothenburg, SE*)

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Joint work of Felix Cherubini, Thierry Coquand, Matthias Hutzler

Algebraic Geometry is about studying shapes given by polynomial equations. Systems of polynomial equations are mapped to some “space” of solutions (the common zeros). It turns out that we can just use sets as “spaces” if we do not insist on the axiom of choice or the law of excluded middle. In their absence we can ask that the SQC axiom (of Blechschmidt) holds: The map $A \rightarrow R^{\text{Spec}(A)}$ is an isomorphism for all finitely presented R -algebras $A = R[X_1, \dots, X_n]/(P_1, \dots, P_\ell)$ and $\text{Spec}(A) := \{x \in R^n \mid \forall i. P_i(x) = 0\}$. Together with two other axioms, this is the start of a well-working synthetic version of algebraic geometry which is modeled by a sheaf topos, the Zariski topos.

In the talk we also mentioned the topic of finite schemes in the synthetic setting, hoping to spark new ideas in connecting the geometric-algebraic notion of a finite scheme with finiteness notions in constructive mathematics.

References

- 1 Felix Cherubini, Thierry Coquand, Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. arXiv:2307.00073, 2023

3.9 Defining subsets of rational numbers using first-order logic

Nicolas Daans (Charles University – Prague, CZ)

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Joint work of Becher, Karim Johannes; Daans, Nicolas; Dittmann, Philip; Fehm, Arno
Main reference Nicolas Daans: “Existential first-order definitions and quadratic forms”. Phd thesis. Universiteit Antwerpen, 2022. Under the supervision of Karim Johannes Becher and Philip Dittmann.
URL <https://hdl.handle.net/10067/1903760151162165141>

In 1970, building on work of Davis, Putnam, and Robinson, Matiyasevich established a complete classification of subsets of the integers which can be described by an existential first-order formula in the language of rings, in terms of their computability by a Turing machine. After Gödel’s (in)famous Incompleteness Theorems, this marks one of the major milestones in the study of computability in number theory, and it has inspired continued research into (existential) first-order definability of subsets of rings.


This evening talk offers a lighthearted initiation into the realm of first-order and existential definability within fields, with a focus on the field of rational numbers. Through various examples, I highlight the intricacies surrounding the classification of definable subsets of the rationals and associated decidability problems, while showcasing the diverse array of methodologies drawn from algebra, number theory, quadratic form theory, and model theory that have been employed to tackle them. To align with the theme of this week’s seminar, particular attention is devoted to contrasting constructive and non-constructive approaches to these problems. Nonetheless, the talk remains accessible to audiences with classical training and assumes no prior background knowledge.

References

- 1 Nicolas Daans, Philip Dittmann, and Arno Fehm. “*Existential rank and essential dimension of diophantine sets*”. Available as arXiv:2102.06941. Oct. 2021. <https://doi.org/10.48550/arXiv.2102.06941>
- 2 Nicolas Daans. “*Existential first-order definitions and quadratic forms*”. Phd thesis. Universiteit Antwerpen, 2022. <https://hdl.handle.net/10067/1903760151162165141>
- 3 Nicolas Daans. “*Universally defining Z in Q with 10 quantifiers*”. In: Journal of the London Mathematical Society 109.2 (2024), e12864. <https://doi.org/10.1112/jlms.12864>
- 4 Yuri R. Matiyasevich. “*Diofantovost’ perechislimykh mnozhestv [Enumerable sets are Diophantine]*”. In: Doklady Akademii Nauk SSSR 191 (1970), pp. 279-282. <http://mi.mathnet.ru/dan35274>
- 5 Julia Robinson. “*Definability and decision problems in arithmetic*”. In: Journal of Symbolic Logic 14 (1949), pp. 98-114.

3.10 Hierarchy of Σ_n -fragments of logical principles over HA and hierarchy of intermediate propositional logics

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Joint work of Makoto Fujiwara, Hajime Ishihara, Takako Nemoto, Nobu-Yuki Suzuki, Keita Yokoyama
Main reference Makoto Fujiwara, Hajime Ishihara, Takako Nemoto, Nobu-Yuki Suzuki, Keita Yokoyama: “Extended Frames and Separations of Logical Principles”, Bull. Symb. Log., Vol. 29(3), pp. 311–353, 2023.
URL <http://dx.doi.org/10.1017/BSL.2023.29>

We investigate seven natural variations of the linearity axiom

$$\text{LIN} : (p \rightarrow q) \vee (q \rightarrow p)$$

and the following principles in the context of intermediate propositional logic and arithmetic:

- PEM (the principle of excluded middle) : $p \vee \neg p$;
- WPEM (the weak principle of excluded middle) : $\neg p \vee \neg\neg p$;
- DML (the De Morgan law) : $\neg(p \wedge q) \rightarrow \neg p \vee \neg q$;
- WDML (the weak De Morgan law) : $\neg(\neg p \wedge \neg q) \rightarrow \neg\neg p \vee \neg\neg q$;
- DNE (the double negation elimination) : $\neg\neg p \rightarrow p$.


In fact, the hierarchy of Σ_n -fragments of the logical principles over Heyting arithmetic HA is quite different from that of the intermediate propositional logics induced by the logical principles. For a separation result on the Σ_n -fragment of

$$\text{LIN}_7 : (p \rightarrow \neg\neg q) \vee (\neg\neg q \rightarrow p)$$

in arithmetic, we employ a meta-theorem with respect to extended frame to an appropriate propositional Kripke model which refutes LIN_7 .

3.11 Proof Mining: Logical Foundations and Recent Applications

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In the Proof Mining paradigm one applies proof-theoretic transformations to extract new computational information (e.g. programs or effective bounds) as well as qualitative improvements (e.g. by weakening of assumptions) from given prima facie noneffective proofs in different areas of core mathematics. In this talk, we will

- discuss recent extensions (mainly due to [8] but see also [5]) of the existing framework for logical bound extraction metatheorems to cover set-valued monotone and accretive operators in Hilbert and Banach spaces;
- indicate the potential of generalizing mined proofs such as the quantitative analysis of a Halpern-type proximal point algorithm in suitable Banach spaces given in [3] to new geodesic settings ([7, 10]) as well as to the context of so-called Bregman strongly nonexpansive mappings ([8], [9]);
- show how a rate of convergence for the Dykstra algorithm in convex optimization which recently has been obtained in [6] under a metric regularity assumption can be explained in terms of a novel extension of the concept of Fejér monotonicity ([4]);

- prove the first linear rates of asymptotic regularity for the Tikhonov-Mann algorithm in geodesic spaces as an application of proof mining ([1]);
- present the first effective rate of convergence for the asymptotic regularity of ergodic averages of iterations of nonexpansive mappings in uniformly convex Banach spaces ([2]).

References

- 1 H. Cheval, U. Kohlenbach, L. Leuştean: On modified Halpern and Tikhonov-Mann iterations. *Journal of Optimization Theory and Applications* vol.197, pp. 233-251 (2023).
- 2 A. Freund, U. Kohlenbach: R.E. Bruck, proof mining and a rate of asymptotic regularity for ergodic averages in Banach spaces. *Applied Set-Valued Analysis and Optimization* vol.4, pp. 323-336 (2022).
- 3 U. Kohlenbach: Quantitative analysis of a Halpern-type proximal point algorithm for accretive operators in Banach spaces. *Journal of Nonlinear and Convex Analysis* vol. 21, no.9, pp. 2125-2138 (2020).
- 4 U. Kohlenbach, P. Pinto: Fejér monotone sequences revisited. Submitted
- 5 U. Kohlenbach, N. Pischke: Proof theory and nonsmooth analysis. *Philosophical Transactions of the Royal Society A* vol.381, Issue 2248, DOI 10.1098/rsta.2022.0015, 21 pages (2023).
- 6 P. Pinto: On the finitary content of Dykstra's cyclic projections algorithm. arXiv:2306.09791, 2023, submitted.
- 7 P. Pinto: Nonexpansive maps in nonlinear smooth spaces. Submitted.
- 8 N. Pischke: Proof-Theoretical Aspects of Nonlinear and Set-Valued Analysis. PhD Thesis, Technische Universität Darmstadt, xvi+338pp., 2024.
- 9 N. Pischke, U. Kohlenbach: Effective rates for iterations involving Bregman strongly nonexpansive mappings. 2024 Submitted.
- 10 A. Sipoş: Abstract strongly convergent variants of the proximal point algorithm. *Computational Optimization and Applications* vol.83, pp. 349-380 (2022).

3.12 Geometric theories from the point of view of constructive mathematics

Henri Lombardi (University of Franche-Comté – Besançon, FR)

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1) Constructive mathematics.

Gauss, Kronecker, Poincaré, Bishop.

Intuitive mathematics, (no need of formalisation) using basic, undefined tools :

- natural numbers, constructions,
- proofs : a proof is a convincing proof,
- predicativity,
- Poincaré: Never lose sight of the fact that every proposition concerning infinity must be the translation, the precise statement of propositions concerning the finite.

2) Why using geometric theories ?

Formalisation is about a piece of intuitive informal mathematics. It can help to understand and analyse this piece of mathematics.

Geometric theories are interesting because they use only very simple kinds of assertions, geometric assertions, managed without using logic: they are purely computational machineries.

Precursor: recursive arithmetic by Goodstein.

My understanding :

- sorts in a geometric theories correspond to sets in Bishop constructive mathematics (BCM)
- geometric theories are to be used and analysed in the external world given by BCM

This is very different of the usual understanding of geometric theories where the external world is a world of classical mathematics, using categorical logic, LEM and Choice, perhaps also power sets and other ugly things à la ZFC.

The underlying logic of geometric theories is a small part of minimal logic and natural deduction: no negation, no connector \Rightarrow , no complicate sentences using forall exists forall.

No conflict between classical and constructive understanding of mathematics used inside geometric theories. Adding formally negation and \Rightarrow does not add new valid rules in the former language.

3) Dynamic versus geometric theories.

The dynamic view of geometric theories is purely computational, logic free

- no new formulas: only atomic formulas given in the language
- no logical implication: only deduction rules
- connector “or” is replaced by “open branches of computations”
- quantification “exists y such that $P(y)$ ” is replaced by “introduce a new fresh variable y satisfying $P(y)$ ”
- a deduction rule is valid when you have constructed a deduction tree:
 - at the root you have your context and your hypothesis,
 - at each node you use axioms as previously explained,
 - and the conclusion is to be valid at each leaf.

4) Strength of geometric theories.

A large part of usual abstract algebra can be formalised in **finitary geometric theories**, which corresponds to usual coherent first order theories in logic.

E.g., groups, lattice groups, local rings, residually discrete local rings, Prüfer domains, finitely presented modules, gcd rings, f-rings, discrete fields, discrete ordered fields, real closed discrete fields, discrete valued fields, henselian fields, ...

Some concepts need **infinitary geometric theories**. E.g. flatness, Krull dimension, coherence, Dedekind domains, Krull rings, projectively resolvable modules, ...

Some concepts, as Noetherianity, are outside the scope of geometric theory.

Infinitary geometric theories allow infinite disjunctions in the conclusion of a dynamic rule. The external world becomes crucial when constructing deduction trees.

5) Grothendieck toposes.

In classical mathematics, Grothendieck toposes are more or less “the same objects” as geometric theories. Grothendieck coherent toposes correspond to finitary geometric theories.

This important fact has to be understood in the context of constructive mathematics with an external world à la BCM.

Particularly important is to interpret geometric functors between Grothendieck toposes as corresponding to some kinds of morphisms between geometric theories, with an external world à la BCM.

3.13 Joyal’s Arithmetic universes via the pure existential completion

Maria Emilia Maietti (University of Padova, IT)

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Joint work of Maria Emilia Maietti, Davide Trotta

Arithmetic universes were built by A. Joyal in the seventies in some lectures, all still unpublished, to provide a categorical proof of Gödel’s incompleteness theorems. They amount to be exact completions of the lex category of predicates of a given Skolem theory. In [1] we proposed the notion of list-arithmetic pretopos as the general abstract notion of arithmetic universe.


In our talk, we report recent results obtained in joint work with Davide Trotta in [3]. We show that each arithmetic universe is the exact completion of the pure existential completion of the doctrine of its Skolem predicates. The notion of existential completion was introduced by D. Trotta in [4] and further analyzed in [2] in terms of existential free objects relative to an existential doctrine. In particular, we show in [3] that the Skolem predicates of an arithmetic universe provide a projective cover of the exact completion of their free pure existential completion. As a byproduct, we conclude that the initial arithmetic universe is the exact completion of the doctrine of recursively enumerable predicates.

References

- 1 M.E. Maietti. Joyal’s arithmetic universe as list-arithmetic pretopos. *Theory Appl. Categ.*, 24(3):39–83, 2010
- 2 M.E. Maietti and D. Trotta. A characterization of generalized existential completions. *Ann. Pure Appl. Logic*, 174(4):103234, 2023.
- 3 M.E. Maietti and D. Trotta. A characterization of regular and exact completions of pure existential completions. *Arxiv*, <https://arxiv.org/abs/2306.13610>, 2023
- 4 D. Trotta. The existential completion. *Theory Appl. Categ.*, 35(43):1576–1607, 2020.

3.14 Structural proof theory for logics of strong negation

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Joint work of Norihiro Kamide, Sara Negri

Gurevich logic is an extended constructive three-valued logic obtained from intuitionistic logic by adding a connective \sim of *strong negation*, with the following axiom schemata, where \neg is intuitionistic negation:

1. $\sim\sim A \supset A$,
2. $\sim\neg A \supset A$,
3. $\sim A \supset \neg A$,
4. $\sim(A \wedge B) \supset \sim A \vee \sim B$,
5. $\sim(A \vee B) \supset \sim A \wedge \sim B$,
6. $\sim(A \supset B) \supset A \wedge \sim B$.

Nelson logic, also known as Nelson’s constructive three-valued logic N3, is the intuitionistic negation-less fragment of Gurevich logic.

The primary formal difficulty in developing a natural deduction system for logics of strong negation lies in the requirement of having rules for \neg and \sim without \perp . This is solved using the rules of *explosion*, of \neg -*introduction*, and of *excluded middle*:

$$\frac{\neg A \quad A}{C} \text{Exp} \qquad \frac{\begin{array}{c} [A] \\ \vdots \\ C \end{array} \quad \begin{array}{c} [A] \\ \vdots \\ \neg C \end{array}}{\neg A} \neg\text{I} \qquad \frac{\begin{array}{c} [\neg A] \\ \vdots \\ C \end{array} \quad \begin{array}{c} [A] \\ \vdots \\ C \end{array}}{C} \text{Em}$$

The following are presented:

- Natural deduction and G3-style sequent calculi systems for Gurevich and Nelson logic.
- A translation between the natural deduction and sequent calculi introduced, with an indirect proof of normalization as a consequence of cut elimination.
- A syntactic embedding of Gurevich logic into intuitionistic logic.
- A classical sequent calculus with strong negation, used as a platform for obtaining classical logics of strong negation, such as Avron and De-Omori logics.
- A Glivenko theorem for embedding classical logic with strong negation into Gurevich logic.

References

- 1 N. Kamide and S. Negri, Unified natural deduction for logics of strong negation, ms., 2023.
- 2 N. Kamide and S. Negri, G3-style sequent calculi for Gurevich logic and its neighbors, ms., 2023.

3.15 König's lemma, Weak König's lemma and Σ_1^0 induction

Takako Nemoto (Tohoku University – Sendai, JP)

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Joint work of Takako Nemoto, Makoto Fujiwara

It is known that weak König's lemma (WKL) is strictly weaker than König's lemma (KL) in Friedman-Simpson's reverse mathematics ([2]). Then a natural question is how we can characterise the difference between WKL and KL. In Simpson's book, it is stated:

- WKL does not imply Σ_1^0 induction.
- WKL is equivalent to KL for trees with height-wise bounded functions over RCA_0 .

It is also known that Fan theorem, a classical contraposition of KL, yields Σ_1^0 induction over RCA_0^* ([1]). In this talk, we consider the difference between WKL and KL in the context of constructive reverse mathematics and the relationship between KL and Σ_1^0 induction.

References

- 1 T. Nemoto and K. Sato, *A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic*, The Journal of Symbolic Logic, 87 (2), pp.437-497 (2022) doi:10.1017/jsl.2018.6
- 2 Subsystems of second order arithmetic, CUP (2009)

3.16 From a Constructive Logic to a Contradictory Logic


Satoru Niki (Ruhr-Universität Bochum, DE)

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The negation in intuitionistic logic has sometimes been criticised as problematic. One reason for this is the alleged lack of witness for negated conjunctions: $\text{not}-(A \text{ and } B)$ may hold without one of $\text{not}-A$ and $\text{not}-B$ being so. As a remedy, David Nelson introduced an alternative notion of “constructible falsity”, which promises such a witness. Nelson’s negation can even be made paraconsistent, which also encouraged the abandonment of intuitionistic negation altogether. However, such a move may obscure the notion of negation from the viewpoint of an intuitionistic logician. In this talk, I am going to give an overview of some different views for constructive negation. Then I will suggest another solution to the problem of negated conjunction. This solution, which leads to a variant of Heinrich Wansing’s logic C, hints that intuitionistic negation and paraconsistent constructible falsity may be seen as complementary, once we accept some contradictions to be provable.

3.17 Topologies of open complemented subsets

Iosif Petrakis (University of Verona, IT)

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We present cs-topologies, or topologies of complemented subsets, as a new approach to constructive topology that preserves the duality between open and closed subsets of classical topology. Complemented subsets were used successfully by Bishop in his constructive formulation of the Daniell approach to measure and integration. We use complemented subsets in topology, in order to describe simultaneously an open set, the first-component of an open complemented subset, and its complement as a closed set, the second component of an open complemented subset. We analyse the canonical cs-topology induced by a metric, and we introduce the notion of a modulus of openness for a cs-open subset of a metric space. Pointwise and uniform continuity of functions between metric spaces are formulated with respect to the way these functions inverse open complemented subsets together with their moduli of openness. The addition of moduli of openness in the concept of an open subset, given a base for the cs-topology, makes possible to define the notion of uniform continuity of functions between csb-spaces, that is cs-spaces with a given base. In this way, the notions of pointwise and uniform continuity of functions between metric spaces are directly generalised to the notions of pointwise and uniform continuity between csb-topological spaces.

References

- 1 I. Petrakis. *Topologies of open complemented subsets*. arXiv:2312.17095v1, 2023.

3.18 Looking for the ideal background theory

Michael Rathjen (University of Leeds, GB)

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Proofs in mathematics often have a narrative quality to them, taking the reader on a long journey. Sometimes the reader has to wait for new mathematical characters (like imaginary numbers) to be created so the journey can be continued. Hilbert called these novel characters ideal elements. His conservation program was the idea that, while being important for the advancement of mathematics, ideal elements should be eliminable from proofs of concrete mathematical theorems.

Investigations by a long list of mathematicians/logicians (e.g. Weyl, Hilbert, Bernays, Lorenzen, Takeuti, Feferman, Friedman, Simpson to name a few) have shown that large swathes of ordinary mathematics can be undergirded by theories of fairly modest consistency strength. This confirms what Hilbert surmised in his program, namely that elementary results (e.g. those expressible in the language of number theory) proved in abstract, non-constructive mathematics can often be proved by elementary means.

The best known program for calibrating the strength of theorems from ordinary mathematics is reverse mathematics (RM). RM's scale for measuring strength is furnished by certain standard systems couched in the language of second order arithmetic. However, its language is not expressive enough to be able to talk about higher order objects, such as function spaces, directly.

Richer formal systems, in which higher order mathematical objects can be directly accounted for, have been suggested. The price for maintaining conservativity over elementary theories, however, is that one has to use different logics for different ontological realms, allowing classical logic to reign at the level of numbers whereas higher type mathematical objects are merely answerable to intuitionistic logic. In the talk, I'd like to present some of these semi-intuitionistic systems, give a feel for carrying out mathematics within them, and relate them to systems considered in RM.

3.19 A topos for continuous logic

Benno van den Berg (University of Amsterdam, NL)

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Joint work of Daniel Figueroa

Main reference Daniel Figueroa, Benno van den Berg: "A topos for continuous logic". *Theory Appl. Categ.* 38, 1108-1135 (2022)

URL <http://www.tac.mta.ca/tac/volumes/38/28/38-28.pdf>

Continuous model theory is an area of model theory in which one studies structures where the truth values of sentences can be any element of the unit interval. This branch of model theory has applications in areas where metric structures are pervasive, such as in analysis and probability theory.

In this talk we propose an analysis of continuous model theory based on ideas from categorical logic, in particular Lawvere's notion of a hyperdoctrine. We propose a hyperdoctrine for continuous model theory and show that this hyperdoctrine can be embedded in the subobject hyperdoctrine of a topos. This means that continuous logic can be understood as the internal logic of a suitable topos.

3.20 Is geometric logic constructive?

Steven J. Vickers (*University of Birmingham, GB*)

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Joint work of Ming NG

For point-free topology (locales, etc.) we can reason validly in a natural manner using points, provided we restrict ourselves to constructions that are geometric – preserved by pullback along geometric morphisms. This style has now been applied in some results of real analysis, including exponentiation and logarithms (joint with Ming Ng) and the Fundamental Theorem of Calculus.

Are such constructions constructive in the strong sense of yielding algorithms? This is very unclear. My talk discusses some aspects of the question.

1. Some constructive taboos are true geometrically! But that is because they have to be interpreted topologically.
2. Geometric maths can be understood via an ontology of observations rather than constructions.
3. Point-free surjections, which embody conservativity principles, allow us to reason as if certain constructions can be done, even when they can't.

3.21 The Fundamental Theorem of Calculus, point-free

Steven J. Vickers (*University of Birmingham, GB*)

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Main reference Steven Vickers. *The Fundamental Theorem of Calculus point-free, with applications to exponentials and logarithms*. arXiv:2312.05228, 2023

URL <https://arxiv.org/abs/2312.05228>

The Fundamental Theorem of Calculus is so fundamental that, once it has been proved point-free, much subsequent development can be more or less standard. In a 5 minute minitalk I summarized the main features in [1] by which this was achieved.

1. Lower and upper integrals, valued as lower and upper reals, were described in [2]. After this, it is necessary only to show how they can be combined to give Dedekind reals.
2. Differentiation is treated in a Carathéodory style, with limits replaced by the existence of continuous slope maps. This was used in [3], where also Rolle's Theorem was proved.
3. For the derivative of an integral, the slope map can be defined as an integral with respect to the *uniform* valuation on $[x_0, x]$. Geometricity guarantees that this is continuous even at the limiting case $x = x_0$.

References

- 1 Steven Vickers. *The Fundamental Theorem of Calculus point-free, with applications to exponentials and logarithms*. arXiv:2312.05228, 2023
- 2 Steven Vickers. *A localic theory of lower and upper integrals*. *Mathematical Logic Quarterly* **54**, 2008
- 3 Steven Vickers. *The connected Vietoris powerlocale*. *Topology and its Applications* **156**, 2009

4 Working groups

4.1 Finiteness in synthetic algebraic geometry

Ingo Blechschmidt (Universität Augsburg, DE), Andrej Bauer (University of Ljubljana, SI), Felix Cherubini (Chalmers University of Technology – Göteborg, SE), Martín H. Escardó (University of Birmingham, GB), and Matthias Hutzler (University of Gothenburg, SE)

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Joint work of Ingo Blechschmidt, Andrej, Felix Cherubini, Martín H. Escardó, Hugo Moeneclaey, Matthias Hutzler, David Wärn
Main reference Ingo Blechschmidt, Felix Cherubini, Hugo Moeneclaey, Matthias Hutzler, David Wärn: “Finite Schemes in Synthetic Algebraic Geometry”, 2024
URL <https://felix-cherubini.de/finite.pdf>

In the related talk of Cherubini and Hutzler, the recent foundations for synthetic algebraic geometry were discussed [1]. One important step in improving synthetic calculations is understanding what notions of finiteness can help to move from geometric problems to numbers of interest. Ingo Blechschmidt presented a proof that finite schemes satisfy the finiteness notion of “Noetharian” from constructive mathematics [1]. This notion captures the intuition that there is no infinite sequence in a set. However, we found out in the group, that there are Noetherian sets in our language, which are not finite schemes. We discussed results on compactness of finite schemes which are currently expanded in [2] and Ingo Blechschmidt found a candidate for the synthetic version of the notion of “quasi-finite” from classical algebraic geometry. In a discussion with Martín Escardó and Andrej Bauer we checked if it is possible that the base ring of our theory is overt (a notion from synthetic topology). This turned out to be true, for example in the case where the external base ring is an algebra over an infinite field. To our surprise, overtness has a strong implications: non-empty open subsets of the base ring are inhabited.

References

- 1 T. Coquand, A. Spiwack. Constructively Finite?
- 2 I. Blechschmidt, F. Cherubini, H. Moeneclaey, M. Hutzler, D. Wärn. *Finite Schemes in Synthetic Algebraic Geometry*.
- 3 Felix Cherubini, Thierry Coquand, Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. arXiv:2307.00073, 2023

4.2 Formalizing infinite time Turing machines

Ingo Blechschmidt (Universität Augsburg, DE), Andrej Bauer (University of Ljubljana, SI), Karim Johannes Becher (University of Antwerp, BE), and Martín H. Escardó (University of Birmingham, GB)

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URL <https://agdapad.quasicoherent.io/Dagstuhl2024/html/ittm.html>

Infinite time Turing machines provide an exotic model of computation where the operation of Turing machines is extended to infinite ordinal time. They have been devised by Joel David Hamkins and Andy Lewis and offer an intriguing connection between computability theory and set theory. Among their special features is that the realizability model they give rise to validates the peculiar statement “there is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$ ”, contradicting both classical mathematics and ordinary Turing realizability.


In the working group, we explored possible avenues for formalizing the notion of infinite time Turing machines in proof assistants based on dependent type theory. To the best of our knowledge, infinite time Turing machines have not been object of constructive scrutiny before.

In a constructive setting, a fundamental question is whether the cell contents should be Booleans or general truth values. The latter seem to force their inclusion as limsups of sequences of Booleans, but are at odds with the envisioned mechanical flavor of the computational model. We tentatively resolved this tension by sticking to Booleans, but weakening the transition function to a transition relation. Arbitrary infinite time Turing machines can then not be expected to have a well-defined execution trace, but many machines of interest will.

The working group concluded with a first sketch of the basic definitions in Agda, parametrized by the choice of ordinals used for indexing the time steps.

4.3 Tutorial on Agda, the dependently typed proof assistant

Ingo Blechschmidt (Universität Augsburg, DE)

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Agda is one of the commonly-used dependently typed proof assistants, facilitating collaborative verification and development of mathematical proofs. The strategic decision to schedule this tutorial on the first day of the seminar allowed participants to explore Agda during the week. The tutorial was a joint session of several people already well-versed in Agda, and profited greatly from numerous questions and comments from people with varying backgrounds, including familiarity with other different proof assistants and those new to the realm of computer formalization of mathematics.

References

- 1 M. Escardó, Introduction to Univalent Foundations of Mathematics with Agda. (2021), <https://www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/>
- 2 P. Wadler, W. Kokke, J. Siek, Programming Language Foundations in Agda. (2020), <https://plfa.inf.ed.ac.uk/20.07/>

4.4 Proof systems for geometric axioms and beyond

Sara Negri (University of Genova, IT), Giulio Fellin (University of Verona, IT), Eugenio Orlandelli (University of Bologna, IT), Edi Pavlovic (LMU München, DE), and Elaine Pimentel (University College London, GB)

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This working group presented an informal overview of the basics of the proof theory for geometric axioms together with recent developments.

Sara Negri presented an introduction to the method of “axioms as rules” applied to coherent and geometric axioms. By the method, cut- and contraction-free sequent calculi of the G3 family are extended while maintaining their structural properties [3, 5, 4]. One can go beyond geometric axioms by a change of basic notions (such as apartness instead of equality), by the method of systems of rules, and by the method of “geometrization” [1].

Elaine Pimentel showed how focusing and polarities can be applied for the construction of synthetic inference rules. In particular, we showed how this method can be applied for generating sequent rules for geometric axioms, advancing the work developed by Sara Negri and colleagues.

Edi Pavlović discussed the application of these methods in formal metaphysics. In particular, he discussed a four-sided sequent calculus for neutral free logic, how it simplifies quantifier rules for quantified weak Kleene, and what that can tell us.

Eugenio Orlandelli presented a terminating calculus for monadic first-order logic. The calculus has all rules invertible without relying on (implicit) contraction. The decision procedure based on this calculus has optimal complexity.

Giulio Fellin presented an infinitary version of Orevkov’s theorems about the Glivenko sequent classes [2].

References

- 1 Dyckhoff, R. and Negri, S. Geometrization of first-order logic. *The Bulletin of Symbolic Logic*, vol. 21, pp. 123–163, 2015.
- 2 Fellin, G., Negri, S. and Orlandelli, E. Glivenko sequent classes and constructive cut elimination in geometric logics. *Archive for Mathematical Logic*, vol. 62, pp. 657–688, 2023.
- 3 Negri, S. Contraction-free sequent calculi for geometric theories, with an application to Barr’s theorem. *Archive for Mathematical Logic*, vol. 42, pp. 389–401, 2003.
- 4 Negri, S. Geometric rules in infinitary logic. In *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics, Outstanding Contributions to Logic*. Springer, pp. 265.-293, 2021.
- 5 Negri, S. and von Plato, J. *Proof Analysis: A Contribution to Hilbert’s Last Problem*. Cambridge University Press, 2011.

4.5 Maps as bundles for point-free spaces

Steven J. Vickers (University of Birmingham, GB)

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Main reference Steven Vickers. Generalized point-free spaces, pointwise. arXiv:2206.01113, 2022

URL <https://arxiv.org/abs/2206.01113>

In the geometric style of point-free reasoning, just from the existence and nature of classifying toposes $\mathcal{S}[\mathbb{T}_X]$, we can say that

- *space* X = geometric theory \mathbb{T}_X (point = model of the theory),
- *map* = geometric construction of points from points.

I explained how one further gets

- *bundle* = geometric construction of spaces from points.

This is the essence of dependent type theory. We also get pullbacks of bundles as substitution. I focused on the case of localic bundles, but it does go through more generally for bounded geometric morphisms.

So: how is a map $p : Y \rightarrow X$ a bundle?

Joyal and Tierney [2] showed that localic maps p are equivalent to internal frames in the topos of sheaves over X . In [3] I showed how to replace an internal frame by a presentation \mathbb{T} of it, and then for a map $f : Z \rightarrow X$ the pullback f^*Z can be obtained from $f^*(\mathbb{T})$, where now f^* is the inverse image functor. This amounts to substitution.

If X already corresponds to a theory \mathbb{T}_X , then the internal \mathbb{T} amounts externally to an extension \mathbb{T}_Y , a process that is easier to work with if one uses “geometric type theory” rather than forcing theories into a standard form such as sites or first order theories.

For more discussion, see [1].


References

- 1 Steven Vickers. Generalized point-free spaces, pointwise. arXiv:2206.01113, 2022
- 2 Joyal and Tierney. An extension of the Galois theory of Grothendieck. *Memoirs AMS* 309, 1984
- 3 Steven Vickers. The double powerlocale and exponentiation Theory and Applications of Categories 12, 2004

5 Open problems

5.1 On the status of Zorn’s lemma

Ingo Blechschmidt (Universität Augsburg, DE)

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Over classical Zermelo-Fraenkel set theory, the axiom of choice (AC) and Zorn’s lemma (ZL) are well-known to be equivalent. Dropping the law of excluded middle (LEM) allows us to distinguish these two principles: A refined analysis shows that $AC = ZL + LEM$. While AC implies LEM and is hence a constructive taboo, ZL can be regarded as constructively neutral.

In fact, assuming ZL in the metatheory, there are plenty of models of constructive mathematics which validate ZL, and even more which validate all bounded first-order consequences of ZL: All localic Grothendieck toposes respectively all Grothendieck toposes.

That said, in mathematical practice, applications of ZL are often followed by an appeal to LEM, and without LEM, ZL loses much of its power. But there are important results which use only ZL and not LEM, such as in commutative algebra the existence of maximal ideals, the equivalence of divisible and injective abelian groups, and the existence of enough injectives.

The talk gave a summary of this circle of ideas and invited discussion on open questions: Is there a way to extract constructive content from ZL-powered results? Are there models of constructive mathematics which validate ZL, have strong ties to a given standard model and which do not require ZL in the metatheory?

5.2 Remarks on predicativity

Stefan Neuwirth (University of Franche-Comté – Besançon, FR)

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The talk by Laura Crosilla on predicativity has triggered a special interest for the “classical approach to predicativity” that culminated in the determination of the so-called limit of predicativity. This approach has much to offer in terms of technical sophistication, but it

leads to the following question, both an epistemological and a sociological one: does the fascination for technique do justice to the very concept of predicativity? Predicativity is about the open-endedness of the process of mathematical creation; how could it be given a definite limit?

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