

Proof Representations: From Theory to Applications

Anupam Das^{*1}, Elaine Pimentel^{*2}, Lutz Straßburger^{*3}, and Robin Martinot^{†4}

1 University of Birmingham, GB. a.das@bham.ac.uk

2 University College London, GB. e.pimentel@ucl.ac.uk

3 INRIA Saclay – Île-de-France, FR. lutz@lix.polytechnique.fr

4 Utrecht University, NL. r.a.martinot@uu.nl

Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 24341 “Proof Representations: From Theory to Applications”. Proof theory is the study of formal proofs as mathematical objects in their own right. The subject has enjoyed continued attention among computer scientists in particular due to its significance for formalization, metalogic, and automation. In recent decades there has been a surge of interest on the representations of formal proofs themselves. The outcomes of these investigations have been remarkable, in particular extending the scope of structural proof theory to novel and richer settings:

Richer line structures (e.g. hypersequents, nested sequents, labelled sequents) have resulted in a uniform treatment of standard modal logics, streamlining their metatheory and providing new tools for metalogical problems. Richer proof structures (e.g., cyclic proofs, annotated systems, infinitely branching proofs) have significantly advanced our understanding of fixed points and (co)induction. Indeed, we are now seeing many of these previously disjoint techniques being combined to push the boundaries of proof theoretic approaches to computational logic. Graphical proof representations (e.g., proof nets, atomic flows, combinatorial proofs) originating from “linear” logics, now not only comprise a well-behaved computational model for resource-sensitive reasoning, but also provide an impressively uniform treatment for logics across the board. In fact, we are now seeing many of these previously disjoint techniques being combined to push the boundaries of proof theoretic approaches to computational logic, which has produced deep and fruitful cross-fertilizations between programming languages and proof theory. Arguably, the most well-known is the Curry-Howard correspondence (“propositions-as-types”) where (functional) programs correspond to formal proofs and their execution to normalization. A complementary tradition, proof-search-as-computation (“propositions-as-processes”), instead interprets (logic) programs to formulas and their execution to proof search.

The goal of this Dagstuhl Seminar was twofold. First and foremost, we aimed to bring together theorists and practitioners exploiting proof representations to identify new directions of application and, simultaneously, distill new theoretical directions from problems “in the wild”. At the same time, this seminar was intended to expose the interface between the proof-normalization and proof-search traditions by probing proof representations from both directions.

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* Editor / Organizer

† Editorial Assistant / Collector



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1 Executive Summary

Anupam Das (University of Birmingham, GB, a.das@bham.ac.uk)

Elaine Pimentel (University College London, GB, e.pimentel@ucl.ac.uk)

Lutz Straßburger (INRIA Saclay – Île-de-France, FR, lutz@lix.polytechnique.fr)

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Dagstuhl Seminar 24341 was organized in response to growing interest in the representation of formal proofs. Its primary aim was to bring together theorists and practitioners who are exploring various proof representations, with the goal of identifying new applications while simultaneously creating new theoretical directions. A key focus of the seminar was to explore the interface between proof normalization and proof search traditions, by examining proof representations from both perspectives.

The seminar focused on the relation between various new developments in the field, including: the more philosophical direction of proof-theoretic semantics; the upcoming unifying research program of universal proof theory; and the study of non-standard proof formats, such as non-well-founded proofs. Individual talks covered a broad array of topics relevant to these developments, such as the particulars of certain variants on standard proof calculi (calculi including witness operators, calculi for inconsistent logics and logics with numerical quantification, calculi for second-order logic, and “exotic” proof calculi, e.g. labelled calculi for modal and intuitionistic logics), but also various connections of proof theory to computational applications (verification logic, propositional dynamic logic, tableaux using SAT solvers, and interactive theorem provers). To do justice to these complex topics, researchers from proof theory, computational logic, and philosophical logic were invited to collaborate and provide insights. The seminar also identified several research gaps across these areas. One important takeaway was the importance of utilizing different representations of proof systems to address emerging open questions in the field, and to prioritize the need for unification.

The seminar itself was structured to encourage extensive interaction among participants, both formally and informally. Participants were given considerable freedom in preparing their contributions and during the seminar itself, including the option to give talks in a variety of formats. Several in-depth special sessions were organized to focus on the most important developments in the field, and longer individual talks by experts covered specific topics in greater detail. Evening sessions, dubbed “beer talks”, provided a relaxed environment for casual discussions, allowing participants to explore topics of shared interest and build connections across different areas of research.

The discussions from the seminar reflect the key interests of the community and provide valuable topics for ongoing research. At the end of the seminar, participants agreed to stay in contact to continue their discussions and foster new collaborations. Already various initiatives are being taken. For instance, Fernando Raymundo Velazquez Quesada, Carlos Olarte, and Elaine Pimentel began a promising collaboration on Epistemic Propositional Dynamic Logic following discussions at Dagstuhl. They recently held an in-person meeting in Bergen, Norway, and plan to apply for a European grant in the near future. Additionally, collaborations have among others been started by Lutz Straßburger and Revantha Ramanayake on modal logic, and by Lutz Straßburger and Matteo Acclavio on linear logic. We look forward to seeing how the results of these efforts will further shape the field.

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3 Overview of Talks

3.1 Even more exotic proof systems

Matteo Acclavio (*University of Southern Denmark – Odense, DK*)

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In this talk, I presented several proof systems that have been underrepresented throughout the week. These systems are categorized into three groups, each with its own strengths and weaknesses: (1) proofs as sequences of rules (e.g., sequent calculi, natural deduction, and deep inference); (2) proofs as argumentation (e.g., dialogical games, Hyland-Ong games); (3) proofs as sound relations between elements of a theorem (e.g., proof nets, combinatorial proofs, and connection method proofs).

3.2 Fantastic connectives and where to find them

Matteo Acclavio (*University of Southern Denmark – Odense, DK*)

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In this talk, I discuss the notion of (multiplicative) logical connectives. I address the limitations of connectives defined by rules, such as synthetic connectives, and generalized connectives from the early works in linear logic (defined as sets of partitions). I then briefly explain how defining connectives as prime graphs enables the development of logics and proof systems with a more robust proof theory.

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- 3 Matteo Acclavio and Roberto Maieli. *Logic Programming with Multiplicative Structures*. <https://doi.org/10.4204/EPTCS.408.3>
- 4 Matteo Acclavio. *Sequent Systems on Undirected Graphs*. https://doi.org/10.1007/978-3-031-63501-4_12

3.3 Demystifying mu

Bahareh Afshari (*University of Gothenburg, SE*)

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Joint work of Bahareh Afshari, Graham E. Leigh, Guillermo Menéndez Turata
Main reference Bahareh Afshari, Graham E. Leigh, Guillermo Menéndez Turata: “Demystifying μ ”, CoRR, Vol. abs/2401.01096, 2024.
URL <https://doi.org/10.48550/ARXIV.2401.01096>

Formal proofs are an important tool in the mathematical study of computational systems, such as certifying the correctness of an abstract model or verifying that an implementation adheres to a specification. The certification and verification role of proofs boils down to

questions of proof existence and proof synthesis: Does a given formula (sequent, judgement, etc.) admit a proof? Can a proof be generated for each provable formula (sequent, etc.)? Both questions rest heavily on a third, often unspoken: what constitutes a proof?

A proof is normally understood as a finite tree with vertices labelled from a class of deduction elements (formulas, sequents, judgements, etc.) where each vertex together with its children match the conclusion and premises of one of a fixed collection of inferences, the decision of which should be complexity-theoretically simple, at worst polynomial-time decidable. This definition is computationally sufficient for many logics such as propositional logic, modal logics, even predicate logic. In each case, soundness and completeness theorems for the corresponding notion of proof confirm that proof existence agrees with semantic validity, and normal form theorems – notably admissibility of cut – give rise to proof synthesis.

Logics and theories incorporating inductive and co-inductive concepts strain the traditional notion of proof, however. Proof existence relies on more complex completeness theorems which coax the infinitary behaviour of the inductive and co-inductive constructions into finitary induction principles. Synthesis likewise suffers, becoming a task in discerning induction invariants from mere provability or elaborating the proof calculus to recover desirable normal forms. The latter is the typical approach of sequent calculi for modal logics where inductive properties – transitivity, factivity, well-foundedness, etc. – are internalised in the inferences rules.

Illfounded proofs offer to alleviate the proof-theorist’s burden not by complicating the notion of proofs but by relaxing it. Proofs are no longer constrained to finite trees; they can be infinite trees, or even graphs, at the cost of infinite branches fulfilling some pre-determined correctness condition. Illfounded proofs provide an alternative to the traditional notion of proof that treats logics and theories of inductive concepts as manifestations of a general infinitary framework rather than diverging extensions of finitary logic. With a change in perspective comes the need to revisit the encompassing theory. Indeed, the inductive construction of formal proof plays a non-trivial role in most – if not all – fundamental results of well-founded proof theory: proof transformations, cut elimination, computational interpretations, syntactic approaches to interpolation and so on.

The talk presented an introduction to the theory and application of illfounded and cyclic proof systems. We focused on two aspects, interpolation and completeness, and saw how results concerning illfounded proofs can be reflected to finitary, inductive, proofs.

3.4 Abella: An Overview

Kaustuv Chaudhuri (INRIA Saclay – Île-de-France, FR)

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Joint work of Kaustuv Chaudhuri, David Baelde, Andrew Gacek, Dale Miller, Gopalan Nadathur, Alwen Tiu, Yuting Wang

Main reference David Baelde, Kaustuv Chaudhuri, Andrew Gacek, Dale Miller, Gopalan Nadathur, Alwen Tiu, Yuting Wang: “Abella: A System for Reasoning about Relational Specifications”, *J. Formaliz. Reason.*, Vol. 7(2), pp. 1–89, 2014.

URL <https://doi.org/10.6092/ISSN.1972-5787/4650>

Abella is an interactive theorem prover for reasoning about higher-order relational specifications. It is particularly well suited for formalizing the meta-theory of deductive formalisms, including proof systems, type systems, operational semantics, process calculi, etc. One of its key features is the ability to define the inductive structure of data with binding constructs such as quantifiers from logic, abstraction from programming languages, and restriction from process calculi.

This talk gives an overview of the system and presents a main example that can be seen as a challenge problem for formalized meta-theory: showing the equivalence of the higher-order and De Bruijn indexed representations of λ -terms.

3.5 Direct Manipulation for Interactive Theorem Proving

Kaustuv Chaudhuri (INRIA Saclay – Île-de-France, FR)

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Main reference Kaustuv Chaudhuri: “Subformula Linking for Intuitionistic Logic with Application to Type Theory”, in Proc. of the Automated Deduction - CADE 28 - 28th International Conference on Automated Deduction, Virtual Event, July 12-15, 2021, Proceedings, Lecture Notes in Computer Science, Vol. 12699, pp. 200–216, Springer, 2021.

URL https://doi.org/10.1007/978-3-030-79876-5_12

Interactive theorem provers commonly have formal proof languages for expressing formal proofs. Such languages are rarely portable across different systems. I propose an alternative interactive proving interface where textual proof languages have a negligible role. Instead, proofs are built by manipulating the theorem using interaction devices such as mice and touch screens, and interaction mechanisms such as clicking, dragging-and-dropping, etc. The proofs can then be extracted in a variety of existing textual languages. This talk covers intuitionistic first-order logic and discusses some open challenges for this technique, particularly for dependent type theory.

A web-based prototype implementation can be found at: <https://chaudhuri.info/research/profint/index-dagstuhl-2024.html>.

3.6 Exotic proof systems


Marianna Girlando (University of Amsterdam, NL)

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In this tutorial we explore the proof theory of modal logic. It is well-known that Gentzen-style sequent calculi fail to meet basic requirements in the case of modal logic: for instance, no cut-free sequent calculus is known for the modal logic S5. In order to overcome these difficulties, several extensions of Gentzen’s formalism, which we refer to as “exotic” proof systems, have been studied in the literature. These can be approximately grouped into two categories: labelled calculi, which extend the language of the calculus with explicit semantic information, and structured calculi, among which nested sequents, which instead enrich the structure of sequents. In this tutorial, we will cover the basics of labelled calculi and nested calculi for the S5-cube of modal logics.

3.7 CEGAR-Tableaux: improved modal satisfiability via modal clause-learning and SAT

Rajeev P. Gore (Turner, AU)

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Joint work of Rajeev Gore, Cormac Kikkert

We present CEGAR-Tableaux, a tableaux-like method for propositional modal logics utilising SAT-solvers, modal clause-learning and multiple optimisations from modal and description logic tableaux calculi. We use the standard Counter-example Guided Abstract Refinement (CEGAR) strategy for SAT-solvers to mimic a tableau-like search strategy that explores a rooted tree-model with the classical propositional logic part of each Kripke world evaluated using a SAT-solver. Unlike modal SAT-solvers and modal resolution methods, we do not explicitly represent the accessibility relation but track it implicitly via recursion. By using “satisfiability under unit assumptions”, we can iterate rather than “backtrack” over the satisfiable diamonds at the same modal level (context) of the tree model with one SAT-solver. By keeping modal contexts separate from one another, we add further refinements for reflexivity and transitivity which manipulate modal contexts once only. Our solver CEGARBox is, overall, the best for modal logics K, KT and S4 over the standard benchmarks, sometimes by orders of magnitude.

3.8 Proof systems for term-forming operators

Andrzej Indrzejczak (University of Lodz, PL)

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Main reference Andrzej Indrzejczak: “Towards Proof-Theoretic Formulation of the General Theory of Term-Forming Operators”, in Proc. of the Automated Reasoning with Analytic Tableaux and Related Methods - 32nd International Conference, TABLEAUX 2023, Prague, Czech Republic, September 18-21, 2023, Proceedings, Lecture Notes in Computer Science, Vol. 14278, pp. 131–149, Springer, 2023.

URL https://doi.org/10.1007/978-3-031-43513-3_8

Complex descriptive terms are commonly used in natural languages but its role in conveying information is rather neglected in formal languages. This short talk presents the possible way of constructing well-behaved proof systems for operators enabling the expression of complex terms (like definite descriptions, set-abstracts) in proof systems, like sequent calculi, natural deduction, tableaux systems etc.¹

¹ This work is sponsored by the ERC advanced grant ERC-2021-ADG, ExtenDD 101054714.

3.9 Epsilon Calculus and LK

Anela Lolic (TU Wien, AT)

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Joint work of Anela Lolic, Matthias Baaz, Alexander Leitsch

Main reference Kaustuv Chaudhuri: “Subformula Linking for Intuitionistic Logic with Application to Type Theory”, in Proc. of the Automated Deduction - CADE 28 - 28th International Conference on Automated Deduction, Virtual Event, July 12-15, 2021, Proceedings, Lecture Notes in Computer Science, Vol. 12699, pp. 200–216, Springer, 2021.

URL https://doi.org/10.1007/978-3-030-79876-5_12

The epsilon calculus, which is the oldest formalism in proof theory, provides important algorithms for proof theoretic properties through its theorems. For example, Herbrand disjunctions of existential formulas can be computed via the extended first epsilon theorem. The proof of the second epsilon theorem provides a direct approach for the elimination of Skolem functions from Herbrand disjunctions of existential formulas. However, the epsilon calculus is not widely used in the literature, and an important reason for this is its complex notation. One way to overcome this problem is to translate the epsilon calculus in a sequent calculus format. In this talk we will present possible translations of the epsilon formalism into a sequent calculus format and discuss their advantages and problems.

3.10 A short note on expressiveness and complexity in Propositional Dynamic Logic

Bruno Lopes (Fluminense Federal University – Rio de Janeiro, BR)

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Joint work of Mario Benevides, Leandro Gomes, Bruno Lopes, Edward Hermann Heusler

Main reference Mario Benevides, Leandro Gomes, Bruno Lopes: “Towards determinism in PDL: relations and proof theory”, Journal of Logic and Computation, p. exae022, 2024.

URL <https://doi.org/10.1093/logcom/exae022>

Propositional Dynamic logic is a sophisticated modal logic tailored to reason about programs. Several worth-of-studying fragments (controlling the complexity) and extensions (aiming to increase expressiveness) exist. We discuss some impacts of restricting to deterministic programs and how to enhance expressiveness using memory.

The first approach [1] relies on the usage of Guarded Kleene Algebras with Tests (GKAT – [2]) to define Strict Deterministic PDL programs (which will compose GPDL). As GKAT programs can be translated to Thompson’s Automata and have its equivalence determined in almost linear time, the proposed algorithm is used to show that in GPDL $\models \langle \pi \rangle p \leftrightarrow \langle \eta \rangle p \Leftrightarrow \pi = \eta$.

A second approach consists on increasing the expressiveness by adding a memory to PDL [3]. Differently from typical Memory Logics, the memory is in the syntax and not only in the model. We show that it is possible to reason about some context-sensitive languages and, for bounded memory, present a [factorial] translation to standard PDL.

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3.11 About Trust and Proof: An experimental framework for heterogeneous verification

Dale Miller (INRIA Saclay – Île-de-France, FR)

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Joint work of Farah Al Wardani, Kaustuv Chaudhuri, Dale Miller

Main reference Farah Al Wardani, Kaustuv Chaudhuri, Dale Miller: “Formal Reasoning Using Distributed Assertions”, in Proc. of the Frontiers of Combining Systems - 14th International Symposium, FroCoS 2023, Prague, Czech Republic, September 20-22, 2023, Proceedings, Lecture Notes in Computer Science, Vol. 14279, pp. 176–194, Springer, 2023.

URL https://doi.org/10.1007/978-3-031-43369-6_10

Information and opinions come to us daily from a wide range of actors, including scientists, journalists, and pundits. Some actors may be biased or malicious, while others rely on physical measurements, statistics, or in-depth research. Some sources may be signed or edited, while others are anonymous and unmoderated. Trusting information from such diverse sources is a serious challenge facing society today. In this paper, we will describe another domain – the world of machine-checked logic and mathematics – in which many similar issues can appear but in which tractable solutions are possible. Many actors (people or software systems) assert that certain logical statements are theorems in this domain. We describe the Distributed Assertion Management Framework (DAMF) that explicitly manages claims by theorem provers that they have proved certain theorems from associated contexts. Provers willing to trust other provers will be able to avoid rechecking proofs.

3.12 Second-order, well-behaved logic

Sara Negri (University of Genova, IT)

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Joint work of Sara Negri, Matteo Tesi

The expressive capabilities of first-order logic are significantly expanded by second-order logic, which allows quantification over sets and properties. This extension addresses the natural demand in mathematics of expressing properties that involve quantification over all subsets or families of subsets of a given structure. Despite these advantages, full second-order logic lacks essential metalogical properties due to its impredicativity, which presents challenges in the development of proof systems. Our study tackles these challenges by introducing a G3-style calculus that admits the predicative comprehension schema, enabling a constructive cut elimination proof.

The calculi are extended to explore the proof theory of mathematical theories, with methods from first-order calculi being adapted, and structural results being established for both classical and intuitionistic calculi. Additionally, extensional equality and apartness are defined in second-order logic, showcasing the reduction of mathematical notions to pure

logic. As an example, the theory of predicative second-order arithmetic is presented, and a variant of Herbrand’s theorem tailored for predicative second-order intuitionistic logic is established, leading to the conservativity of predicative second-order Heyting arithmetic over its first-order counterpart. Furthermore, the interpolation theorem and the modal embedding of intuitionistic logic are extended to predicative second-order logic.

3.13 A rewriting logic approach to specification, proof-search, and meta-proofs in sequent systems

Carlos Olarte (Université Sorbonne Paris Nord – Villetaneuse, FR), Elaine Pimentel (University College London, GB)

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Joint work of Carlos Olarte, Elaine Pimentel, Camilo Rocha

Main reference Carlos Olarte, Elaine Pimentel, Camilo Rocha: “A rewriting logic approach to specification, proof-search, and meta-proofs in sequent systems”, *J. Log. Algebraic Methods Program.*, Vol. 130, p. 100827, 2023.

URL <https://doi.org/10.1016/J.JLAMP.2022.100827>

In this talk, we demonstrate how inductive properties of propositional sequent systems can be automatically proved using the rewriting logic framework and its implementation in Maude. The properties of interest include the admissibility of weakening and contraction, rule invertibility, cut-admissibility, and identity expansion. We present examples of the L-Framework tool (<https://carlosolarte.github.io/L-framework/>) applied to various logical systems, including single-conclusion and multi-conclusion intuitionistic logic, classical logic, classical linear logic (and its dyadic system), intuitionistic linear logic, and normal modal logics.

3.14 Is, Ought, and cut

Edi Pavlovic (LMU München, DE)

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Joint work of Norbert Gatzl, Edi Pavlovic

Main reference Norbert Gatzl, Edi Pavlovic: “Is, Ought, and Cut”, *J. Philos. Log.*, Vol. 52(4), pp. 1149–1169, 2023.

URL <https://doi.org/10.1007/S10992-023-09701-Y>

We demonstrate how to use proof-theoretic methods, specifically for a series of philosophically motivated deontic logics, to show that the principle of not deriving an “ought” from an “is” holds.

3.15 Proof Methods for Logics with Numerical Quantification

Ian Pratt-Hartmann (University of Manchester, GB)

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In this talk, I ask about the limitations on various types of deductive systems for logics featuring numerical quantification. By “numerical quantification” I understand the use of constructions such as “There exist at most/at least/exactly n x such that ...”, where

n is a natural number. It is known that many familiar fragments of first-order logic for which satisfiability is decidable can be extended with numerical quantification without losing decidability of satisfiability. At issue in this talk is what the decision methods in question have to look like. I begin with a negative result concerning the extension of the Aristotelian syllogistic with numerical quantification: I show that there exist no sound and complete collections of Aristotelian-like syllogisms for such logics. I then switch to the extension of the two-variable fragment of first-order logic with numerical quantification: I present a high-level account of decision procedures for this logic based on reduction to integer linear programming. Finally, I turn my attention to the one-variable fragment of first-order logic extended with numerical quantification: I ask whether there is a numerical extension of propositional resolution which is sound and complete for this logic.

3.16 Ultimate Glivenko?

Peter M. Schuster (University of Verona, IT)

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Joint work of Sara Negri, Giulio Fellin, Peter Schuster

A simple criterion for the validity of an abstract Glivenko master theorem facilitates to instantiate the latter not only in conventional contexts such as Kuroda's but always when an axiom, e.g. stability, is added to a basic provability relation.

3.17 What is proof theory?

Lutz Straßburger (INRIA Saclay – Île-de-France, FR)

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In this short talk I will discuss the problem of proof identity and explain how it is related to Hilbert's 24th problem. I will also argue that not knowing when two proofs are “the same” has embarrassing consequences not only for proof theory but also for certain areas of computer science where formal proofs play a fundamental role, in particular the formal verification of software.

3.18 Negation inconsistent logics

Heinrich Wansing (Ruhr-Universität Bochum, DE)

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Main reference Heinrich Wansing: “Beyond paraconsistency. A plea for a radical breach with the Aristotelean orthodoxy in logic”, in: *Walter Carnielli on Reasoning, Paraconsistency, and Probability*, A. Rodrigues, H. Antunes, and A. Freire (eds), 2022, Springer, to appear 2025.

The talk contains material from a programmatic paper written in the context of an ERC grant, ERC-2020-ADG, ConLog, Contradictory Logics: A Radical Challenge to Logical Orthodoxy, H. Wansing, Beyond paraconsistency. A plea for a radical breach with the Aristotelean

orthodoxy in logic, in: *Walter Carnielli on Reasoning, Paraconsistency, and Probability*, A. Rodrigues, H. Antunes, and A. Freire (eds), 2022, Springer, to appear 2024, and some comments on Gentzen-style proof systems for a certain non-trivial negation inconsistent logic.

After some terminological preliminaries on the notions of contradiction, inconsistency, and negation, I will make comments on paraconsistency, classical logic, and consistency. Then I will go beyond paraconsistency and will present a list of non-trivial negation inconsistent logics. Finally, I will introduce the non-trivial negation inconsistent connexive logic **C** and bilateral proof systems for **C**.

The references below may be considered *selected references* on this topic.

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3.19 The Epsilon Calculus in Non-classical Logics: Recent Results and Open Questions

Richard Zach (University of Calgary, CA)

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The epsilon operator [1, 3] is mainly studied in the context of classical logic. It is a term forming operator: if $A(x)$ is a formula, then $\varepsilon x A(x)$ is a term – intuitively, a witness for $A(x)$ if one exists, but arbitrary otherwise. Its dual $\tau x A(x)$ is a counterexample to $A(x)$ if one exists. Classically, it can be defined as $\varepsilon x \neg A(x)$. Epsilon and tau terms allow the classical quantifiers to be defined: $\exists x A(x)$ as $A(\varepsilon x A(x))$ and $\forall x A(x)$ as $A(\tau x A(x))$.

Epsilon operators are closely related to Skolem functions, and the fundamental so-called epsilon theorems to Herbrand’s theorem. Recent work with Matthias Baaz [2] investigates the proof theory of $\varepsilon\tau$ -calculi in superintuitionistic logics. In contrast to the classical ε -calculus, the addition of ε - and τ -operators to intuitionistic and intermediate logics is not conservative, and the epsilon theorems hold only in special cases. However, it is conservative as far as the propositional fragment is concerned.

Despite these results, the proof theory and semantics of $\varepsilon\tau$ -systems on the basis of non-classical logics remains underexplored.


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4 Panel discussions

4.1 Special Session: Proof-theoretic Semantics

Alexander Gheorghiu (University College London, GB), Sara Ayhan (Ruhr-Universität Bochum, DE), and Victor Nascimento (Universidade do Estado do Rio de Janeiro, BR)

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In *model-theoretic semantics* (M-tS), logical consequence is defined in terms of models; that is, abstract mathematical structures in which propositions are interpreted and their truth is judged. This includes, in particular, denotational semantics and Tarski’s conception of logical consequence: a proposition ϕ follows model-theoretically from a context Γ iff every model of Γ is a model of ϕ ,

$$\Gamma \models \phi \quad \text{iff} \quad \text{for all models } \mathcal{M}, \text{ if } \mathcal{M} \models \psi \text{ for all } \psi \in \Gamma, \text{ then } \mathcal{M} \models \phi$$

Proof-theoretic semantics (P-tS) [16] is an alternative approach to meaning and validity in which they are characterized in terms of *proofs* – understood as objects denoting collections of acceptable inferences from accepted premisses. It also concerns the semantics *of* proofs, understood as “valid” arguments.

To be clear, P-tS is not about providing a proof system. As Schroeder-Heister [15] observes, since no formal system is fixed (only notions of inference) the relationship between semantics and provability remains the same as it has always been: soundness and completeness are desirable features of formal systems.

The semantic paradigm supporting P-tS is *inferentialism* – the view that meaning (or validity) arises from rules of inference (see Brandom [2]). This may be viewed as a particular instantiation of the *meaning-as-use* paradigm by Wittgenstein [20] in which “use” in logic is understood as inferential rôle.

Heuristically, what differs is that (pre-logical) *proofs* in P-tS serve the rôle of *truth* in M-tS. This shift has substantial and subtle mathematical and conceptional consequences, as discussed below.

To illustrate the paradigmatic shift from M-tS to P-tS, consider the proposition “Tammy is a vixen”. What does it mean? Intuitively, it means, somehow, “‘Tammy is female’ and ‘Tammy is a fox’”. On inferentialism, its meaning is given by the rules,

$$\frac{\text{Tammy is a fox} \quad \text{Tammy is female}}{\text{Tammy is a vixen}} \quad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}} \quad \frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}}$$

These merit comparison with the laws governing conjunction (\wedge), which justify the sense in which the above proposition is a conjunction,

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \quad \frac{\phi \wedge \psi}{\phi} \quad \frac{\phi \wedge \psi}{\psi}$$

There are several branches of research within P-tS – see, for example, the discussion on proof-theoretic validity in the Dummett-Prawitz tradition by Schroeder-Heister [14] – see also Gheorghiu and Pym [6]. Here, we concentrate on two topics: *base-extension semantics* and *bilateralism*.

Before proceeding to the topic details, we outline some important questions for the development of P-tS:

- *To what extent should P-tS depend on paradigms of proof?* On the one hand, different logics are more naturally expressed in some format of proofs than others (e.g., substructural logics typically favour sequent presentations more than natural deduction) and their P-tS may be influenced by this bias. Moreover, P-tS gives up an opportunity to challenge the current foundations and received dogma of the very concept of “proof” in logic. On the other hand, semantics ought to be syntax independent (in some sense). This may mean that a given notion of P-tS should be instantiable to different paradigms of proof, if none is taken as conceptually prior to the others (e.g., one may view a sequent calculus as providing “constructions” and natural deduction as providing the genuine article).
- *What might we expect of the relationship to M-tS?* Since M-tS is a powerful way of looking at logics, one may strive to show that the usual properties of M-tS are not lost if one transitions to P-tS. In particular, one may desire that the behaviour of models be represented in P-tS in some way that remains to be made precise. On the contrary, P-tS may offer an entirely different meta-theory that gives access to entirely distinct understandings of logics while forbidding other, perhaps useful, features of their extant semantics.

- *What is the real value of P-tS?* Developing the last point in a particular direction, we should consider what mathematical and computational value P-tS holds beyond its philosophical significance. To this end, one may begin by investigating how meta-theoretic properties of logics (e.g., compactness, categoricity, decidability, and so on) may be proved from the point of view of P-tS.

There are, of course, many more questions that one could ask. For example, P-tS may lead us to consider entirely new logics that have an obscure M-tS (if they have one at all). We defer further discussion of these matters to another time.

Base-extension Semantics

This section is concerned with a formalism in P-tS called *base-extension semantics* (B-eS). It follows the tradition of Piecha et al. [9] and Sandqvist [13].

The idea of B-eS begins with the notion of an *atomic system*. An atomic system is a collection of inferential relationships between *atoms*. They represent some beliefs that an agent may possess about the inferential relationship between thoughts. Piecha and Schroeder-Heister [17] and Sandqvist have given an analysis of them based on earlier work by Prawitz [10] and Schroeder-Heister.

Presently, we shall consider three types of atomic rules. Let C, P_1, \dots, P_n be atoms and $\mathbb{P}_1, \dots, \mathbb{P}_n$ be finite, possibly empty, sets of atoms. The following are zero-, first-, and second-level atomic rules, respectively

$$\frac{}{C} \quad \frac{P_1 \quad \dots \quad P_n}{C} \quad \frac{\frac{[\mathbb{P}_1]}{P_1} \quad \dots \quad \frac{[\mathbb{P}_n]}{P_n}}{C}$$

The rules governing Tammy and her vixenhood above are atomic rules; specifically, they are *first-level* rules. Sandqvist [13] provides the following example of a second-level rule:

$$\frac{\text{A is a sibling of B} \quad \frac{[\text{A is a brother of B}]}{P} \quad \frac{[\text{A is a sister of B}]}{P}}{P}$$

Whether atomic rules correspond to “knowledge” or “definition” is a debated topic.

Atomic rules are read essentially as natural deduction rules in the sense of Gentzen. However, they are taken *per se* so that no substitution is allowed. Thus, they are intuitively related to hereditary Harrop formulae in the sense of Miller.

A collection of atomic rules is an *atomic system*. We may restrict attention to certain atomic systems, in which case we call them *bases* (\mathcal{B}). Their reading as natural deduction rules (without substitution) determines a notion of *derivability in a base* ($\vdash_{\mathcal{B}}$).

Relative to a notion of derivability in a base ($\vdash_{\mathcal{B}}$), a B-eS is determined by a judgement called *support* ($\Vdash_{\mathcal{B}}$) defined inductively according to the structure of formulae with the base case (i.e., the support of atoms) given by *provability in a base*. This induces a validity judgement by quantifying our bases,

$$\Gamma \Vdash \phi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{B}} \phi \text{ for any base } \mathcal{B}$$

We illustrate this idea below.

Define a base \mathcal{B} to be an atomic system that only contains zero- and first-level rules,

$$\frac{}{C} \quad \frac{P_1 \quad \dots \quad P_n}{C}$$

Let $\overline{\mathbb{A}}$ denote the set of closed atoms and $\overline{\mathbb{T}}$ denote the set of closed terms. Relative to this notion of base, define a support relation (\Vdash) as follows:

$$\begin{array}{lll} \Vdash_{\mathcal{B}} P & \text{iff} & \vdash_{\mathcal{B}} P & (\text{At}) \\ \Vdash_{\mathcal{B}} \perp & \text{iff} & \Vdash_{\mathcal{B}} P \text{ for any } P \in \overline{\mathbb{A}} & (\perp) \\ \Vdash_{\mathcal{B}} \phi \wedge \psi & \text{iff} & \Vdash_{\mathcal{B}} \phi \text{ and } \Vdash_{\mathcal{B}} \psi & (\wedge) \\ \Vdash_{\mathcal{B}} \phi \rightarrow \psi & \text{iff} & \phi \Vdash_{\mathcal{B}} \psi & (\rightarrow) \\ \Vdash_{\mathcal{B}} \forall x \phi & \text{iff} & \Vdash_{\mathcal{B}} \phi[x \mapsto t] \text{ for any } t \in \overline{\mathbb{T}} & (\forall) \\ \Gamma \Vdash_{\mathcal{B}} \phi & \text{iff} & \text{for any } \mathcal{C} \supseteq \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \psi \text{ for } \psi \in \Gamma, \text{ then } \Vdash_{\mathcal{C}} \phi & (\text{Inf}) \end{array}$$

Sandqvist (see also Makinson [7]) have shown that this characterises classical logic; that is,

$$\Gamma \Vdash \phi \quad \text{iff} \quad \phi \text{ follows classically from } \Gamma$$

Interestingly, $\Gamma \Vdash \phi$ is equivalent to $\Gamma \Vdash_{\emptyset} \phi$, suggesting that logical validity corresponds to *analytic* knowledge.

To express intuitionistic logic, we require extending the language with disjunction (\vee) and the existential quantifier (\exists). To this end, we may propose the following clauses:

$$\begin{array}{lll} \Vdash_{\mathcal{B}} \phi \vee \psi & \text{iff} & \Vdash_{\mathcal{B}} \phi \text{ or } \Vdash_{\mathcal{B}} \psi & (\vee) \\ \Vdash_{\mathcal{B}} \perp & \text{iff} & \Vdash_{\mathcal{B}} P \text{ for any } P \in \overline{\mathbb{A}} & (\exists) \end{array}$$

Piecha et al. [9] have shown that, surprisingly, intuitionistic logic is *incomplete* for this semantics. Subsequently, Stafford [18] showed that, in the propositional case, it corresponds to an intermediate logic known as (*general*) *inquisitive logic*.

We now observe that in the B-eS above, absurdity (\perp) is defined by *ex falso quodlibet*. This is quite unlike its treatment in more traditional M-tS. A philosophical motivation for this clause has been given by Dummett [3]. Following this motivation, Sandqvist [13] suggests the following alternative clauses:

$$\begin{array}{lll} \Vdash_{\mathcal{B}} \phi \vee \psi & \text{iff} & \text{for any } \mathcal{C} \supseteq \mathcal{B} \text{ and } P \in \overline{\mathbb{A}}, \\ & & \text{if } \phi \Vdash_{\mathcal{C}} P \text{ and } \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P & (\vee) \\ \Vdash_{\mathcal{B}} \exists x \phi & \text{iff} & \text{for any } \mathcal{C} \supseteq \mathcal{B} \text{ and } P \in \overline{\mathbb{A}}, \\ & & \text{if } \phi[x \mapsto t] \Vdash_{\mathcal{C}} P \text{ for any } t \in \overline{\mathbb{T}}, \text{ then } \Vdash_{\mathcal{C}} P & (\exists) \end{array}$$

Here $\phi[x \mapsto t]$ is the result of replacing every free occurrence of x in ϕ by t .

To capture intuitionistic logic, some modification must be required at this point. To see this, consider Peirce's Law, $((P \rightarrow Q) \rightarrow P) \rightarrow P$. This formula is classically but not intuitionistically valid. Since it only contains implications and atoms, it is valid in the B-eS before the clauses for disjunction (\vee) and existential quantifier (\exists) were added, but that corresponds to classical logic. Hence, this intuitionistic logic is not complete for this B-eS.

We require only a small but significant change for intuitionistic logic: we now permit second-level rules in bases,

$$\frac{\begin{array}{ccc} [\mathbb{A}_1] & \dots & [\mathbb{A}_n] \\ P_1 & & P_n \end{array}}{c}$$

Sandqvist (see also Gheorghiu) have shown that the result indeed corresponds to intuitionistic logic,

$$\Gamma \Vdash \phi \quad \text{iff} \quad \phi \text{ follows intuitionistically from } \Gamma$$

Though B-eS appears to be closely related to M-tS (esp. possible world semantics in the sense of Beth and Kripke), the formal connection remains an enigma. Indeed, while Makinson [7] (resp. Eckhardt and Pym [4]) have made formal connections between the M-tS and B-eS of classical logic (resp. normal modal logic), the analogous connections for intuitionistic logics are currently unknown. Part of the challenge is in the considerably different ways that disjunctive structures (i.e., \perp , \vee , \exists) are treated.

The above work on the B-eS of classical and intuitionistic logic has been extended by Eckhardt and Pym [4] to modal logic, by Gheorghiu et al. [5] and Buzoku to substructural logic (namely, *intuitionistic Linear Logic* and *the logic of Bunched Implications*), and by Nascimento et al. to ecumenical logic. Closely related approaches have also been developed by Goldfarb and Nascimento and Stafford [8].

Bilateralism

Logical bilateralism can be very generally described as an approach to meaning and consequence on the grounds of a symmetry between certain notions, like assertion and denial, proof and refutation or truth and falsity, in that both are taken as primitive and not, as in conventional “unilateralist” approaches, merely reducing the latter to the former, more primary notion. In recent years, the field of logical bilateralism has seen significant development with various systems being developed that showcase a range of orientations within this framework. In Rumfitt’s seminal paper [12], in which the term “bilateralism” was introduced, he means to give a motivation for how the natural deduction rules of classical logic lay down the meaning of the connectives once we consider a calculus containing introduction and elimination rules determining not only the assertion conditions for the connectives but also the denial conditions.

This is realized by using signed formulas in the form of “+A” and “-A” where “+” and “-” are used as force indicators. Smiley developed a similar approach and there are also other and earlier works promoting general bilateralist ideas. While several works explore and refine this approach to bilateralism in that the main focus is on natural deduction style proof systems with assertion and denial conditions, there have been developments in other directions, in which bilateralist considerations play an equally central role. Some propose a new way of reading a (classical) sequent calculus with multiple conclusions, namely by way of defining an inference, represented by a sequent, as valid if and only if it is incoherent to assert all the premises (i.e., the formulas on the left side of the sequent sign), while simultaneously denying all the conclusions (i.e., the formulas on the right side of the sequent sign) – see, for example, Restall [11]. Here, the bilateralist considerations do not arise in the design of a distinctive proof system, but in the interpretation of an already existing proof calculus by way of taking assertion and denial as dual notions.

The approach presented in the special session focuses not so much on the speech acts of assertion and denial but on a duality between different inferential relationships, which in turn give rise to motivating proof systems with dual derivability relations. Such proof systems displaying provability and refutability can be represented both in natural deduction and in sequent calculus style (see, for example, Wansing [19] and Ayhan [1]). On such a


view it can be asked, then, how these dual derivability relations can be implemented on a meta-level. In a sequent calculus setting, for example, this would mean not only to have signed sequents, displaying provability and refutability within sequents, but also displaying the dual relations between sequents.

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4.2 Special Session: Universal Proof Theory

Raheleh Jalali (The Czech Academy of Sciences – Prague, CZ), Timo Lang (University College London, GB), and Iris van der Giessen (University of Birmingham, GB)

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Universal Proof Theory is a recent research program that aims to study the mathematical properties of proof systems in a generic manner (analogous to the generic study of algebras in Universal Algebra). The term Universal Proof Theory was first introduced in preprint [1]. The program consists of three fundamental problems: (1) the existence problem to investigate the existence of certain kinds of proof systems for a given logic, (2) the equivalence problem to study the natural notions of equivalence of proof systems, and (3) the characterization problem to investigate the possible characterizations of proof systems. In the session, we provide an overview of the results which are mainly achieved for the existence problem (i.e., [3, 1, 4, 5, 2]). We discuss a small unsolved problem to provide a flavor of what this research involves. We invite the audience to connect their work to this program so that we create a better understanding of Universal Proof Theory within the broader community of Proof Theory.

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4.3 Special Session: Circular and Non-wellfounded Proofs

Alexis Saurin (CNRS – Paris, FR), Anupam Das (University of Birmingham, GB), and Abhishek De

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Special session held on 22nd August 2024

Contributors: Anupam DAS, Abhishek DE & Alexis SAURIN

Non-wellfounded proofs and their regular fragment – cyclic, or circular proofs – appear in a very natural way when studying inductive and coinductive reasoning [5, 8, 12, 13] (such as in variants of the μ -calculus, or in logics with inductively-defined predicates). While the complex inference rules for induction and coinduction are replaced by simpler fixed-point unfolding rules, the existence of infinite branches in proof trees requires that a global condition is imposed on derivations in order to ensure soundness and to allow a structured proof-theoretical analysis as shown below:

$$\frac{\frac{\vdots}{\vdash \mu X.X} (\mu) \quad \frac{\vdots}{\vdash \nu X.X, \Gamma} (\nu)}{\vdash \mu X.X} (\mu) \quad \frac{\vdots}{\vdash \nu X.X, \Gamma} (\nu)}{\vdash \Gamma} (\text{cut})$$

The special session presented some complements on Bahareh AFSHARI's invited talk along the following lines and with a specific focus on the Curry-Howard correspondence between proofs and programs.

- Abhishek DE presented the logical frameworks under study as well as the thread progress-condition for non-wellfounded and regular proofs (illustrated below, with $F = \nu X.(a \vee \bar{a}) \wedge (X \wedge \mu Y.X)$) and discussed its decidability and complexity. He also explained how the condition ensures the soundness of the proof system [5, 7, 8, 12].

$$\frac{\frac{\frac{\vdash a, \bar{a}}{\vdash a \vee \bar{a}} (\text{Ax}) \quad \frac{\frac{\vdash F}{\vdash \mu Y.F} (\mu)}{\vdash F \wedge \mu Y.F} (\wedge)}{\vdash (a \vee \bar{a}) \wedge (F \wedge \mu Y.F)} (\wedge)}{\vdash F} (\text{v})$$

- Anupam DAS compared finitary (co)induction inference rules [1, 5] and the provability in cyclic proofs [5, 8]: he showed that, under a very general condition, circular proofs subsume finitary induction and coinduction [8] (see figure below) and reviewed the literature studying the converse property [3, 4, 8, 11].

$$\frac{\frac{\frac{\pi_1}{\vdash \Gamma, S} \quad \frac{\pi_2}{\vdash S^\perp, F[S]}}{\vdash \Gamma, \nu X.F} (\text{v}) \quad \frac{\frac{[\pi_2]}{\vdash S^\perp, F[S]} \quad \frac{\vdash S^\perp, \nu X.F}{\vdash F[S]^\perp, F[\nu X.F]} (\exists F)}{\vdash S^\perp, F[\nu X.F]} (\text{cut})}{\vdash \Gamma, \nu X.F} (\text{cut})$$

- Finally, Alexis SAURIN reviewed cut-elimination theorems for those proof systems [2, 8, 9, 10] including non-wellfounded derivations and the challenge raised by the non-wellfounded proof trees. On the way, he showed, on some examples, how inductive and coinductive data can be represented as non-wellfounded and circular proofs [6, 8].

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Participants

- Matteo Acclavio
University of Southern Denmark – Odense, DK
- Bahareh Afshari
University of Gothenburg, SE
- Sara Ayhan
Ruhr-Universität Bochum, DE
- Eben Blaisdell
University of Pennsylvania – Philadelphia, US
- Yll Buzoku
University College London, GB
- Kaustuv Chaudhuri
INRIA Saclay – Île-de-France, FR
- Zhibo Chen
Carnegie Mellon University – Pittsburgh, US
- Anupam Das
University of Birmingham, GB
- Abishek De
University of Birmingham, GB
- Amy Felty
University of Ottawa, CA
- Alexander Gheorghiu
University College London, GB
- Marianna Girlando
University of Amsterdam, NL
- Rajeev P. Gore
Turner, AU
- Tao Gu
University College London, GB
- Andrzej Indrzejczak
University of Lodz, PL
- Raheleh Jalali
The Czech Academy of Sciences – Prague, CZ
- Timo Lang
University College London, GB
- Anela Lolic
TU Wien, AT
- Bruno Lopes
Fluminense Federal University – Rio de Janeiro, BR
- Robin Martinot
Utrecht University, NL
- Dale Miller
INRIA Saclay – Île-de-France, FR
- Victor Nascimento
Universidade do Estado do Rio de Janeiro, BR
- Sara Negri
University of Genova, IT
- Carlos Olarte
Université Sorbonne Paris Nord – Villetaneuse, FR
- Edi Pavlovic
LMU München, DE
- Elaine Pimentel
University College London, GB
- Ian Pratt-Hartmann
University of Manchester, GB
- Revantha Ramanayake
University of Groningen, NL
- Alexis Saurin
CNRS – Paris, FR
- Peter M. Schuster
University of Verona, IT
- Sana Stojanovic-Djordjevic
University of Belgrade, RS
- Lutz Straßburger
INRIA Saclay – Île-de-France, FR
- Iris van der Giessen
University of Birmingham, GB
- Fernando Velázquez Quesada
University of Bergen, NO
- Heinrich Wansing
Ruhr-Universität Bochum, DE
- Richard Zach
University of Calgary, CA

