Report from Dagstuhl Seminar 24062

Beyond-Planar Graphs: Models, Structures and Geometric Representations

Vida Dujmović*[1], Seok-Hee Hong*[2], Michael Kaufmann*[3], János Pach*[4], and Henry Förster†[5]

1 **University of Ottawa, CA.** vida@cs.mcgill.ca
2 **The University of Sydney, AU.** seokhee.hong@sydney.edu.au
3 **Universität Tübingen, DE.** michael.kaufmann@uni-tuebingen.de
4 **Alfréd Rényi Institute – Budapest, HU & EPFL – Lausanne, CH.** pach@renyi.hu
5 **Universität Tübingen, DE.** henry.foerster@uni-tuebingen.de

Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 24062 "Beyond-Planar Graphs: Models, Structures and Geometric Representations". The seminar investigated beyond-planar graphs, in particular, their combinatorial and topological structures, computational complexity and algorithmics for recognition, geometric representations, and their applications to real-world network visualization. Compared to the previous two editions of the seminar, we focus more on aspects of combinatorics and geometry.

The program consists of four invited talks on beyond planar graphs, open problem session, problem solving sessions and progress report sessions. Specific open problems include questions regarding the combinatorial structures and topology (e.g., k^+-real face graphs, beyond upward planar graphs, sparse universal geometric graphs, local-crossing-critical graphs), the geometric representations (e.g., constrained outer string graphs, rerouting curves on surface), and applications.

The details of the invited talks and progress reports from each working groups are included in this report.

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1 Executive Summary

Vida Dujmović (University of Ottawa, CA, vida@cs.mcgill.ca)
Seok-Hee Hong (The University of Sydney, AU, seokhee.hong@sydney.edu.au)
Michael Kaufmann (Universität Tübingen, DE, michael.kaufmann@uni-tuebingen.de)
János Pach (Alfréd Rényi Institute – Budapest, HU & EPFL – Lausanne, CH, pach@renyi.hu)

Many big data sets in various application domains have complex relationships, which can be modeled as *graphs*, consisting of entities and relationships between them. Consequently, graphs are extensively studied in both mathematics and computer science. In particular,

* Editor / Organizer
† Editorial Assistant / Collector

planar graphs, which can be drawn without edge crossings in the plane, form a distinguished role in graph theory and graph algorithms. Many structural properties of planar graphs are investigated in terms of excluded minors, low density, and small separators, leading to efficient planar graph algorithms. Consequently, fundamental algorithms for planar graphs have been discovered.

However, most real-world graphs, such as social networks and biological networks, are *nonplanar*. For example, the scale-free networks, which are used to model web graphs, social networks, and biological networks, are globally sparse nonplanar graphs with locally dense clusters and low diameters. To understand such real-world networks, we must solve fundamental mathematical and algorithmic research questions on *beyond-planar graphs*, which generalize the notion of planar graphs regarding topological constraints or forbidden edge crossing patterns.

This Dagstuhl Seminar investigated beyond-planar graphs, in particular, their combinatorial and topological structures (i.e., density, thickness, crossing pattern, chromatic number, queue number, and stack number), computational complexity and algorithmics for recognition, geometric representations (i.e., straight-line drawing, polyline drawing, intersection graphs), and their applications to real-world network visualization.

Compared to the previous two editions of the seminar, we focus more on aspects of combinatorics and geometry. Therefore, we included one new organizer and more participants from the corresponding fields. Thirty-two participants accepted the invitation to participate and arrived on Sunday afternoon.

On Monday morning, the program started with an introduction of all participants, followed by four invited talks to provide fundamental background knowledge on related research fields. We organized an open problems session on Monday afternoon and formed new working groups for research collaboration.

Many new problems related to combinatorics and geometry of beyond-planar graphs have been proposed. Specific open problems include questions regarding the combinatorial structures and topology (e.g., k^+-real face graphs, beyond upward planar graphs, sparse universal geometric graphs, local-crossing-critical graphs), the geometric representations (e.g., constrained outer string graphs, rerouting curves on the surface), and applications.

Two progress report sessions were organized on Tuesday and Thursday afternoons to report progress and plans for future publications and follow-up meetings among researchers. From the participants' feedback, the seminar has initiated new research collaboration and led to new research ideas and directions.

Taking this opportunity, we thank Schloss Dagstuhl for providing an environment for fruitful research collaboration.

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3 Overview of Talks

3.1 Crossing numbers of crossing-critical graphs

Géza Tóth (Alfréd Rényi Institute of Mathematics – Budapest, HU, geza@renyi.hu)

Joint work of Géza Tóth, János Barát

A graph G is k-crossing-critical if $\mathrm{cr}(G) \geq k$, but for any edge e of G, $\mathrm{cr}(G-e) < k$. In 1993 Richter and Thomassen conjectured that for any k-crossing-critical graph G, $\mathrm{cr}(G) \leq k + c\sqrt{k}$ and proved that $\mathrm{cr}(G) \leq 5k/2 + 16$. We improve it to $\mathrm{cr}(G) \leq 2k + 6\sqrt{k} + 47$.

3.2 The Density Formula for Beyond-Planar Graph Classes

Torsten Ueckerdt (Karlsruhe Institute of Technology, DE, torsten.ueckerdt@kit.edu)

Joint work of Michael Kaufmann, Boris Klemz, Kristin Knorr, Meghana M. Reddy, Felix Schröder, Torsten Ueckerdt
Main reference Michael Kaufmann, Boris Klemz, Kristin Knorr, Meghana M. Reddy, Felix Schröder, Torsten Ueckerdt: "The Density Formula: One Lemma to Bound Them All", CoRR, Vol. abs/2311.06193, 2023.
URL https://doi.org/10.48550/ARXIV.2311.06193

We introduce the Density Formula for drawings of graphs on the sphere, which can be used to derive tight upper bounds for the density (maximum number of edges for given number of vertices) of several beyond-planar graph classes, such as 1- and 2-planar graphs, fan-planar graphs, k-bend RAC graphs, and quasiplanar graphs. While in some cases we even obtain the first tight upper bounds, the real strength of the Density Formula is its simplicity and versatility. In this talk, I showcase the Density Formula with a few examples and mention a few open problems that seem worth investigating next.

3.3 Connected Dominating Sets in Triangulations

Pat Morin (Carleton University – Ottawa, CA, morin@scs.carleton.ca)

Joint work of Prosenjit Bose, Vida Dujmovic, Hussein Houdrouge, Pat Morin, Saeed Odak
Main reference Prosenjit Bose, Vida Dujmovic, Hussein Houdrouge, Pat Morin, Saeed Odak: "Connected Dominating Sets in Triangulations", CoRR, Vol. abs/2312.03399, 2023.
URL https://doi.org/10.48550/ARXIV.2312.03399

We show that every n-vertex triangulation has a connected dominating set of size at most $10n/21$. Equivalently, every n vertex triangulation has a spanning tree with at least $11n/21$ leaves. Prior to the current work, the best known bounds were $n/2$, which follows from work of Albertson, Berman, Hutchinson, and Thomassen (J. Graph Theory **14**(2):247–258). One immediate consequence of this result is an improved bound for the SEFENOMAP graph drawing problem of Angelini, Evans, Frati, and Gudmundsson (J. Graph Theory **82**(1):45–64). As a second application, we show that for every set P of $\lceil 11n/21 \rceil$ points in $\mathbb{R}^2$ every n-vertex planar graph has a one-bend non-crossing drawing in which some set of $11n/21$

vertices is drawn on the points of P. The main result extends to n-vertex triangulations of genus-g surfaces, and implies that these have connected dominating sets of size at most $10n/21 + O(\sqrt{gn})$.

3.4 Beyond-planar Euclidean spanners

Csaba D. Tóth (California State University – Northridge, US, csaba.toth@csun.edu)

Main reference Csaba D. Tóth: "Minimum weight Euclidean $(1+\varepsilon)$-spanners", European Journal of Combinatorics, Vol. 118, p. 103927, 2024.
URL https://doi.org/10.1016/j.ejc.2024.103927

For a set P of n points in the plane and a parameter $t \geq 1$, a t-spanner is a geometric graph G such that for all pairs $u, v \in P$, the shortest path distance in G (with Euclidean edge weights) approximates the Euclidean distance between u and v up to a factor of at most t; the parameter t is the *stretch* of H. For example, the Delaunay triangulation is 1.998-spanner, but in general plane graphs on P cannot achieve a stretch less than $\pi/2$. If edge crossings are allowed, the stretch can be arbitrarily close to 1: For every $\varepsilon > 0$ there are $(1+\varepsilon)$-spanners with $O(\varepsilon^{-1}n)$ edges and $\tilde{O}(\varepsilon^{-2}) \cdot MST(P)$ weight. These bounds are the best possible, and such spanners also have separators of size $\varepsilon^{-O(1)}\sqrt{n})$. However, it remains an open problem to quantify, in terms of $\varepsilon > 0$, how much $(1+\varepsilon)$-spanners are beyond planar graphs.

4 Working Groups

4.1 Constrained Outerstring Graphs

Therese Biedl (University of Waterloo, CA, biedl@uwaterloo.ca)
Sabine Cornelsen (University of Konstanz, DE, sabine.cornelsen@uni-konstanz.de)
Jan Kratochvíl (Charles University Prague, CZ, honza@kam.mff.cuni.cz)
Ignaz Rutter (Universität Passau, DE, ignaz.rutter@uni-passau.de)

In an *outer-string representation* [6] (implicitly defined and first results obtained in [9]) of a graph each vertex is drawn as a simple curve $\partial(v)$ within a simple closed region D such that one end point of $\partial(v)$, called the anchor of v, is on the simple closed curve bounding D. The curves $\partial(v)$ and $\partial(w)$ of two vertices v and w intersect if and only if v and w are adjacent. It is NP-hard to decide whether a graph has an outer-string representation [8]. Unfortunately, outer-string representations sometimes need exponentially many crossings [1]. So it is interesting to investigate which graphs allow an outer-string representation with a restricted number of crossings. In an *outer-1-string representation*, it is additionally required that the curves $\partial(v)$ and $\partial(w)$ of two vertices v and w intersect at most once. This is similar to the intersection graph of pseudosegments [4], however, with the additional constraint that the anchors still have to be on the boundary of a simple closed region containing all pseudosegments. Representing chordal graphs as intersections of pseudosegments was considered in [3].

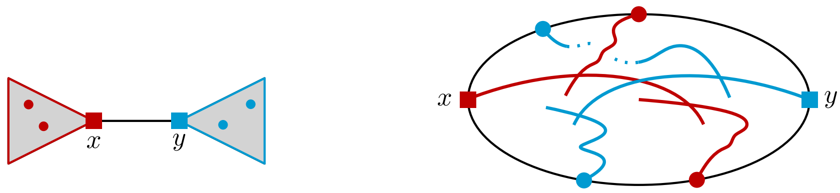


Figure 1 Triply interleaved bridge.

In [2], the order of crossings along a string was constrained. We focus on the constrained version where the cyclic order of the anchors is fixed, i.e., an instance of constrained outer-(1)-string representation consists of a graph and a cyclic order of the vertices. In addition to general outer-string and outer-1-string representations, we also consider *L-shaped* [7, 5] and *U-shaped representations* in which the anchors are on a horizontal line and the vertices are 1- or 2-bend orthogonal polylines below that line. I.e., in particular, we also allow ⅃s. See Figures 3b and 3c. In the constrained version, the *linear* order of the anchors is fixed.

▶ **Theorem 1** ([9]). *The complement of a simple cycle with at least four vertices does not have a constrained outer-string representation, i.e., if the cyclic order of the vertices is* $v_1, \ldots, v_n$ *then the graph with edge set* $E = \{\{v_i, v_j\}; |i - j| \notin \{1, n-1\}\}$ *does not have a constrained outer-string representation.*

4.1.1 Summary of Results

We say that two sets V_1 and V_2 of vertices are *interleaved* if in the (cyclic) order no two vertices of V_1 nor two vertices of V_2 are consecutive. Observe that the complement of a 4-cycle consists of two *interleaved independent edges.*

▶ **Theorem 2.** *A chordal graph with a fixed cyclic order of the vertices admits a constrained outer-string representation if and only if it contains no two interleaved independent edges.*

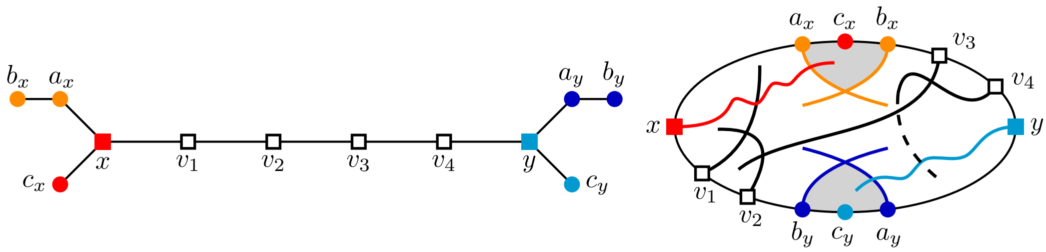
The following instances do not admit a constrained outer-1-string representations: (a) a *triply interleaved bridge*, i.e., a bridge e of the graph G, such that the two connected components of $G - e$ containing the end vertices of e each contain a set X and Y of three vertices such that X and Y are interleaved. See Figure 1. (b) An *X-obstruction*; see Figure 2.

▶ **Theorem 3.** *A tree with a fixed cyclic order of the vertices admits a constrained outer-1-string representation if and only if it contains no two interleaved independent edges, no triply interleaved bridge, nor an X-obstruction. Moreover, for trees there is a certifying polynomial-time recognition algorithm, which either outputs a constrained outer-1-string representation or an obstruction.*

An *extended complement of a 5-cycle* is either the complement of a 5-cycle or a subpath $w_1v'vuu'w_2$ of a cycle whose anchors are in the order $w_1uv'u'vw_2$. See Figure 3a.

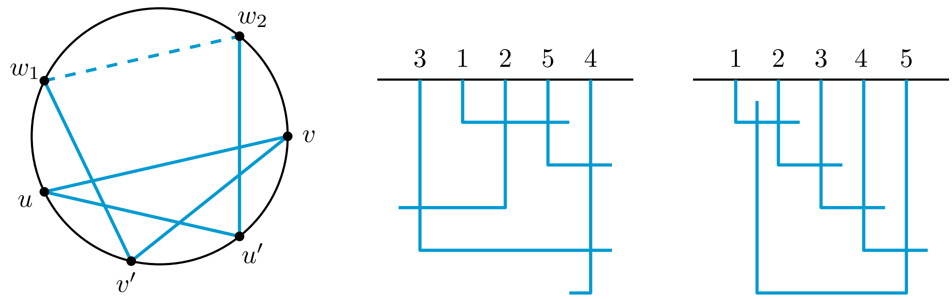
▶ **Theorem 4.** *Let* $G = (V, E)$ *be a simple cycle and let* $\prec$ *be a cyclic order of* V*. Then the following are equivalent.*

1. $(G, \prec)$ *has a constrained outer-1-string representation*
2. *For every edge* $\{u, v\}$ *of* G *one of the following sequences* uv, $uu'v'v$, $uu'v$, *or* $uv'v$, *or their reverse is a subsequence of* $\prec$, *where* u' *and* v' *are the neighbors of* u *and* v *other then* v *and* u, *respectively.*



(a) graph **(b)** forbidden cyclic ordering

Figure 2 An *X-obstruction* contains the vertices i, a_i, b_i, c_i, d_i and the edges ia_i, a_ib_i, ic_i for $i = x, y$ as well as an x-y path $x = v_0, v_1, \ldots, v_{\ell-1}, v_\ell = y$ of arbitrary length, including zero. For any $k = -1, \ldots, \ell$, the set $\{v_0, \ldots, v_k, a_x, b_x, c_x\}$ of vertices appears consecutive (not necessarily in this order) in the cyclic order $\prec$ and for $i = x, y$ the pairs $\{a_i, b_i\}$ and $\{i, c_i\}$ are interleaved.



(a) extended compl. of 5-cycle **(b)** L-shaped **(c)** U-shaped

Figure 3 An obstruction and two representations of a 5-cycle.

3. $(G, \prec)$ *does not contain two interleaving independent edges nor an extended complement of a 5-cycle.*

Observe that a path has a constrained L-shaped outer-1-string representation if there are no two independent edges that are interleaved. Every simple cycle with a fixed linear order of the vertices that admits a constrained outer-1-string representation also admits a constrained U-shaped outer-1-string representation.

▶ **Theorem 5.** *It can be tested in polynomial time whether a graph with a given ordering of the vertices admits a constrained L-shaped outer-1-string representation.*

4.1.2 Open Problems

What is the complexity of testing whether a graph has an outer-1-string, a constrained outer-1-string, or a constarined outer-string representation? What if the instances are restricted to graphs with bounded treewidth?

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4.2 Universal Geometric Graphs

Vida Dujmović (University of Ottawa, CA, vida.dujmovic@uottawa.ca)
Fabrizio Frati (Roma Tre University, IT, fabrizio.frati@uniroma3.it)
Michael Hoffmann (ETH Zürich, CH, hoffmann@inf.ethz.ch)
Pat Morin (Carleton University, CA, morin@scs.carleton.ca)

4.2.1 Problem statement

A *drawing* of a graph is a mapping of each vertex to a point in the plane and of each edge to a Jordan arc between its endvertices. A drawing is *straight-line* if each edge is represented by a straight-line segment and it is *planar* if no two edges intersect, except at common endvertices. A *planar graph* is a graph that admits a planar drawing. An *embedding* of a graph is a planar straight-line drawing of it. Every planar graph admits an embedding [15, 23].

A *geometric graph* is a graph whose vertices are points and whose edges are straight-line segments. A geometric graph is *planar* if it defines an embedding of the underlying (abstract) graph. A geometric graph is *universal* for a family $\mathcal{F}$ of planar graphs if it contains an embedding of every graph in $\mathcal{F}$. That is, for every graph $G \in \mathcal{F}$, there exists a subgraph of the universal graph which is isomorphic to G and is planar.

The question we study is the following.

▶ Problem 1. Let $f(n)$ be the minimum number of edges of any geometric graph that is universal for the family of the n-vertex planar graphs. What is the asymptotic growth of $f(n)$?

4.2.2 Related results

Universality has long been studied from a graph-theoretic perspective, starting from a paper by Rado in the 1960s [21]. An (abstract) graph is *universal* for a family $\mathcal{F}$ of graphs if it contains every graph in $\mathcal{F}$ as a subgraph. Clearly, the complete graph K_n with n vertices is universal for any family $\mathcal{H}$ of n-vertex graphs. Henceforth, research has been conducted on determining upper and lower bounds on the number of edges that universal graphs for notable families of n-vertex sparse graphs must have. Babai et al. [4] proved that a universal graph with $O(m^2 \log\log m/\log m)$ edges exists for the family of all graphs with m edges, whereas any such a universal graph has $\Omega(m^2/\log^2 m)$ edges. Alon et al. [1, 2] proved that there exists a universal graph with $O(n^{2-2/k})$ edges for the n-vertex graphs with maximum degree k; such a bound is tight in the worst case.

A special attention has been devoted to planar graphs and their subclasses. It has long been known [4] that there exists a graph with $O(n^{3/2})$ edges that is universal for the family of the n-vertex planar graphs. This bound was recently improved to $n \cdot 2^{O(\sqrt{\log n \cdot \log\log n})}$ by Esperet et al. [14]. For bounded-degree planar graphs, there exists an (optimal) $O(n)$ bound, due to Capalbo [9]. Böttcher et al. [7, 8] proved that every n-vertex graph with minimum degree $\Omega(n)$ is universal for the n-vertex planar graphs of bounded degree. Chung and Graham [11, 12] constructed a universal graph with $O(n \log n)$ edges for the n-vertex trees. This bound is the best possible, apart from constant factors.

Universal geometric graphs were first defined and studied by Frati, Hoffmann, and Tóth [17]. They strengthened Chung and Graham result [11, 12] by proving that there exists an n-vertex geometric graph with $O(n \log n)$ edges that is universal for the n-vertex trees. They also proved that every n-vertex convex geometric graph that is universal for the n-vertex outerplanar graphs has $\Omega_h(n^{2-1/h})$ edges, for every positive integer h, which almost matches the trivial $O(n^2)$ upper bound given by a convex complete geometric graph.

The study of universal geometric graphs has a strong relationship with the study of universal point sets. A set $\mathcal{P}$ of points is *universal* for a family $\mathcal{F}$ of planar graphs if every graph in $\mathcal{F}$ has an embedding in which the vertex set is mapped to a subset of $\mathcal{P}$. The question is then, for a family $\mathcal{F}$ of n-vertex planar graphs, what is the asymptotic growth of the function representing the minimum number of points of a universal point set for the graphs in $\mathcal{F}$. Answering such a question for the family of all n-vertex planar graphs is perhaps the most famous graph drawing open problem. It is has been known for a long time that there exists a universal point set for the n-vertex planar graphs with $O(n^2)$ points [13], see also [5], while the currently best known lower bound is only linear, namely $(1.293 - o(1))n$ [22]; see also [10, 20]. Universal point sets with sub-quadratic size are known for the 2-outerplanar graphs and the simply nested graphs [3], and for the n-vertex stacked triangulations [18]. Linear-size universal point sets are known for the n-vertex outerplanar graphs [6, 19], as well as for the cubic planar graphs and the bipartite planar graphs [16].

Consider a point set $\mathcal{P}$ which is universal for a family $\mathcal{F}$ of planar graphs. Then the complete geometric graph with vertex set $\mathcal{P}$ is universal for $\mathcal{F}$. This connection, together with the existence of a quadratic-size universal point set for the n-vertex planar graphs, gives us an $O(n^4)$ upper bound on the number of edges of a universal geometric graph for the n-vertex planar graphs, which is the best known upper bound we are aware of for Problem 1. On the other hand, the best known lower bound is only $\Omega(n \log n)$, which comes from the described graph-theoretic setting [11, 12].

4.2.3 Our research

Our research aimed at finding an upper bound better than $O(n^4)$ for Problem 1. We now explain the strategy we pursued in order to achieve such a goal.

As already mentioned, de Fraysseix, Pach, and Pollack proved the existence of a universal point set $\mathcal{P}$ with $O(n^2)$ points (in fact, a $2n \times n$ section of the integer lattice) for the n-vertex planar graphs [13]. The embedding of any n-vertex planar graph G on $\mathcal{P}$ can be constructed incrementally as follows. First, one can assume without loss of generality that G is a *maximal plane graph*. Indeed, maximality can be guaranteed by an initial edge-augmentation. Furthermore, a maximal planar graph has a unique combinatorial embedding (this is the circular order of the incident edges in an embedding); this, together with a choice of the outer face, enhances G to a maximal plane graph. Second, every maximal plane graph G with $n \geq 3$ vertices and with outer face (u, v, z) admits a *canonical ordering*. This is a labeling of the vertices $v_1 = u, v_2 = v, v_3, \ldots, v_{n-1}, v_n = z$ meeting the following requirements for every $k = 3, \ldots, n$:

- The plane subgraph $G_k \subseteq G$ induced by $v_1, v_2, \ldots, v_k$ is 2-connected; let C_k be the cycle bounding its outer face;
- v_k is in the outer face of G_{k-1}, and its neighbors in G_{k-1} form an (at least 2-element) subinterval of the path $C_{k-1} - (u, v)$.

A *canonical drawing* of G can be constructed from a canonical ordering of G in $n - 2$ steps. At step 1, a planar straight-line drawing Γ_3 of G_3 is constructed with v_1 at $(0, 0)$, with v_2 at $(2, 0)$, and with v_3 at $(1, 1)$. Auxiliary sets $M_3(v_1) := \{v_1, v_2, v_3\}$, $M_3(v_3) := \{v_2, v_3\}$, and $M_3(v_2) := \{v_2\}$ are also defined. For $k = 4, \ldots, n$, at step $k - 2$, a planar straight-line drawing Γ_k of G_k is constructed from Γ_{k-1}, as follows. Let $w_1 = u, w_2, \ldots, w_r = v$ be the clockwise order of the vertices along the outer face of G_{k-1}, where $w_p, w_{p+1}, \ldots, w_q$ are the neighbors of v_k in G_{k-1}, for some $1 \leq p < q \leq r$. Then Γ_k is constructed from Γ_{k-1} by "shifting" the vertices in $M_{k-1}(w_{p+1})$ by one unit to the right, by shifting the vertices in $M_{k-1}(w_q)$ by one additional unit to the right, and by placing v_k at the intersection point of the line through w_p with slope $+1$ and of the line through w_q with slope -1. Step $k - 2$ is completed by defining the sets:

- $M_k(w_i) = M_{k-1}(w_i) \cup \{v_k\}$, for $i = 1, \ldots, p$;
- $M_k(v_k) = M_{k-1}(w_{p+1}) \cup \{v_k\}$; and
- $M_k(w_i) = M_{k-1}(w_i)$, for $i = q, \ldots, r$.

Note that the above described construction maintains the x-monotonicity of the boundary of the drawing at every step. The shifting of the vertices in the sets $M_{k-1}(w_{p+1})$ and $M_{k-1}(w_q)$ makes room for drawing the edges incident to the newly inserted vertex v_k in a planar way.

The starting observation of our approach is that a canonical ordering of G can be used in a much simpler way to obtain a planar straight-line drawing of G, entirely avoiding the shifting phase and the definition of the sets $M.(\cdot)$. Indeed, because of the x-monotonicity of the boundary of the drawing, one can simply place v_k at a "sufficiently high" point in the interior of the x-interval spanned by its neighbors $w_p, w_{p+1}, \ldots, w_q$. This ensures planarity and maintains the x-monotonicity of the boundary of the drawing. We call *generalized canonical drawing* a drawing constructed in this way.

Now, consider an $n \times n$ *stretched grid*. This is a point set obtained from an $n \times n$ section of the integer lattice by translating grid rows upwards, in such a way that each point is above the line through any two points in lower rows that are not vertically aligned. Stretched grids were used in [18]. It can be proved that every n-vertex maximal plane graph G has a generalized canonical drawing in which the vertex set is mapped to a subset of any $n \times n$ stretched grid $\mathcal{S}$; thus, $\mathcal{S}$ is a universal point set for the n-vertex planar graphs. This can be proved as follows. First, compute a canonical ordering $v_1, v_2, \ldots, v_n$ of G. Second, define a partial order Y of the vertices of G iteratively, so that each vertex v_k follows all its neighbors $w_p, w_{p+1}, \ldots, w_q$ in G_{k-1}. Third, define a partial order X of the vertices of G iteratively, so that each vertex v_k follows its first neighbor w_p and precedes its last neighbor w_q in G_{k-1}. It is easy to see that any assignment of the vertices of G to the points of $\mathcal{S}$ such that:

- if a vertex u precedes a vertex v in Y, then u is assigned to a lower row than v; and
- if a vertex u precedes a vertex v in X, then u is assigned to a column to the left of the one of v

results in a generalized canonical drawing of G whose vertex set lies at $\mathcal{S}$.

Our intuition is that a universal geometric graph $\mathcal{G}$ with $o(n^4)$ edges that is universal for the n-vertex planar graphs can be constructed so that its vertex set is an $n \times n$ stretched grid $\mathcal{S}$, possibly slightly perturbed so that each row defines a convex point set. Our approach for defining $\mathcal{G}$ consists of connecting the points on each column of $\mathcal{S}$ to all the points on a number of adjacent columns which depends on the index of the column. More specifically, consider the sequence π_i which is inductively defined as follows: (i) $\pi_0 := 1$; (ii) $\pi_i := \pi_{i-1} \circ 2^i \circ \pi_{i-1}$. For example, $\pi_3 = 1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1$. Let i be sufficiently large so that π_i has at least n elements. For $j = 1, \dots, n$, assign the j-th element of π_i to the j-th column of $\mathcal{S}$. Then the points on the j-th column of $\mathcal{S}$ are connected to all the points on a number of adjacent columns which is equal to the element of π_i assigned to the column times some integer constant $c > 0$. Since the sum of the elements assigned to the columns of $\mathcal{S}$ is in $O(n \log n)$, the number of edges of the resulting geometric graph $\mathcal{G}$ is in $O(n^3 \log n)$. Whether $\mathcal{G}$ is actually a universal geometric graph for the n-vertex planar graphs however remains to be proved.

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4.3 Recognizing k^+-real Face Graphs

Michael A. Bekos (University of Ioannina, GR, bekos@uoi.gr)
Giuseppe Di Battista (Università Roma Tre, IT, giuseppe.dibattista@uniroma3.it)
Emilio Di Giacomo (University of Perugia, IT, emilio.digiacomo@unipg.it)
Walter Didimo (University of Perugia, IT, walter.didimo@unipg.it)
Michael Kaufmann (Universität Tübingen, DE, michael.kaufmann@uni-tuebingen.de)
Fabrizio Montecchiani (University of Perugia, IT, fabrizio.montecchiani@unipg.it)

Abstract. A nonplanar drawing Γ of a graph G divides the plane into topologically connected regions, called *faces* (or *cells*). The boundary of each face is formed by vertices/crossings and edges. Given a positive integer k, we say that Γ is a *k^+-real face drawing* of G if the boundary of each face of Γ contains at least k vertices of G. The study of k^+-real face drawings started in a paper by Binucci et al. (WG 2023), where edge density bounds and results about the relationship with other beyond-planar graph classes are given. In this seminar we have investigated the complexity of recognizing k^+-real face graphs, i.e., graphs that admit a k^+-real face drawing. We have studied both the general unconstrained scenario and the 2-layer scenario in which the graph is bipartite, the vertices of the two partite sets are placed on two distinct horizontal layers, and the edges are drawn as straight segments (or equivalently as vertical monotone curves).

4.3.1 Introduction

The study of k^+-real face drawings of (nonplanar) graphs started in a recent paper by Binucci et al. [1]. In a k^+-real face drawing, the boundary of each face contains at least k vertices of the graph, where $k \geq 1$ is a given integer. In particular, for any positive integer k, a k^+-real face drawing forbids faces formed only by crossing points and edges. From the practical side, the interest in k^+-real face graphs is motivated by the intuition that faces mostly consisting of crossing points make the graph layout less readable. From the theoretical side, k^+-real face drawings can be regarded as a generalization of planar drawings whose face sizes are above a desired threshold [2, 3, 4].

Basic Notations and Terminology. Let G be a graph. We assume that G is simple, that is, it contains neither multiple (i.e., parallel) edges nor self-loops. We also assume, without loss of generality, that G is connected, as otherwise we can just consider each connected component of G independently. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. A *drawing* Γ of G is a geometric representation of G that maps each vertex $v \in V(G)$ to a distinct point of the plane and each edge $(u, v) \in E(G)$ to a simple Jordan arc between the points corresponding to u and v. We always assume that Γ is a *simple* drawing, that is: (i) adjacent edges do not intersect, except at their common endpoint; (ii) two independent (i.e., non-adjacent) edges intersect in at most one of their interior points, called a *crossing point*; and (iii) no three edges intersect at a common crossing point.

A *vertex* of Γ is either a point corresponding to a vertex of G, called a *real-vertex*, or a point corresponding to a crossing point, called a *crossing-vertex*. Since the drawing is simple, a crossing-vertex has always degree four. We denote by $V(\Gamma)$ the set of vertices of Γ. An *edge* of Γ is a curve connecting two vertices of Γ; an edge of Γ whose endpoints are both real-vertices coincides with an edge of G; otherwise it is just a proper portion of an edge of G. We denote by $E(\Gamma)$ the set of edges of Γ. Drawing Γ subdivides the plane into topologically connected regions, called *faces* (or *cells*). The boundary of each face consists of a circular sequence of vertices and edges of Γ. The set of faces of Γ is denoted by $F(\Gamma)$. Exactly one face in $F(\Gamma)$ corresponds to an infinite region of the plane, called the *external face* (or *outer face*) of Γ; the other faces are the *internal faces* of Γ. When the boundary of a face f of Γ contains a vertex v (or an edge e), we also say that f contains v (or e).

Given an integer $k \geq 1$, a *k^+-real face drawing* of a graph G is such that each face contains at least k real-vertices. If G admits such a drawing, then we call G a *k^+-real face graph*. If G is bipartite, then a *2-layer k^+-real face drawing* of G is a k^+-real face drawing Γ of G such that the vertices of the two parts of its vertex partition are drawn on two distinct horizontal lines, called *layers*, and each edge is a straight-line segment. If G admits such a drawing, then we call G a *2-layer k^+-real face graph*.

4.3.2 Contribution

During the seminar we investigated the complexity of recognizing k^+-real face graphs, that is, the complexity of testing whether, given a graph G and a positive integer k, there exists a k^+-real face drawing of G. We studied both the general (unconstrained) scenario and the 2-layer drawing scenario. A summary of the main contributions is given below.

- In the general case, we are able to show that recognizing k^+-real face graphs for values of $k \in \{1, 2\}$ is NP-complete. For the hardness proof we exploit a reduction from the well-known 3-Partition problem. Note that, for $k \geq 3$, *optimal* k^+-real face graphs (i.e., k^+-real face graphs with the maximum possible edge density) are always planar graphs with all faces of degree k (see [1]). Hence, recognizing optimal k^+-real face graphs when $k \geq 3$ is equivalent to testing whether the graph admits a planar embedding where all faces have size at least k, a problem studied in [5].
- We proved tights upper bounds on the maximum number of edges in a 2-layer k^+-real face graph, for every value of k. These types of results can help in the design of recognition

algorithms. Specifically, we established that 1^+-real face and 2^+-real face graphs with n vertices have at most $2n-4$ and $1.5n-2$ edges, respectively. Also, for $k \geq 3$, optimal 2-layer k^+-real face graphs are caterpillar graphs, and therefore have $n-1$ edges.

- We believe that it is possible to efficiently recognize 2-layer 2^+-real face graphs. In particular, during the seminar we designed a testing algorithm that seems to work in linear time in the size of the graph. We plan to give a formal description and a proof of correctness of this algorithm in a near future article.
- For 2-layer 1^+-real face graphs, we characterized the structure of optimal graphs (i.e., 2-layer 1^+-real face graphs with exactly $2n-4$ edges) and of biconncted graphs. These characterizations should lead to efficient recognition algorithms. Recognizing 2-layer 1^+-real face graphs that are not biconnected seems to be more difficult; we are still working on establishing whether a polynomial-time algorithm exists in this case, even if the graph is a tree.

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4.4 Local-crossing-critical graphs and covering complete geometric graphs

János Pach (Rényi Institute – Budapest, HU & EPFL – Lausanne, CH, pach@renyi.hu)
Géza Tóth (Rényi Institute, Budapest, HU, geza@renyi.hu)
Pavel Valtr (Charles University, Prague, CZ, valtr@kam.mff.cuni.cz)

4.4.1 Local-crossing-critical graphs

The crossing number of a graph G, $\mathrm{cr}(G)$ is the minimum number of edge crossings of G over all its drawings on the plane. G is called k-crossing-critical if $\mathrm{cr}(G) \geq k$, but for any edge e of G, $\mathrm{cr}(G \setminus e) < k$. Richter and Thomassen [6] proved that the crossing number of a k-crossing-critical graph cannot be arbitrarily large, if G is a k-crossing-critical graph, then $\mathrm{cr}(G) \leq 5k/2 + 16$. It was improved by Barát and Tóth [2], [4] to $\mathrm{cr}(G) \leq 2k + 6\sqrt{k} + 47$. It is conjectured that for any such graph G we have $\mathrm{cr}(G) \leq k + c\sqrt{k}$.

We worked on the following related problem. The local crossing number of a graph G, $\mathrm{lcr}(G)$ is the minimum number l with the property that G can be drawn in the plane with at most l crossings on each edge. In other words, $\mathrm{lcr}(G)$ is the minimum number l such that G is l-planar. A graph G is k-local-crossing-critical if $\mathrm{lcr}(G) \geq k$, but for any edge e of G, $\mathrm{lcr}(G \setminus e) < k$.

Is there a function $f(k)$ with the property that for any k-local-crossing-critical graph G, $\mathrm{lcr}(G) \leq f(k)$?

1-local-crossing critical graphs are easy to describe, removing any edge we get a planar graph, but the graph itself is not planar. It follows from Kuratowski's theorem, that these graphs are the topological K_5 and $K_{3,3}$ graphs, therefore, $f(1) = 1$.

Observation. *If $f(2)$ exists, then $f(k)$ exists for all k, and $f(k) \leq (k-1)f(2)$.*

Proof. Suppose that we know that $f(2) = f$ exists. That is, if G is a graph with the property that for any edge e of G, the graph $G - e$ is 1-planar, then G is f-planar.

Let $k > 2$ and suppose that G is a k-local-crossing-critical graph. Replace each edge of G by a path of length $k - 1$ (that is, ($k - 1$ edges, $k - 2$ subdividing vertices), let H be the resulting graph. Remove an edge e from H. It follows from the assumption on G that that $H - e$ an be drawn such that each path that replaces an edge of G contains at most $k - 1$ crossings. But then the subdividing vertices can be arranged so that there is at most one crossing on each edge. Therefore, H is 2-local-crossing-critical. Consequently, H is f-planar. Consider an f-planar drawing of H. In the corresponding drawing of G, there are at most $f(k - 1)$ crossings on each edge. This finishes the proof.

We are left with the case $k = 2$: Is there an $f > 0$ so that the following satement holds? Suppose that G is a graph with the property that for any edge e of G, the graph $G - e$ is 1-planar. Is there a number f then G is f-planar.

We tried to use the ideas of Richter and Thomassen and other related papers on crossing-critical graphs, but there were some unexpected and very exciting difficulties.

4.4.2 Covering complete geometric graphs with plane trees and forests

Definition. A geometric graph is a graph drawn in the plane with possibly crossing straight-line edges. A plane star-forest is a geometric graph in which each component is a star (a tree with exactly one non-leaf vertex) and no two edges the graph cross. A complete convex geometric graph is a geometric graph whose vertex set is a set of points in the plane in strictly convex position, where every pair of vertices are connected by an edge.

Answering a question of Dujmović and Wood [3] Pach, Saghafian, and Schnider [5] proved that the edge set of a complete *convex* geometric graph on n vertices cannot be covered by fewer than $n - 1$ plane star-forests. This bound is tight. They made the following

Conjecture. No complete geometric graph can be covered with less than $3n/4$ plane star forests.

This was proved to be *false* [1]

Theorem. (Antić, Gliǐć, Milivojčević): *There are infinitely many even values of n, for which there exists a complete geometric graph with n vertices whose edges set can be covered by $n/2 + 1$ plane star-forests.*

We studied the analogous problem where instead of star-forests, we are allowed to use any plane trees. It appears to be true that there exists a constant $c > 0$ such that the edge set of every complete geometric graph can be covered by $(1 - c)n$ plane trees. We verified this conjecture in some special cases.

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4.5 Rerouting Curves on Surfaces

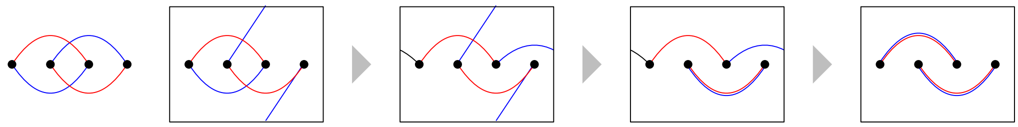
Stefan Felsner (Technische Universität Berlin, DE, felsner@math.tu-berlin.de)
Henry Förster (Universität Tübingen, DE, henry.foerster@uni-tuebingen.de)
Stephen Kobourov (University of Arizona, US, kobourov@cs.arizona.edu)
Anna Lubiw (University of Waterloo, CA, alubiw@uwaterloo.ca)
Yoshio Okamoto (The University of Electro-Communications, JP, okamotoy@uec.ac.jp)
János Pach (Rényi Institute – Budapest, HU & EPFL – Lausanne, CH, pach@renyi.hu)
Csaba D. Tóth (California State University Northridge, US, csaba.toth@csun.edu)
Géza Tóth (Rényi Institute of Mathematics, HU, geza@renyi.hu)
Torsten Ueckerdt (Karlsruhe Institute of Technology, DE, torsten.ueckerdt@kit.edu)
Pavel Valtr (Charles University Prague, CZ, valtr@@kam.mff.cuni.cz)

4.5.1 The Problem

We study the problem of reconfiguring graph embeddings on an orientable surface, where the vertices are fixed and each reconfiguration step redraws one edge curve. Consider a set of points $\mathcal{S}$ on an orientable surface Σ, and two embeddings $\mathcal{P}$ and $\mathcal{Q}$ of the same graph G on vertices $\mathcal{S}$. Here an *embedding* means that each edge is drawn as a curve, which we call an *edge curve*, on the surface and no two edge curves intersect except at a common endpoint. Note that the edge curves of $\mathcal{P}$ may cross the edge curves of $\mathcal{Q}$. We assume that the correspondence between edge curves of $\mathcal{P}$ and $\mathcal{Q}$ is given. A *reconfiguration step* or *move* replaces one edge curve γ of an embedded graph G by a new curve γ' to obtain a new embedding of G – in other words, γ' may not cross any of the other edge curves of the embedded graph, though we allow γ and γ' to intersect. The question we address is whether $\mathcal{P}$ can be reconfigured to $\mathcal{Q}$ via a sequence of moves.

The special case where the graph is a matching consisting of two disjoint edges was considered by Ito, Iwamasa, Kakimura, Kobayashi, Maezawa, Nozaki, Okamoto, and Ozeki [8]. In this restricted situation, they showed that reconfiguration is not always possible in the plane (see Fig. 4), but is always possible on a surface Σ of genus $g \geq 1$. (Note that their paper is primarily about reconfiguration in a more discrete setting where $\mathcal{P}$ and $\mathcal{Q}$ consist of disjoint paths in a fixed graph.)

Our main result is that if the graph G is a matching and the surface Σ is a torus, then reconfiguration is always possible. This immediately extends to any orientable surface of genus $g \geq 1$ and nonorientable surface of genus $g \geq 2$. The only open case remains the projective plane. We extend the result to the case where G is a tree. The result does not extend to general embeddings of a graph on the torus, as we show by an example. However, we conjecture that reconfiguration is possible if we restrict to plane embeddings, and we prove this for the special case of series-parallel graphs.



(a) **(b)**

Figure 4 (a) Two embeddings of a matching of two edges (red and blue) that cannot be reconfigured on the plane. (b) Reconfiguration of the two embeddings on the torus using 4 steps.

4.5.2 Related Work

The problem of morphing graph drawings on a torus [1, 7] is different in that the vertices are allowed to move but the edges must remain straight segments on the flat torus. The problem of tightening or untangling curves on a surface [2, 3, 4, 6] is also different in that they consider drawings with possible crossings (i.e., immersions rather than embeddings), and deform the edge curves continuously via so-called homotopy moves (local moves that modify the topology of the immersion). We also point the interested reader to Colin de Verdière's survey [5] on graphs on surfaces.

4.5.3 Rerouting of Matchings on the Torus

We have a set $\mathcal{P}$ of n non-crossing blue paths on the torus that form a matching of $2n$ points, and we have a set $\mathcal{Q}$ of n non-crossing red paths that form the same matching of the points. Our algorithm consists of the following three steps:

1. Draw the torus as a flat torus with all the red paths inside (i.e., none of them cross the torus boundary).
2. Re-draw the blue paths so that none of them cross the torus boundary.
3. Use the top/bottom boundary of the flat torus which now forms a clean handle (i.e., a closed non-separating curve not crossed by any red or blue path) to solve the problem.

We remark that for the third step, it would be enough to re-draw the blue paths so that none of them cross the top/bottom boundary. Curiously, however, our proof for the second step will establish the stronger property of clearing the entire boundary.

4.5.3.1 Draw the torus as a flat torus with all the red paths inside

In this step, we begin with an arbitrary projection of the red paths on the flat torus. We now seek a closed non-separating curve σ that avoids all red paths. Note that σ necessarily exists as the red paths form a non-crossing matching. We use σ as the new horizontal boundary of the flat torus. The argument can be repeated to obtain a new vertical boundary of the flat torus; see Fig. 5.

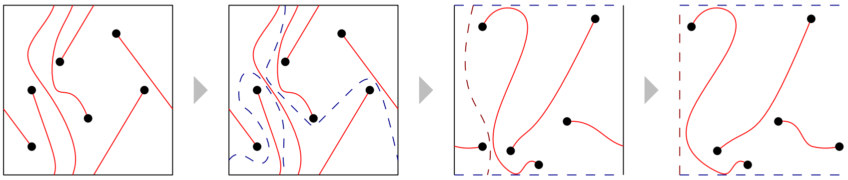


Figure 5 Illustration for Step 1.

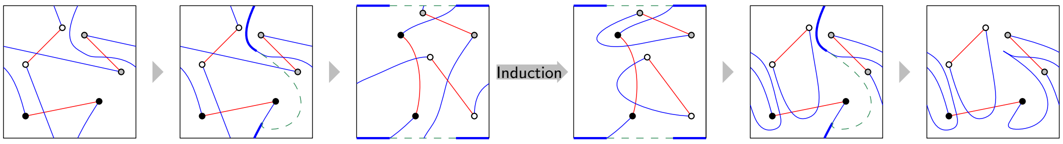


Figure 6 Illustration for Step 2.

4.5.3.2 Re-draw the blue paths so that none of them cross the torus boundary

In this step, we focus on the blue paths. The high-level idea (also see Fig. 6) is to pick one of the blue paths $p \in \mathcal{P}$ that crosses the flat torus boundary and reduce the number of times that it crosses the flat torus boundary. To this end, we find a *shortcut* γ such that

- γ lies in the torus boundary,
- the endpoints of γ lie in p, and there are no other intersections between γ and p,
- let p' be the piece of p that makes a cycle with γ,
- rerouting p' to γ reduces the number of times the path crosses the torus boundary,
- the cycle $p' \cup \gamma$ is a non-separating cycle on the torus.

The proof that γ exists requires a careful argument. Our goal is to reroute p' to γ but note that γ may cross other blue paths. Thus we must first clear all the crossings where other blue curves cross γ. This is done by induction on an appropriate (different) flat torus. After that we can reroute p along γ which reduces the number of times that p crosses the flat torus boundary. Observe that this comes at the expense of possibly increasing the number of times that other blue curves cross the torus boundary.

4.5.3.3 Use the clean handle to solve the problem

Once the clean handle is established, we can solve the problem similarly to the example shown in Fig. 4. To this end, we redraw one blue path $p \in \mathcal{P}$ at a time and resolve one of the crossings of its corresponding red path $q \in \mathcal{Q}$ which is closest to one of its endpoints in each step; see Fig. 7. We can use one boundary (say the vertical) to reroute p from its first crossing. Then, we reroute the crossing path c such that it avoids the crossing using the other boundary (say the horizontal). Now we can redraw p so to follow the trajectory of q until q's second crossing. Finally, we redraw c so that it does not cross the boundary of the flat torus, avoiding p.

4.5.3.4 Extension to forests

Finally, we remark that our result can be generalized to the case where $\mathcal{P}$ and $\mathcal{Q}$ are toroidal embeddings of a forest. While our strategy remains the same, this requires a slightly more careful analysis.

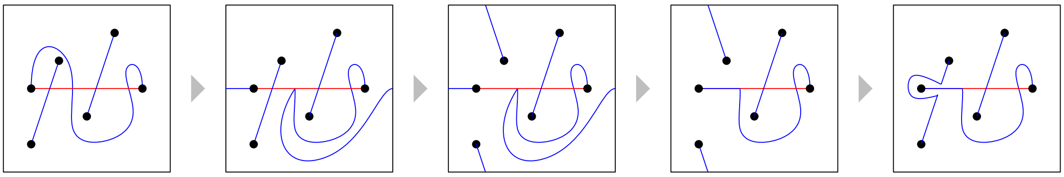


Figure 7 Illustration for Step 3.

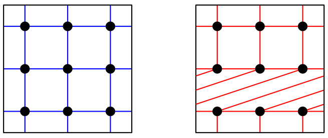


Figure 8 Two toric embeddings $\mathcal{P}$ (blue) and $\mathcal{Q}$ (red) that cannot be reconfigured into each other using a sequence of moves.

4.5.4 Non-Reroutable Toric Graph Embeddings

Following our previous positive result, one may wonder if it is always possible to reconfigure a given toric embedding $\mathcal{P}$ with a sequence of moves into another given toric embedding $\mathcal{Q}$. Unfortunately, this is not always possible as the example in Fig. 8 demonstrates.

For this example, it can be easily verified that a single curve of $\mathcal{P}$ can only be replaced by a topologically equivalent curve, i.e., it is impossible to change the embedding by replacing a single edge per move. Moreover, observe that both embeddings correspond to a quadrangulation of the torus where the embedding $\mathcal{Q}$ differs from $\mathcal{P}$ by a twist of the torus. This observation implies that we can generalize this result easily to a surface Σ of higher genus, i.e., one can use a suitably rigid tessellation of Σ for $\mathcal{P}$ and then perform a twist along a non-separating curve to obtain another embedding $\mathcal{Q}$ into which $\mathcal{P}$ cannot be reconfigured.

4.5.5 Rerouting of Plane Graphs on the Torus

While we showed that not all toric graphs can be reconfigured on the torus one may still wonder what happens if we restrict the embeddings $\mathcal{P}$ and $\mathcal{Q}$ of the graph to be plane. We remark that one may ask the same question for a surface Σ of higher genus g by requiring embeddings $\mathcal{P}$ and $\mathcal{Q}$ to be embeddable on a surface of genus $g-1$.

In particular, we can show that a plane embedding $\mathcal{P}$ can be reconfigured into another plane embedding $\mathcal{Q}$ on the torus if the input graph G is *series-parallel*. To this end, recall that the family of series-parallel graphs can be defined recursively as follows:

1. The graph consisting of a single edge st is a series-parallel graph with poles s and t.
2. Given two series-parallel graphs G_1 with poles s_1 and t_1 and G_2 with poles s_2 and t_2, the series composition obtained by identifying t_1 and s_2 is a series-parallel graph with poles s_1 and t_2.
3. Given two series-parallel graphs G_1 with poles s_1 and t_1 and G_2 with poles s_2 and t_2, the parallel composition obtained by identifying s_1 and s_2 as well as t_1 and t_2 is a series-parallel graph with poles $s_1 = s_2$ and $t_1 = t_2$.

Notably, all plane embeddings of series-parallel graphs differ only in the order in which parallel subgraphs are sorted at their common poles. We schematically show in Fig. 9 how two consecutive parallel components can be resorted at their common poles. Transforming $\mathcal{P}$ into $\mathcal{Q}$ then reduces to a sequence of such reorderings.

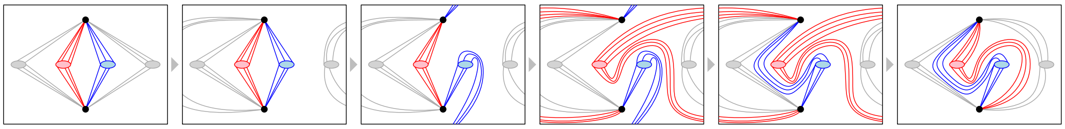


Figure 9 Reordering of two parallel components (red and blue) in a plane embedding on the torus.

4.5.6 Next Steps

As a follow-up to the above results found at the Dagstuhl Seminar, we intend to work on the following aspects:

1. Most importantly, we want to formalize our approaches further and provide reasonable bounds on their run times.
2. Our result on plane embeddings on series-parallel may be generalizable to plane embeddings of general planar graphs on the torus. The missing link is the analysis of triconnected planar graph whose embedding we have to be able to mirror.
3. Finally, we want to consider additional types of surfaces. With respect to our results on matchings, we want to attempt to achieve a similar result on the projective plane. Moreover, our results on plane embeddings motivates to study embeddings embeddable on surfaces of genus $g-1$ on a surface of genus g.

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4.6 Upward Drawings Beyond Planarity

Patrizio Angelini (John Cabot University, Rome, IT, pangelini@johncabot.edu)
Therese Biedl (University of Waterloo, Canada, biedl@uwaterloo.ca)
Markus Chimani (Universität Osnabrück, DE, markus.chimani@uni-osnabrueck.de)
Sabine Cornelsen (University of Konstanz, DE, sabine.cornelsen@uni-konstanz.de)
Giordano Da Lozzo (Roma Tre University, IT, giordano.dalozzo@uniroma3.it)
Seok-Hee Hong (University of Sydney, AU, seokhee.hong@sydney.edu.au)
Giuseppe Liotta (University of Perugia, IT, giuseppe.liotta@unipg.it)
Maurizio Patrignani (Roma Tre University, IT, maurizio.patrignani@uniroma3.it)
Sergey Pupyrev (Meta, Menlo Park, USA, spupyrev@gmail.com)
Ignaz Rutter (University of Passau, DE, rutter@fim.uni-passau.de)
Alexander Wolff (Universität Würzburg, DE, alexander.wolff@uni-wuerzburg.de)

4.6.1 Summary of Results

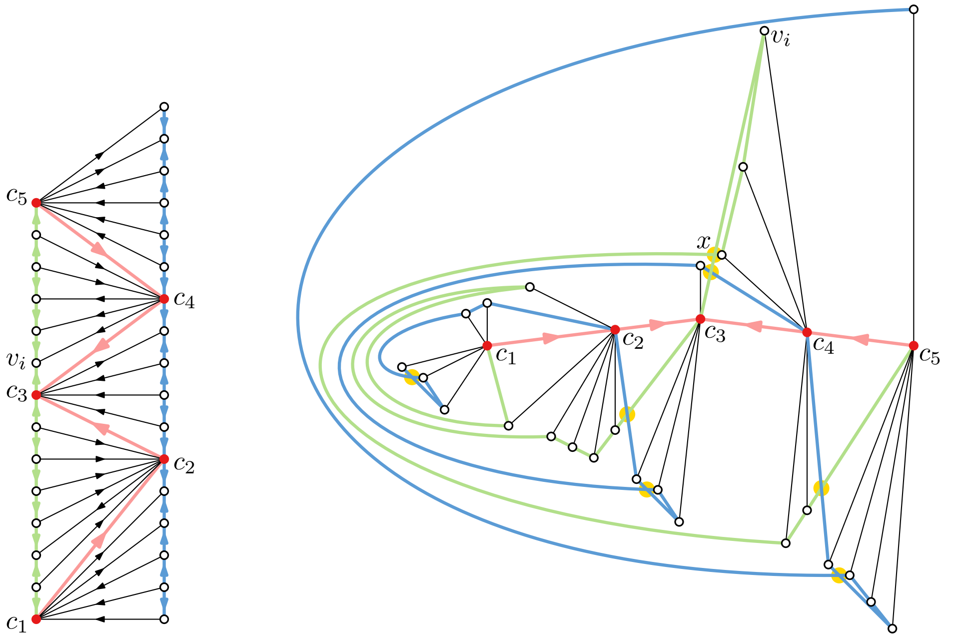
It is known that not every directed acyclic graph whose underlying undirected graph is planar admits an upward planar drawing. We are interested in pushing the notion of upward drawings beyond planarity. We investigate the "price of upwardness" for drawing planar directed acyclic graphs upwards – in terms of the maximum number of crossings per edge. More formally, we say that the drawing of a directed graph is *upward k-planar* if each edge is a y-monotone curve that is crossed at most k times by other edges. Our aim is to give good bounds on this parameter k for classes of planar directed acyclic graphs. For example, it is easy to see that every tree, no matter how its edges are directed, admits a planar upward drawing. On the other hand, Papakostas [2] showed that there is a directed acyclic 8-vertex outerpath that does not admit a planar upward drawing. (An outerpath is an outerplanar graph whose weak dual is a path.)

We have studied the problem both from a combinatorial and an algorithmic perspective. While this is still work in progress, we briefly summarize our results below. Let a *fan* be an outerpath in which there is a vertex, the *apex* of the fan, that is adjacent to all other vertices. We first show that every directed fan has an upward 2-planar drawing with specific properties (see Theorem 1). We then use this to show that every outerpath has an upward 2-planar drawing (see Theorem 2).

The edges incident to the apex are the *inner edges* of the fan. The other edges are the *outer edges* of the fan. Observe that the outer edges of a fan induce a path.

▶ **Lemma 1.** *Let c be the apex of a directed acyclic fan G, and let $P = \langle v_1, v_2, \ldots, v_{n-1}\rangle$ be the path of the remaining vertices in G. Let $P_1, P_2, \ldots, P_k$ be an ordered partition of P into maximal subpaths such that, for every $i \in \{1, 2, \ldots, k\}$, the edges between P_i and c are either all directed towards c or are all directed away from c. Then there is an upward 2-planar drawing of G with the following properties.*

1. *No inner edge is crossed.*
2. *Vertex v_1 has x-coordinate 1, the apex c and the vertex v_{n-1} have x-coordinate $n-1$, and the x-coordinates of $v_2, v_3, \ldots, v_{n-2}$ are distinct values in the set $\{2, 3, \ldots, n-2\}$.*
3. *For all edges all x-coordinates of the curves are at most $n-1$. All inner edges and all edges of the subpaths $P_1, \ldots, P_k$ are in the vertical strip between 1 and $n-1$.*
4. *The edge between P_1 and P_2 is crossed at most once if P_1 is a directed path.*



(a) a directed outerpath G' **(b)** upward drawing of G' with at most two crossings per edge

Figure 10 Example input and output of our algorithm for drawing outerpaths upward (edge crossings are highlighted in yellow).

We use Theorem 1 to prove the following.

▶ **Theorem 2.** *Every directed acyclic outerpath admits a upward 2-planar drawing.*

Proof. We assume that the given outerpath is maximal. If the outerpath has interior faces that are not triangles, we triangulate them using additional edges, which we direct such that they do not induce directed cycles. After drawing the resulting maximal outerpath, we remove the additional edges.

Let G' be such a graph; see Figure 10a. Let $c_1, c_2, \ldots, c_k$ be the vertices of degree at least 4 in G' (marked red in Figure 10). These vertices form a path (light red in Figure 10); let them be numbered along this path, which we call the *backbone* of G'. We draw the backbone in an x-monotone fashion, with very small slopes, going up and down as needed; see Figure 10b. For $i \in \{1, 2, \ldots, k-1\}$, we set $x(c_{i+1})$ to $x(c_i)$ plus the number of inner edges incident to c_{i+1}. For $i \in \{1, 2, \ldots, k\}$, we place the vertices incident to backbone vertex c_i using the algorithm for drawing a fan as detailed in the proof of Theorem 1. The vertices above (below) c_i are placed above (below) the backbone. If $i < k$, then the last vertex in the fan of c_i is connected to c_{i+1} and c_i is connected to the first vertex v_i in the fan of c_{i+1}. These two edges may cross each other. If the edge $c_i v_i$ goes, say, up but the following outer edges go down until a vertex v_k below c_{i+1} is reached, then the edge e_i between c_i and v_i may be crossed a second time by the edge e between v_{k-1} and v_k – as the crossing labeled x on the edge $c_3 v_3$ in Figure 10b – but, due to our invariant for drawing fans, e had been crossed only once within its fan. Also, the edge e_i cannot have a third crossing. Thus, in total no edge is crossed three times. ◀

Theorem 2 naturally raises the question about whether we can extend the proof to any graph having pathwidth 2. This is not the case, as we can prove the following.

▶ **Lemma 3.** *For every $k \geq 1$, there exists a directed acyclic graph with pathwidth 2 and $O(k)$ vertices that does not admit an upward k-planar drawing.*

Another research direction motivated by Theorem 2 is whether the result about outerpaths can be extended to any outerplanar graph. Also this question has a negative answer. Namely, we can prove the following.

▶ **Lemma 4.** *For every $k \geq 1$, there exists an outerplanar directed acyclic graph that does not admit an upward k-planar drawing.*

We have also studied the complexity of testing upward k-planarity of directed acyclic graphs. An st-graph is a directed acyclic graph with only one source and only one sink. Every *planar* st-graph with the source and the sink on the same face is upward planar, that is, it admits an upward drawing where no edge is crossed [1]. Leaving the domain of planar st-graphs, we can prove the following.

▶ **Theorem 5.** *Testing upward 1-planarity is NP-complete even for st-graphs both with and without a fixed rotation system.*

On the positive side, we are working on proving the following recognition result concerning outer upward 1-planar graphs, that is, graphs that admit an upward 1-planar drawing where all vertices lie on the outer face.

▶ **Theorem 6.** *Outer upward 1-planarity can be tested in polynomial time for single-source graphs.*

4.6.2 Open Problems

The research activity in Dagstuhl has also identified a list of related problems that can be the subject of future studies. Among them are the following questions.

1. Is there a directed outerpath that does not admit an upward 1-planar drawing?
2. Consider the class $\mathcal{O}_\Delta$ of outerplanar graphs (or even 2-trees) of maximum degree Δ. Is there a function f such that every graph in $\mathcal{O}_\Delta$ admits an upward $f(\Delta)$-planar drawing?
3. For which families of biconnected directed acyclic graphs is testing upward 1-planarity tractable?

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Participants

- Patrizio Angelini
John Cabot University – Rome, IT
- Michael A. Bekos
University of Ioannina, GR
- Therese Biedl
University of Waterloo, CA
- Markus Chimani
Universität Osnabrück, DE
- Sabine Cornelsen
Universität Konstanz, DE
- Giordano Da Lozzo
University of Rome III, IT
- Giuseppe Di Battista
University of Rome III, IT
- Emilio Di Giacomo
University of Perugia, IT
- Walter Didimo
University of Perugia, IT
- Vida Dujmovic
University of Ottawa, CA
- Stefan Felsner
TU Berlin, DE
- Henry Förster
Universität Tübingen, DE
- Fabrizio Frati
University of Rome III, IT
- Michael Hoffmann
ETH Zürich, CH
- Seok-Hee Hong
The University of Sydney, AU
- Michael Kaufmann
Universität Tübingen, DE
- Stephen G. Kobourov
University of Arizona – Tucson, US
- Jan Kratochvil
Charles University – Prague, CZ
- Giuseppe Liotta
University of Perugia, IT
- Anna Lubiw
University of Waterloo, CA
- Fabrizio Montecchiani
University of Perugia, IT
- Pat Morin
Carleton University – Ottawa, CA
- Yoshio Okamoto
The University of Electro-Communications – Tokyo, JP
- János Pach
Alfréd Rényi Institute – Budapest, HU & EPFL – Lausanne, CH
- Maurizio Patrignani
University of Rome III, IT
- Sergey Pupyrev
Facebook – Menlo Park, US
- Ignaz Rutter
Universität Passau, DE
- Csaba Tóth
California State University – Northridge, US
- Géza Tóth
Alfréd Rényi Institute of Mathematics – Budapest, HU
- Torsten Ueckerdt
KIT – Karlsruher Institut für Technologie, DE
- Pavel Valtr
Charles University – Prague, CZ
- Alexander Wolff
Universität Würzburg, DE