

# Behavioural Metrics and Quantitative Logics

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## Abstract

This report documents the program and the outcome of Dagstuhl Seminar “Behavioural Metrics and Quantitative Logics” (24432).

Behavioural metrics and quantitative logics specify quantitative aspects of systems. A metric measures how far apart two systems are in their behaviour while a quantitative logic evaluates the degree to which a state satisfies a formula. They are often intimately connected via a Hennessy-Milner theorem stating that the distance induced by a quantitative logic coincides with behavioural distance. There are various applications in model-checking, differential privacy, hybrid systems and learning. Several challenges in this area have been identified: studying suitable metrics and their corresponding logics, generalizing to the setting of coalgebras by parameterizing the branching type of the system under consideration, developing methods of quantitative algebraic reasoning, finding efficient methods for computing behavioural methods. This Dagstuhl Seminar provided a forum to researchers working in this area, to discuss the state-of-the-art and further developments, and in particular address applications in various domains.

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## 1 Executive summary

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Behavioural equivalences are central in the analysis and verification of concurrent systems. They come with a very well-developed meta-theory including logical characterisations, efficient algorithms, game-theoretic perspectives, algebraic characterisations, and generalisations to a wide variety of state-based systems. Two processes or system states are behaviourally equivalent if they are indistinguishable from the point of view of an external observer. Behavioural equivalences have for instance been used for comparing a system with its

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specification, for the analysis of cryptographic protocols, for the verification of model transformations, for efficient comparison of non-deterministic automata and in programming language semantics.

In recent years, behavioural metrics and quantitative logics specifying quantitative aspects of systems have received considerable attention. Previous approaches are qualitative and address the question “do they match?” whereas metrics address the question “to what degree? how much?”. Indeed, a metric measures how far apart two systems are in their behaviour, while a quantitative logic evaluates the degree to which a state satisfies a formula. For quantitative systems such as Markov chains, Markov decision processes and probabilistic automata, a behavioural equivalence is often too strict: small deviations in quantitative information, such as probabilities, can cause two states that intuitively behave very much alike to be inequivalent in a formal sense. Indeed, behavioural equivalence misses *robustness* under small perturbations in such systems. Metrics and quantitative logics provide a more nuanced approach than qualitative behavioural equivalence.

As behavioural equivalences, behavioural metrics can be characterized via fixpoint equations. In addition there are alternative characterizations of metrics, either via modal logics or via spoiler-duplicator games. The focus of this Dagstuhl Seminar is in particular on quantitative logics, where evaluating a formula on a state does not yield a boolean value, but a (real) number, intuitively indicating to which degree a state satisfies a formula. This is connected to a quantitative Hennessy-Milner property which is supposed to hold for the behavioural distance.

There are various applications in model-checking, differential privacy, hybrid systems and learning. Several challenges in this area have been identified: studying suitable metrics and their corresponding logics, generalizing to the setting of coalgebras by parameterizing the branching type of the system under consideration, developing methods for quantitative algebraic reasoning, and finding efficient methods for computing behavioural metrics.

This Dagstuhl Seminar provided a forum for researchers working in all areas of behavioural metrics and quantitative logics, to discuss the state-of-the-art and further developments, and in particular to address applications in various domains, including machine learning.

The topics discussed at the seminar included all aspects of behavioural metrics and quantitative logics. In particular:

- Various approaches to define behavioural metrics, including characterizations via fixpoint equations, logics and games and their relations.
- Quantitative logics and their expressiveness.
- Methods and theories for quantitative equational reasoning.
- Algorithms for (compositionally) computing behavioural metrics or distinguishing formulas and their efficiency.
- Applications of behavioural metrics and quantitative logics, including – but not limited to – model-checking, differential privacy, hybrid systems and learning.

The seminar programme left room to address the above individual topics, but the main aim was to connect these topics, which previously have been studied by different research groups and published in different venues. Establishing such connections proved to be the main way forward, both to solidify the field and to identify new research problems and opportunities for applications. In particular, a key challenge was to connect recent approaches to quantitative logics, game characterisations, and quantitative equational theories, and to generalise these connections and algorithmic perspectives to enable their application to a wide range of models. We also focused on practical applications and promoted the usefulness of behavioural metrics. The seminar stimulated interactions between more theoretical scientists with researchers working on application-oriented aspects.

We invited four representatives of the different communities to give tutorial talks in order to introduce fundamental concepts and techniques. Specifically, the following four tutorial talks took place on the first day of the seminar:

- Franck van Breugel: Behavioural metrics
- Clemens Kupke: Coalgebras
- Giorgio Bacci: Quantitative equational reasoning
- Pablo Castro: Behavioural metrics for Reinforcement Learning

On Tuesday and Thursday we organized the following working groups in order to discuss more specific topics which were of interest to a substantial part of the participants:

- Generalised Kantorovich-Rubinstein Duality
- Quantitative Algebras/Coalgebras/Modal Logic
- Applications of Behavioural Metrics
- Fixpoints
- Locksteps

The seminar made it possible for many participants to connect and discuss joint collaborations and research projects. The seminar at Dagstuhl was also the opportunity for Matteo Mio, Lutz Schröder, Henning Urbat and Paul Wild to coordinate between the pre-proposal and the full proposal submission for an ANR-DFG project. The full proposal will be submitted in March 2025. This project also involves other applicants from FAU Erlangen-Nürnberg and ENS-Lyon and it aims at furthering our understanding of the theory of quantitative algebras. This topic was the main subject of many talks of this seminar. Further ideas for follow-up events and outreach activities were developed in the wrapup session (for the report on this session see Section 5.6).

The organizers would like to thank all the participants and speakers for their inspiring talks and many interesting discussions. Furthermore we would like to acknowledge Florence Clerc who helped to write and prepare this report. A special thanks goes to the Dagstuhl staff who were a great help in organizing this seminar.

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## 3 Overview of Tutorials

### 3.1 Behavioural Metrics

*Franck van Breugel (York University – Toronto, CA)*

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In this tutorial, we motivate the need for behavioural metrics. We show how the notion of probabilistic bisimilarity, due to Kim Larsen and Arne Skou, can be generalized. For this generalization, we consider the characterization of probabilistic bisimilarity in terms of couplings due to Bengt Jonsson and Kim Larsen. The coupling captures how to match transitions such that the target states are probabilistic bisimilar, that is, behave the same. By matching transitions the targets of which behave similarly yet not necessarily the same, we arrive at probabilistic bisimilarity distances. We present the logical characterization due to Josee Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden. Furthermore, we briefly discuss some algorithms to approximate and compute the distances.

### 3.2 Coalgebras

*Clemens Kupke (University of Strathclyde – Glasgow, GB)*

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In my tutorial I will introduce coalgebra as a general framework to model cyclic structures and various types of transition systems. I will discuss central notions of the coalgebraic theory such as bisimulation and behavioural equivalence. Finally I will survey different logics for reasoning about coalgebras. While the talk will focus on Boolean-valued logics I will also sketch how the framework can be extended to logics for quantitative reasoning.

### 3.3 Quantitative Equational Reasoning

*Giorgio Bacci (Aalborg University, DK)*

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In this talk, I review the basic definitions, constructions and key results presented in seminal work by Mardare, Panangaden and Plotkin (LICS'16), Quantitative Algebraic Reasoning, which constitutes the first step of the more ambitious program of understanding the algebraic properties of computational effects on categories enriched over (extended) metric spaces. I provide several motivating examples to offer a general pragmatic picture of how algebraic reasoning works and how it can be used. Finally, I conclude by presenting some influential work in this area of research and discussing some open problems.

### 3.4 Behavioural Metrics for Reinforcement Learning

Pablo Castro (*Google DeepMind – Montreal, CA*)

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In this tutorial we will provide an introduction to reinforcement learning (RL), along with many of the longstanding challenges in the field; one of these is the question of how to deal with very large state spaces. While there are a variety of ways to tackle this problem, we explore the use of behavioural metrics for shaping representations so as to maximize generalizability without sacrificing accuracy. This is of particular interest when combining RL with modern function approximators like deep neural networks. We present a number of recent approaches based on the theory of bisimulation metrics, and provide pointers to open research problems in this area.

## 4 Overview of Talks

### 4.1 Behavioural Metrics via Functor Lifting – A Coalgebraic Approach

Barbara König (*Universität Duisburg-Essen, DE*)

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Behavioural distances of transition systems modelled via coalgebras for endofunctors generalize traditional notions of behavioural equivalence to a quantitative setting, in which states are equipped with a measure of how (dis)similar they are. Endowing transition systems with such distances essentially relies on the ability to lift functors describing the one-step behavior of the transition systems to the category of pseudometric spaces. We consider the category theoretic generalization of the Kantorovich/Wasserstein lifting from transportation theory to the case of lifting functors to pseudo-metric spaces.

We also consider compositionality results which are essential ingredients for adapting up-to-techniques to the case of behavioural distances. Up-to techniques are a well-known coinductive technique for efficiently showing lower bounds for behavioural metrics. We illustrate the method with a case study on probabilistic automata.

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## 4.2 Drawing from an Urn is Isometric

*Bart Jacobs (Radboud University – Nijmegen, NL)*

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Drawing (a multiset of) coloured balls from an urn is one of the most basic models in discrete probability theory. Three modes of drawing are commonly distinguished: multinomial (draw-replace), hypergeometric (draw-delete), and Polya (draw-add). These drawing operations are represented as maps from urns to distributions over multisets of draws. The set of urns is a metric space via the Wasserstein distance. The set of distributions over draws is also a metric space, using Wasserstein-over-Wasserstein. It is shown that these three draw operations are all isometries, that is, they exactly preserve the Wasserstein distances. Further, drawing is studied in the limit, both for large urns and for large draws. First it is shown that, as the urn size increases, the Wasserstein distances go to zero between hypergeometric and multinomial draws, and also between Pólya and multinomial draws. Second, it is shown that, as the draw size increases, the Wasserstein distance goes to zero (in probability) between an urn and (normalised) multinomial draws from the urn. These results are known, but here, they are formulated in a novel manner as limits of Wasserstein distances. We call these two limit results the law of large urns and the law of large draws.

## 4.3 On the Challenges of Quantitative Reasoning

*Radu Mardare (Heriot-Watt University – Edinburgh, GB)*

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We present the main principles of Lawvere Logic, a logic interpreted on top of the semiring of extended positive reals taken with reverse order. This realizes and extends some of the ideas of Lawvere for developing the theory of metric spaces. Lawvere logics are proposed both as a way to extend the quantitative equational logics, and as a way of approaching approximation theories.

## 4.4 Beyond Metrics and Non Expansive Operations in Quantitative Algebra

*Matteo Mio (ENS – Lyon, FR)*

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I will present some recent generalisations of the theory of quantitative algebras, originally introduced by Mardare Panangaden Plotkin in a LICS 2016 paper, which can accommodate quantitative algebras whose distance and operations are not restricted to be metrics and nonexpansive maps, respectively.



## 4.5 Explainability of Probabilistic Bisimilarity Distances

*Franck van Breugel (York University – Toronto, CA)*

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Probabilistic bisimilarity distances measure the similarity of behaviour of states of a labelled Markov chain. The smaller the distance between two states, the more alike they behave. Their distance is zero if and only if they are probabilistic bisimilar. Recently, algorithms have been developed that can compute probabilistic bisimilarity distances for labelled Markov chains with thousands of states within seconds. However, say we compute that the distance of two states is 0.125. How does one explain that 0.125 captures the similarity of their behaviour?

In this talk, we discuss two approaches to explainability: a logical one and a game-theoretic one. In the logical approach, which is joint work with Amgad Rady, we address the question of explainability by returning to the definition of probabilistic bisimilarity distances proposed by Josee Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden more than two decades ago. We use a slight variation of their logic to construct for each pair of states a sequence of formulas that explains the probabilistic bisimilarity distance of the states. We have developed and implemented an algorithm that computes those formulas and each formula can be computed in polynomial time.

The game-theoretic approach is joint work with Anto Nanah Ji, Emily Vlasman, and James Worrell. We demonstrate that the probabilistic bisimilarity distances can be explained by a policy of a game. We start from a characterization of probabilistic bisimilarity distances provided in joint work with Di Chen and James Worrell. We identify properties of policies that make them suitable for explaining the probabilistic bisimilarity distances and present algorithms to generate policies with those properties.

## 4.6 Approximating Fixpoints of Approximated Functions

*Paolo Baldan (University of Padova, IT)*

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Fixpoints are ubiquitous in computer science and when dealing with quantitative semantics and verification one is commonly led to consider least fixpoints of (higher-dimensional) functions over the non-negative reals. We show how to approximate the least fixpoint of such functions, focusing on the case in which they are not known precisely, but represented by a sequence of approximating functions that converge to them. We concentrate on monotone and non-expansive functions, for which uniqueness of fixpoints is not guaranteed and standard fixpoint iteration schemes might get stuck at a fixpoint that is not the least. Our main contribution is the identification of an iteration scheme, a variation of Mann iteration with a dampening factor, which, under suitable conditions, is shown to guarantee convergence to the least fixpoint of the function of interest. We then argue that these results are relevant in the context of model-based reinforcement learning for Markov decision processes (MDPs), showing that the proposed iteration scheme instantiates to MDPs and allows us to derive convergence to the optimal expected return. More generally, we show that our results can be used to iterate to the least fixpoint almost surely for systems where the function of interest can be approximated with given probabilistic error bounds, as it happens for probabilistic systems, which can be explored via sampling.

## 4.7 Quantitative Graded Semantics, Modal Logics (and Games)

Lutz Schröder (Universität Erlangen-Nürnberg, DE)

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Joint work of Lutz Schröder, Jonas Forster, Paul Wild, Harsh Beohar, Sebastian Gurke, Barbara König, Karla Messing

Behavioural metrics provide a robust measure of the behaviour of systems with quantitative data, such as metric or probabilistic transition systems. In analogy to the linear-time/branching-time spectrum of two-valued behavioural equivalences on transition systems, behavioural metrics vary in granularity, and are often characterized by fragments of suitable modal logics. In the latter respect, the quantitative case is, however, more involved than the two-valued one; as a salient negative examples, we show that probabilistic metric trace distance cannot be characterized by any compositionally defined modal logic with unary modalities, in a well-delineated but broadly understood sense. We go on to provide a unifying treatment of spectra of behavioural metrics in the emerging framework of graded monads (Milius, Schröder, Pattinson, CALCO 2015), working in coalgebraic generality, that is, parametrically in the system type. In the ensuing development of *quantitative graded semantics*, we introduce algebraic presentations of graded monads on the category **Met** of metric spaces, generalizing previous systems aimed at presenting plain monads on **Met** (Mardare, Panangaden, Plotkin, LICS 2016). Moreover, we provide a general criterion for a given real-valued modal logic to characterize a given behavioural distance, phrased in terms of a separation property that depends both on the modalities and on the propositional operators of the logic. We apply this criterion to a number of case studies, notably including a new characteristic modal logic for trace distance in fuzzy metric transition systems. In a concluding discussion, we give an outlook on generic game characterizations of behavioural metrics and indeed of more general behavioural conformances, such as behavioural preorders or bisimulation topologies.

## 4.8 A complete Quantitative Axiomatisation of Behavioural Distance of Regular Expressions

Wojciech Różowski (University College London, GB)

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Deterministic automata have been traditionally studied from the point of view of language equivalence, but another perspective is given by the canonical notion of shortest-distinguishing word distance quantifying the difference of states. Intuitively, the longer the word needed to observe a difference between two states, then the closer behaviour is. In this talk, I discuss a sound and complete axiomatisation of shortest-distinguishing-word distance between regular languages. The axiomatisation relies on a recently developed analogue of equational logic for approximate, quantitative reasoning, allowing to manipulate rational-indexed judgements of the form  $e =_{\epsilon} f$ , meaning term  $e$  is approximately equal to term  $f$  within the error margin of  $\epsilon$ . The completeness argument draws techniques from order theory and Banach spaces to simplify the calculation of behavioural distance to the point it can be then mimicked by axiomatic reasoning.

## 4.9 Contextual Distances for Probabilistic Higher-Order Languages

*Raphaëlle Crubillé (Aix-Marseille University, FR)*

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Joint work of Raphaëlle Crubillé, Houssein Mansour

Contextual equivalence considers two higher-order programs as equivalent when they behave the same in any given context, and contexts are modelled as programs written in the same language, making this definition self-contained. In a probabilistic setting, this definition can be made quantitative in a natural way: the distance between two programs  $M$  and  $N$  is defined as the supremum at the difference that a context can observe in the behaviour of  $M$  and  $N$ . However, it has been shown by Crubillé and Dal Lago that when all programs terminate with probability 1, this definition of contextual distance actually leads to a trivial pseudometric: two programs are either equivalent, or they are as far away as possible of each other. The original proof of this fact used techniques from real analysis with Bernstein's theorem as a pivotal tool. In this talk, I presented a new proof that highlights the deep connection of this trivialisation result with the law of large numbers.

## 4.10 Behavioural Metrics for Higher-Order Coalgebras

*Henning Urbat (Universität Erlangen-Nürnberg, DE)*

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Higher-order coalgebras are a convenient abstract setting for modelling higher-order languages and their operational semantics. In this talk, I introduce a fibrational approach to reasoning about congruence properties of higher-order coalgebras. It uniformly applies to various notions and applicative bisimilarity and applicative distances known in the literature, as well as to new settings such as higher-order languages combining nondeterministic and probabilistic effects.

## 4.11 Minimising the Probabilistic Bisimilarity Distance

*Qiyi Tang (University of Liverpool, GB)*

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A labelled Markov decision process (MDP) is a labelled Markov chain with nondeterminism; i.e., together with a strategy a labelled MDP induces a labelled Markov chain. The model is related to interval Markov chains. Motivated by applications to the verification of probabilistic noninterference in security, we study problems of minimising probabilistic bisimilarity distances of labelled MDPs, in particular, whether there exist strategies such that the probabilistic bisimilarity distance between the induced labelled Markov chains is less than a given rational number, both for memoryless strategies and general strategies. We show that the distance minimisation problem is  $\text{ExTh}(\mathbb{R})$ -complete for memoryless strategies and undecidable for general strategies. We also study the computational complexity of the qualitative problem about making the distance less than one. This problem is known to be NP-complete for memoryless strategies. We show that it is EXPTIME-complete for general strategies.

## 4.12 Bisimulation and Metrics for Continuous-Time Systems

*Florence Clerc (Heriot-Watt University – Edinburgh, GB)*

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**Joint work of** Florence Clerc, Linan Chen, Prakash Panangaden

In this talk, I will be looking at extending the notion of bisimulation and bisimulation metrics from discrete-time to continuous-time. The whole discrete-time approach relies entirely on the step-based dynamics: the process jumps from state to state. What we mean by continuous-time is that processes evolve continuously through time, such as Brownian motion or involve jumps or both.

I will first recall some previous work on how to extend the notion of bisimulation to continuous-time by considering trajectories and I will discuss the limits of this approach.

I will then generalize the concept of bisimulation metric in order to metrize the behaviour of continuous-time processes. Similarly to what is done for discrete-time systems, we follow two approaches and show that they coincide: as a fixpoint of a functional and through a real-valued logic.

## 4.13 Polytopal Stochastic Games

*Pedro d'Argenio (National University – Córdoba, AR)*

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Polytopal stochastic games are an extension of two-player, zero-sum, turn-based stochastic games. In these games the uncertainty over the probability distributions is captured via linear (in)equalities whose space of solutions forms a polytope. The idea has its root in a previous work (Castro et al. EXPRESS/SOS 2023) that explores a distance to quantify robustness of fault-tolerant algorithms. Such distance is the value of a total accumulative reward objective in a stochastic game where one of the players needs to choose transitions from a polytope of distributions. Polytopal stochastic games extend this idea so that both players can choose distribution from a finite set of polytopes of distributions. In the presentation I discuss reachability and different reward objectives and present results of determinacy, decidability and complexity. This presentation is based on a joint work with Pablo Castro (UNRC, CONICET) that has been recently submitted for publication.

## 4.14 The Kantorovich Pseudometric Is Not The Only Nice One

*Josée Desharnais (Université Laval – Québec, CA)*

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The most studied and accepted pseudo metric for probabilistic processes is one based on the Kantorovich distance between distributions. It comes with many theoretical and motivating results. With all these seals of approval, it has imposed itself as the one to study.

Other notions of pseudometrics have been proposed, one of them (epsilon-distance) based on epsilon-bisimulation. Its advantages over the Kantorovich distance are that it is intuitively easier to understand, it relates systems that have close probabilities even if these differences can imply very different behavior in the long run. For example an imperfect implementation can be considered close to its specification. Finally, it is easier to compute.

However, this approximate equivalence has only been defined on Markov chains in a structural way. We show that epsilon-distance is also the greatest fixpoint of a functor. The latter is obtained by replacing the Kantorovich distance in the lifting functor with the Lévy-Prokhorov distance.

## 4.15 Type Systems for Numerical Error Analysis

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Joint work of Justin Hsu, Ariel Kellison

Programs in numerical analysis and scientific computing make heavy use of floating-point numbers to approximate ideal real numbers. Operations on floating-point numbers incur roundoff error, and an important practical problem is to bound the overall magnitude of the error for complex computations. Existing work employs a variety of strategies such as interval arithmetic, SMT encodings, and global optimization; however, current methods are not compositional and are limited in scalability, precision, and expressivity.

Today, I'll talk about some of our recent work on NumFuzz, a higher-order language that can bound forward error using a linear, coefficient type system for sensitivity analysis combined with a novel quantitative error type. Subject to certain restrictions, our type inference procedure derives sound relative error bounds for programs that are several orders of magnitude larger than previously possible, while deriving relative error bounds that are several orders of magnitude more precise than prior work.

## 4.16 Dissimilarity for Linear Dynamical Systems

*Giovanni Bacci (Aalborg University, DK)*

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We introduce backward dissimilarity (BD) for discrete-time linear dynamical systems (LDS), which relaxes existing notions of bisimulations by allowing for approximate comparisons. BD is an invariant property stating that the difference along the evolution of the dynamics governing two state variables is bounded by a constant, which we call dissimilarity. We demonstrate the applicability of BD in a simple case study and showcase its use concerning: (i) robust model comparison; (ii) approximate model reduction; and (iii) approximate data recovery. Our main technical contribution is a policy-iteration algorithm to compute BDs. Using a prototype implementation, we apply it to benchmarks from network science and discrete-time Markov chains and compare it against a related notion of bisimulation for linear control systems.

## 5 Summary of Working Groups

### 5.1 Session on Generalised Kantorovich-Rubinstein Duality

*Barbara König (Universität Duisburg-Essen, DE), Jurriaan Rot (Radboud University – Nijmegen, NL)*

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We discussed generalisations of the classical Kantorovich-Rubinstein duality to a categorical level. While the primal and dual formulation coincide in the probabilistic case, this is not true any more when these liftings are generalized. In this case one type of lifting (sometimes called Wasserstein lifting) is based on couplings and takes the infimum while the other (sometimes called Kantorovich or codensity lifting) is based on non-expansive price functions and takes the supremum.

We started by discussing whether we assume the same types of predicate liftings (aka modalities) for both liftings, since allowing to vary the modalities would provide easy possibilities to represent one type of lifting as the other.

We continued with the assumption that the modalities are the same for both liftings. In the sequence, particular emphasis was on the subdistribution functor. The common generalisation of probabilistic couplings here enforces too strong a condition, suggesting that the functor lifting based on it does not yield the right notion; we discussed possible variants and Florence Clerc suggested some relevant literature.

A potential challenge to a general formulation of duality is that it should already include arguments for the well-known case of the distribution functor, which even in the discrete case is not immediate. Barbara König presented a concrete proof of this result and we discussed its possible adaptation to the general case.

### 5.2 Session on Quantitative Algebras/Coalgebras/Modal Logic

*Clemens Kupke (University of Strathclyde - Glasgow, GB)*


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The first point of the discussion was how to relate process algebras (an algebraic theory) with processes (coalgebras). One such connection can be obtained as follows: generate the free algebra (for the operations and equations of the process algebra) and turn this into coalgebra. In order to get all coalgebras (also circular ones) one would need to consider the Cauchy completion. Uniqueness of fixpoints was identified as very useful to be able to approximate this completion. A key problem for moving from processes to process algebra is to axiomatise equational theory generated by bisimulation.

The second point of the discussion concerned what a quantitative coequation is. One proposed idea was to use Stone duality to obtain the canonical model satisfying the “co-equation”. Expressiveness of coequations is fundamentally linked to HM theorems (examples are relational vs event bisimulation for Markov processes). An identified direction for further discussions is the following simple idea: coequations correspond to subcoalgebras of the final coalgebra. To obtain a quantitative notion of coequation we need to define an approximate/metric version of the notion of subcoalgebra.

### 5.3 Session on Applications of Behavioural Metrics

*Jurriaan Rot (Radboud University – Nijmegen, NL)*

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In this breakout session, we discussed possible application domains of behavioural metrics, and had an extensive broader discussion on relations between the theoretical work that receives a lot of emphasis in our community and broader applications and multi-disciplinary work.

We started with a general discussion of the role of theoretical work in computer science. A natural process is that we stay within a “theoretical bubble”, which limits potential applications and more long-term the viability and relevance of the field. We discussed whether everyone should “step out of the bubble” or not, what the challenges are for doing so, and how taking on this challenges can be rewarded in a better way than it is currently. We discussed how to enable and support interaction between theory and practice (both ways) by suitable ways of meeting and communicating as a broader scientific community, and had a brainstorm session about possibilities. Ideas that were mentioned included scientific meetings/conferences that focus on short talks and integrating the community rather than on proceedings; newsletters and social media; colocating workshops with bigger venues; and discussed several related areas where this seems to be going well.

More specifically, we identified application areas of behavioural metrics, including in explainability, differential privacy, machine learning and planning, reinforcement learning, and (model-based) software testing.

### 5.4 Session on Fixpoints

*Barbara König (Universität Duisburg-Essen, DE)*

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This breakout session was concerned with the topic of fixpoints in various contexts. Already in the introduction round on Monday many seminar participants stated an interest in fixpoint theory in general and for behavioural metrics in particular. Hence, the aim of the session was to reflect on the state of the art, exchange ideas and identify new research directions.

The discussion started with fixpoint games, in particular generalized fixpoint games in complete lattices. We also discussed the connection to mu-calculus and parity games and later to simple stochastic and coalgebraic games and techniques for computing behavioural metrics via solution procedures for games.

This triggered an exchange about algorithms and heuristics for computing behavioural metrics, including exploiting dependencies between variables, the worklist algorithm and chaotic iteration.

Subsequently Radu Mardare presented an axiomatization of fixpoints based on so-called Banach patterns and Banach theory. He also gave a comparison to the metric coinduction principle. The following discussion was rather broad, but for a while centered on the question of extending this theory to a setting with non-unique fixpoints (unlike the case of the Banach theorem that relies on contractions with unique fixpoints).

Afterwards the focus of the participants was on linear time (or trace) semantics and the question whether there is an increase in difficulty wrt. the qualitative case. We discussed a negative result on the existence of expressive logics from a previous talk by Lutz Schröder. In this context we also reviewed coalgebraic approaches for representing linear time such as coalgebras living in Kleisli and Eilenberg-Moore categories.

## 5.5 Session on Locksteps

*Florence Clerc (University of Strathclyde - Glasgow, GB)*

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We discussed about continuous-time Markov chains, continuous-time Markov decision processes and Markov automata and we explored the possibility of behavioural “scores” that consider the relative speed of the processes associating to the more standard bisimulation metric. We intend to compare this “score” to previously defined bisimulations on continuous-time Markov chains, continuous-time Markov decision processes and Markov automata.

## 5.6 Wrapup Session

*Barbara König (Universität Duisburg-Essen, DE)*

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The wrapup session gave all participants the opportunity to draw attention to discussions that took place in separate breakout sessions, reflect on the seminar and discuss followup activities. Only very few participants had already left, so that discussion took place with almost all of the participants present.

The session started with a report from the algebra/coalgebra session on Tuesday, where the participants pointed out that defining a quantitative notion of subcoalgebras (respectively coequations) is still an open question and would merit further investigation.

In addition there was a report from the participants of the application session on Thursday where the discussion had centered about strategies for escaping from the “theoretical bubble” as well as observations that theorems (also from category theory) are now starting to become useful in many application areas, such as in machine learning. It was emphasized that researchers should be encouraged to leave their special field and solve problems in other domains. Furthermore, some conferences (such as TACAS and FMICS) have tracks on applications that allow to publish corresponding papers on practical results.

This led to a discussion on followup activities. Ideas for concrete activities that were mentioned included:

- staying in contact via social media; establish a discussion forum (via Zulip?); announcements of new papers and developments
- organize a one-day workshop affiliated with a conferences such as CONCUR, ETAPS or LICS; such a workshop need not have proceedings, but should serve as a forum for staying in contact in the community
- organize/initiate more sessions and tutorials on behavioural metrics at conferences
- projects (in particular projects that provide money for exchange activities)
- mailing lists



In general the participants perceived the emergence of a new community triggered – among other activities – by the Dagstuhl Seminar.

The session and the entire seminar were concluded by a short presentation by Bart Jacobs where he suggested to include more intuition when talking about and teaching probability theory. In particular he reviewed the analogy to water pipes and pumps.

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