Report from Dagstuhl Seminar 25131

Weihrauch Complexity: Structuring the Realm of Non-Computability

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 25131 "Weihrauch Complexity: Structuring the Realm of Non-Computability". It includes an abstract of every talk given during the seminar, as well as summaries of all presentations from the sessions on open problems and new research directions. At the end is the latest version of a bibliography on Weihrauch complexity which was originally started a decade ago at the first Dagstuhl Seminar on the topic (15392).

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1 Executive Summary

Vasco Brattka (Universität der Bundeswehr – München, DE)
Alberto Marcone (University of Udine, IT)
Arno Pauly (Swansea University, GB)

This Dagstuhl Seminar is dedicated to the investigation of two active areas of research, one in theoretical computer science, the other in mathematical logic. These are computable analysis on the one hand, and reverse mathematics and applied computability theory on the other. That there is a deep connection between these areas was first suggested by Gherardi and Marcone (2008) and later independently by Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2016) and Hirschfeldt and Jockusch (2016). The past decade has seen this connection blossom into a rich and productive area of research, with by now many papers and several Ph.D. theses dedicated to it. Results in this area fall into two intertwined groups: Some clarify the structure of the degrees of non-computability; some further our understanding of the precise nature of non-computability of particular computational tasks of interest. Grasping

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the nature of non-computability is a profound goal mirroring the quest to understand the nature of computation. Knowing the degree of non-computability of a computational task brings with it answers as to whether weaker or approximate versions of it might be solvable. This interdisciplinary development was fostered not least by the two precursor Dagstuhl Seminars on this topic.[1]

The current seminar explored recent trends and results, open questions, and new directions of this fascinating field of research that has become known as Weihrauch complexity. The main part of each day was taken up by regular talks, with extra time set aside for two sessions devoted to open questions and new research directions, as well as plenty of opportunities for less structured socialization and collaboration. Although the ratio of number of talks to number of open questions (as represented in the sessions and this report) was nominally greater than in the previous seminar from 2018, a number of the talks themselves focused heavily on enumerating open questions and outlining future work, and indeed the field has only widened in the intervening years. To mention just a few highlights: investigations of the Weihrauch complexity of reverse-mathematical principles have continued to spur new developments, and this was reflected accordingly in many of the talks here, representing the study of "new" principles as well as new light still being shed on old ones. Important progress has also been made in our understanding of the properties of the Weihrauch lattice itself, such as the existence of uncountable chains and antichains and the density of the Weihrauch degrees above the identity map. Operators on Weihrauch degrees were a prominent theme during the seminar, featuring in the sessions on open problems and new research directions as well as being central to several talks. A few talks concerned recent work to place Weihrauch reducibility in context as an instance of a more general sort of object in category or topos theory.

Last but not least, underscoring the increasingly interdisciplinary interest in this subject, a well-attended joint evening session was spontaneously planned with the concurrent Dagstuhl Seminar[2] in which a speaker from each seminar gave an expository talk aimed at the other's participants: Kevin Schewior spoke about approximate sampling algorithms for stochastic function evaluation, and Arno Pauly about the non-computability of finding Nash equilibria.

This report includes the abstracts of all talks and other presentations given during the seminar (except for the joint talks), along with the most recent version of a bibliography on Weihrauch complexity which was begun during the first Dagstuhl Seminar on the topic in 2015. Altogether, this report reflects the high degree of productivity of our seminar, and we would like to use this opportunity to thank all participants for their valuable contributions and the Dagstuhl staff for their excellent support!

[1] Seminars 15392 and 18361; see `https://doi.org/10.4230/DagRep.5.9.77` and `https://doi.org/10.4230/DagRep.8.9.1`.

[2] Approximation Algorithms for Stochastic Optimization (25132; see `https://www.dagstuhl.de/25132`).

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3 Overview of Talks

3.1 A category-theoretic account of generalized Weihrauch degrees

Andrej Bauer (University of Ljubljana & Institute for Mathematics, Physics, and Mechanics – Ljubljana, SI)

Joint work of Danel Ahman, Andrej Bauer

In joint work with Danel Ahman [1] we developed and investigated a general theory of representations of second-order functionals, based on a notion of a right comodule for a monad on the category of containers. The theory can be used to give a type-theoretic account of instance reducibility [2] and, through their realizability interpretation, generalized Weihrauch degrees.

A *container* $A \triangleleft P$ is given by a type A and a type family $P \colon A \to \mathsf{Type}$. A morphism $f \triangleleft g \colon (A \triangleleft P) \to (B \triangleleft Q)$ is given by a map $f \colon A \to b$ and a map $g \colon \prod_{a:A} Q\,(f\,a) \to P\,a$. Containers have been studied extensively in type theory and functional programming.

A special case is a *propositional container* $A \triangleleft^{\mathsf{p}} P$, which is given by a type A and a predicate $P : A \to \mathsf{Prop}$. A morphism of propositional containers $f \colon (A \triangleleft^{\mathsf{p}} P) \to (B \triangleleft^{\mathsf{p}} Q)$ is a map $f \colon A \to B$ such that

$$\forall a{:}A.\, Q\,(f\,a) \Rightarrow P\,a.$$

In terms of instance degrees, such a map f is a *functional instance reduction*. The majority of instance reductions seen in mathematical practice (both classical and constructive) are of this kind.

The notion of *instance reducibility*, which states that $A \triangleleft^{\mathsf{p}} P$ is reducible to $B \triangleleft^{\mathsf{p}} Q$ when

$$\forall a{:}A.\, \exists b{:}B.\, (Q\,b \Rightarrow P\,a),$$

can be accounted for in terms of the general theory of representations of second-order functionals. Namely, it corresponds to the preorder reflection of the Kleisli category for the inhabited powerset monad on the category of propositional containers [1, Prop. 8.5].

These observations open up the possibility for generalizations of Weihrauch degrees, and application of type-theoretic and category-theoretic techniques to the topic. They also show how Weihrauch reducibility is situated in the wider context of representations of second-order functionals.

References

1 Danel Ahman and Andrej Bauer. Comodule representations of second-order functionals. *Journal of Logical and Algebraic Methods in Programming*, 146:101071, 2025.
2 Andrej Bauer. Instance reducibility and Weihrauch degrees. *Logical Methods in Computer Science*, 18(3), 2022.

3.2 Survey on Weihrauch Complexity: Scaffolding, Operators, Dichotomies

Vasco Brattka (Universität der Bundeswehr – München, DE)

Main reference Vasco Brattka: "The discontinuity Problem", J. Symb. Log., Vol. 88(3), pp. 1191 – 1212, 2023.
URL https://doi.org/10.1017/JSL.2021.106

We give a survey on basic problems, operators and dichotomies in Weihrauch complexity. In particular, we describe how LPO and LLPO can be used together with operators such as jump, parallelization, diamond, first-order part, and deterministic part to generate a whole class of very basic and important Weihrauch degrees. We describe how these degrees give natural classes of computable problems and how they match with systems in reverse mathematics. We also briefly discuss the role of closure and interior operators. Finally, we show how some of these degrees also lead to dichotomies for continuous problems with respect to continuous Weihrauch reducibility and different codomains. We close with a brief demonstration of how such dichotomies can be de-uniformized with the help of parallelization in order to obtain dichotomies for computable reducibility.

3.3 Effective Reducibility Notions with Transfinite Machine Models

Merlin Carl (Europa-Universität – Flensburg, DE)

Joint work of Merlin Carl, Lorenzo Galeotti, Robert Passmann

In recent years, various notions of effectivity and effective reducibility, such as Weihrauch reducibility and realizability, have been adapted to work on sets of arbitrary size by replacing Turing computability with computability by transfinite machine models, such as Koepke's Ordinal Turing Machines. In this talk, we will give an overview of this area with some of the central results, in particular concerning the mutual effective reducibility between the axioms and axiom schemes of ZFC usually regarded as non-constructive or impredicative, such as the power set axiom, the axiom of choice, and the schemes of separation and replacement.

References

1. Merlin Carl. Effectivity and reducibility with ordinal Turing machines. *Computability* 10(4) (2021), 289-304. doi: doi:10.3233/COM-210307.
2. Robert Passmann. The first-order logic of CZF is intuitionistic first-order logic. *Journal of Symbolic Logic* 89(1) (2022), 308-330. doi:10.1017/jsl.2022.51.
3. Merlin Carl, Lorenzo Galeotti, and Robert Passmann. Realisability for infinitary intuitionistic set theory. *Annals of Pure and Applied Logic* 174(6):103259 (2023). doi: doi:10.1016/j.apal.2023.103259.
4. Merlin Carl. Full generalized effective reducibility. Submitted (2025). arXiv: 2411.19386.

3.4 A well-quasi-order for continuous functions

Raphaël Carroy (University of Torino, IT)

Joint work of Raphaël Carroy, Yann Pequignot
Main reference Raphaël Carroy, Yann Pequignot: "A well-quasi-order for continuous functions", CoRR, Vol. abs/2410.13150, 2024.
URL https://arxiv.org/abs/2410.13150

We prove that continuous reducibility – or topological strong Weihrauch reducibility – on continuous functions from a 0-dimensional analytic domain to a separable metrizable space is a well-quasi-order, or more precisely, a better-quasi-order. To do so, we introduce and describe the class of scattered continuous functions with a 0-dimensional domain.

3.5 The category of quasi-Polish spaces as a represented space

Matthew de Brecht (Kyoto University, JP)

Main reference Matthew de Brecht: "The category of quasi-Polish spaces as a represented space", 2021
URL https://www.mathsoc.jp/section/topology/topsymp/2021/ts2021Brecht.pdf

We construct the category of quasi-Polish spaces as a represented space, which allows us to investigate the computability aspects of some category theoretical constructions, such as functors and limits, within the framework of Type-Two Theory of Effectivity. As an example, we demonstrate the computability of the lower, upper, double, and valuation powerspace endofunctors on the category of quasi-Polish spaces. (This talk was originally presented at the 68th Topology Seminar, August 2021: `https://www.mathsoc.jp/section/topology/topsymp.html`)

3.6 The tree pigeonhole principle in the Weihrauch degrees

Damir D. Dzhafarov (University of Connecticut – Storrs, US)

Joint work of Damir D. Dzhafarov, Reed Solomon, Manlio Valenti
Main reference Damir D. Dzhafarov, Reed Solomon, Manlio Valenti: "The Tree Pigeonhole Principle In The Weihrauch Degrees", The Journal of Symbolic Logic, p. 1–23, 2025.
URL https://doi.org/10.1017/jsl.2025.11

I will discuss recent work studying versions of the tree pigeonhole principle, TT^1, in the context of Weihrauch-style computable analysis. The principle has previously been the subject of extensive research in reverse mathematics, an outstanding question of which investigation is whether TT^1 is Π^1_1-conservative over the ordinary pigeonhole principle, RT^1. Using the recently introduced notion of the first-order part of an instance-solution problem, we formulate the analogue of this question for Weihrauch reducibility, and give an affirmative answer. In combination with other results, we use this to show that unlike RT^1, the problem TT^1 is not Weihrauch requivalent to any first-order problem. Our proofs develop new combinatorial machinery for constructing and understanding solutions to instances of TT^1. This is joint work with Reed Solomon and Manlio Valenti.

3.7 No dilator characterizes Ramsey's theorem for pairs

Anton Freund (Universität Würzburg, DE)

Main reference Anton Freund: "Dilators and the reverse mathematics zoo", Journal of Mathematical Logic, p. 2550010, 0.
URL https://doi.org/10.1142/S0219061325500102

Dilators are particularly uniform transformations of well-orders. Above ACA_0, every Π^1_2 statement corresponds to a dilator, by a classical result of Girard. In contrast, we show that no dilator corresponds to Ramsey's theorem for pairs and two colours (and the same is true for many other principles from the reverse mathematics zoo). Our proof involves a new principle of slow transfinite Π^0_2-induction, which admits a recursive counterexample but seems to lie below the Turing jump (though the latter is an open conjecture).

3.8 Formalization of Weihrauch reducibility in second-order arithmetic between existence statements

Makoto Fujiwara (Tokyo University of Science, JP)

Joint work of Makoto Fujiwara, Yudai Suzuki
Main reference Makoto Fujiwara and Yudai Suzuki. Formalization of Weihrauch reducibility in second-order arithmetic between existence statements. Accepted to *Computability*.

We formalize the notion of Weihrauch reducibility between existence statements in terms of second-order arithmetic [1], which is a standard framework of reverse mathematics. This formalization enables us to determine the strength of verification theories needed for Weihrauch reducibility between existence statements. As an example, we show that for any second-order theory T which is an extension of RCA_0, weak König's lemma with a uniqueness hypothesis is Weihrauch reducible to the identity map in T if and only if T proves weak König's lemma. This is joint work with Yudai Suzuki.

References

1 S. G. Simpson. *Subsystems of Second Order Arithmetic*, 2nd ed. Cambridge University Press, 2009.

3.9 Reverse Math of Regular Countable Second Countable Spaces

Giorgio G. Genovesi (University of Leeds, GB)

Main reference Giorgio G. Genovesi: "Reverse mathematics of regular countable second countable spaces", CoRR, Vol. abs/2410/22227, 2024.
URL https://arxiv.org/abs/2410.22227

One approach to studying theorems of general topology in second order arithmetic is to consider the countable second countable spaces, or CSC spaces. There are several classical theorems in general topology which characterize the regular CSC spaces. We go over the strength of some of these theorems in relation to the Big Five systems of second order arithmetic. We also outline how ATR_0 proves that regular Hausdorff CSC spaces are a well-quasi-order under embedding.

3.10 Pigeonhole principles for countable structures

Kenneth Gill (La Salle University – Philadelphia, US)

Joint work of Kenneth Gill Damir Dzhafarov, Reed Solomon
Main reference Kenneth Gill: "Indivisibility and uniform computational strength", Log. Methods Comput. Sci., Vol. 21(2), 2025.
URL https://doi.org/10.46298/LMCS-21(2:22)2025

A countable structure is said to be indivisible if for every presentation and every bounded coloring of the presentation, there is a monochromatic substructure isomorphic to the whole structure. Examples include the natural numbers, Rado and Henson graphs, and nonscattered linear orders. This notion naturally gives rise to an instance-solution problem which outputs such a substructure given a presentation and coloring. We discuss the Weihrauch degrees of these problems in general and for some specific structures, surveying what is known and highlighting current investigations. This is (in part) joint ongoing work with Damir Dzhafarov and Reed Solomon.

3.11 Forests Describing Topological Weihrauch Degrees of Functions with Discrete Range

Peter Hertling (Universität der Bundeswehr – München, DE)

We show that a certain initial segment of the degree structure of functions with discrete, possibly infinite, range under continuous Weihrauch reducibility is isomorphic to a hierarchy of labeled forests with respect to a suitable reducibility relation. We also present an explicit calculation of the degree structure of the topological Weihrauch degrees of functions of level of discontinuity at most 4.

References

1 Peter Hertling. *Unstetigkeitsgrade von Funktionen in der effektiven Analysis*. PhD thesis. Fachbereich Informatik, FernUniversität Hagen, 1996.
2 Peter Hertling. Forests describing Wadge degrees and topological Weihrauch degrees of certain classes of functions and relations. *Computability* 9 (2020), 249–307. doi:10.3233/COM-190255.

3.12 Basis theorems: Reverse mathematics and Weihrauch reductions

Jeffry L. Hirst (Appalachian State University – Boone, US)

Joint work of Caleb Davis, Silva Keohulian, Brody Miller, and Jessica Ross, and separately, with Carl Mummert
Main reference Caleb Davis, Jeffry Hirst, Silva Keohulian, Brody Miller, Jessica Ross: "Reverse mathematics of a pigeonhole basis theorem". To appear in *Computability* (2025).
URL https://hirstjl.github.io/bib/pdf/cb111024LargePrint.pdf

There are a number of basis theorems that are equivalent to Σ^0_2 induction in the reverse mathematics framework. For example, the color basis theorem and the basis theorem for finite dimensional e-matroids are provably equivalent. They are not Weihrauch equivalent. See

[1] and [2]. Insights from Weihrauch analysis can motivate interesting reformulations of the reverse mathematics results. Other examples of statements equivalent to Σ^0_2 induction with various Weihrauch strengths can be found in the recent work of Pauly, Pradic, and Soldà [3].

References

1 Caleb Davis, Jeffry Hirst, Silva Keohulian, Brody Miller, and Jessica Ross. Reverse mathematics of a pigeonhole basis theorem. To appear in *Computability* (2025).
2 Jeffry Hirst and Carl Mummert. Reverse mathematics of matroids. In Adam Day et al. (editors), *Computability and Complexity*, Lecture Notes in Computer Science vol. 10010, 143-159. Cham: Springer, 2017. doi: doi:10.1007/978-3-319-50062-1_12.
3 Arno Pauly, Cécilia Pradic, and Giovanni Soldà. On the Weihrauch degree of the additive Ramsey theorem. *Computability* 13(3-4) (2024), 459–483. doi:10.3233/COM-230437.

3.13 Generalized Weihrauch reducibility

Takayuki Kihara (Nagoya University, JP)

I will give an overview of generalized Weihrauch reducibility from the perspectives of computability theory, reverse mathematics, and realizability topos theory, with concrete examples and applications. This talk will cover the following topics: compositional product, reduction game, Weihrauch-oracle realizability, constructive reverse mathematics, realizability topos, Lawvere-Tierney topology, subtopos, and extended generalized Weihrauch reducibility.

References

1 Takayuki Kihara. Lawvere-Tierney topologies for computability theorists. *Trans. Amer. Math. Soc. Series B* 10 (2023), 48-85.
2 Takayuki Kihara. Rethinking the notion of oracle: A prequel to Lawvere-Tierney topologies for computability theorists. Preprint (2022). arXiv: 2202.00188.

3.14 Recent applications of proof mining to splitting algorithms

Ulrich Kohlenbach (TU Darmstadt, DE)

Splitting methods play a central role in nonsmooth optimization in the design of algorithms for the computation of zeros of maximally monotone set-valued operators in Hilbert spaces which can be written as the sum $A + B$ of two such operators. The main point here is to avoid the use of the resolvent of $A + B$ and to involve only the individual resolvents J_A, J_B of A and B respectively, which may be easier to compute (note that to compute the resolvents of an operator amounts to solving in inverse problem). The most well-studied such algorithms are (i) Tseng's Splitting Algorithm, (ii) the Forward-Backward Splitting Algorithm, (iii) the Douglas-Rachford Splitting Algorithm and, as the limiting case of (iii), (iv) the Peaceman-Rachford Algorithm (see e.g. [1]).

In [7] and [5], the logic-based proof mining methodology ([2]) is used to extract rates of convergence in certain quantitative forms of uniform monotonicity which give rise to moduli of uniqueness and hence moduli of regularity in the sense of [6]. The existence of such moduli has been studied in terms of reverse mathematics and Weihrauch complexity in [3] and in terms of intuitionistic reverse mathematics recently in [4].

References

1 H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd ed. CMS Books in Mathematics. Cham: Springer, 2017.
2 Ulrich Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Heidelberg-Berlin: Springer, 2008.
3 Ulrich Kohlenbach. On the reverse mathematics and Weihrauch complexity of moduli of regularity and uniqueness. *Computability* 8 (2019), 377-387.
4 Ulrich Kohlenbach. On the computational content of moduli of regularity and their logical strength. Submitted.
5 Ulrich Kohlenbach and Nicholas Pischke. Quantitative results for a Tseng-type primal-dual method for composite monotone inclusions. submitted.
6 Ulrich Kohlenbach, Genaro Lopéz-Acedo, and Adriana Nicolae. Moduli of regularity and rates of convergence for Fejér monotone sequences. *Israel Journal of Mathematics* 232 (2019), 261-297.
7 Jacqueline Treusch and Ulrich Kohlenbach. Rates of convergence for splitting algorithms. To appear in *Israel Journal of Mathematics*.

3.15 Better quasi-orders on labelled trees

Davide Manca (Universität Würzburg, DE)

Main reference Davide Manca. *At the limits of predicativity: the reverse mathematics of ordering relations.* Ph.D. dissertation. To appear (2025).

Kruskal's theorem states that finite trees with labels in a well quasi-order (wqo) form a wqo under infima-preserving embeddings. Nash-Williams proved a version of that theorem for infinite trees, which relies on the stronger notion of better quasi-order [3] (see [1] for the result for labelled trees). That version has not yet been analyzed in an appropriate context, such as that of reverse mathematics. On the other hand, a number of weaker results about the structure of trees with labels in a better quasi-order have been studied, often in relation to open problems such as the strength of Fraïssé's conjecture [2]. We review the currently available results from the point of view of reverse mathematics and discuss some new ones, as well as some ideas for future research.

References

1 Richard Laver. On Fraïssé's order type conjecture. *Annals of Mathematics* 93(1) (1971).
2 Antonio Montalbán. Fraïssé's conjecture in Π^1_1-comprehension. *Journal of Mathematical Logic* 17(2) (2017).
3 C. St. J. A. Nash-Williams. On well-quasi-ordering infinite trees. *Mathematical Proceedings of the Cambridge Philosophical Society* 61(3) (1965).

3.16 The Galvin-Prikry theorem in the Weihrauch lattice

Alberto Marcone (University of Udine, IT)

Joint work of Alberto Marcone, Gian Marco Osso
Main reference Alberto Marcone, Gian Marco Osso: "The Galvin-Prikry Theorem in the Weihrauch lattice", CoRR, Vol. abs/2410/06928, 2024.
URL https://doi.org/10.48550/arXiv.2410.06928

We address the classification of different fragments of the Galvin-Prikry theorem in terms of their uniform computational content. We show that functions related to the Galvin-Prikry theorem for Borel sets of rank n are strictly between the $(n+1)$th and nth iterate of the hyperjump operator. To this end we establish the following result: a Turing jump ideal containing homogeneous sets for all $\Delta^0_{n+1}(X)$ sets must also contain the nth hyperjump of X. Similar results also hold for Borel sets of transfinite rank. These findings yield a partial refinement of previous results in the reverse mathematics of the Galvin-Prikry theorem. Moreover, in combination with previous results of Marcone and Valenti, they allow us to obtain a fairly complete picture of the Weihrauch degrees of the functions studied.

3.17 Indices, Computable Discontinuities and the Recursion Theorem

Daniel Mourad (Nanjing University, CN)

Consider a problem P with at least one computable instance. Let P' be the problem whose instances are indices n such that the nth computable partial function ϕ_n is an instance of P and such that $P'(n) = P(\phi_n)$. We investigate the relationship between discontinuity of P and computability of P'. We show that if P has a computable discontinuity (which we will define) then P' is not computable. This fact generalizes many applications of the recursion theorem, such as showing that P' is not computable when $P = \mathsf{WKL}$ or $P = \mathsf{RT}^1_1$. We also pose some questions about how having the index of a solution rather than the set that the index encodes influences Weihrauch reductions.

3.18 The equational theory of the Weihrauch degrees

Arno Pauly (Swansea University, GB)

Joint work of Arno Pauly, Eike Neumann, Cécilia Pradic
Main reference Eike Neumann, Arno Pauly, Cécilia Pradic: "The equational theory of the Weihrauch lattice with multiplication", CoRR, Vol. abs/2403.13975, 2024.
URL https://doi.org/10.48550/ARXIV.2403.13975

The algebraic structure of the Weihrauch degrees has long been a subject of study. It is linked to the "inherent logic of computability". Identifying the Weihrauch degrees as an instance of a previously studied class of structures, in particular one with a logical flavour, could significantly advance our understanding.

Here we study the equational theory of the Weihrauch lattice with multiplication, meaning the collection of equations between terms built from variables, the lattice operations $\sqcup$ and $\sqcap$, the product $\times$, and the finite parallelization $(\cdot)^*$ which are true however we substitute Weihrauch degrees for the variables. We provide a combinatorial description of these in terms of a reducibility between finite graphs, and moreover, show that deciding which equations are true in this sense is complete for the third level of the polynomial hierarchy. Pradic has similarly studied the equational structure of the Weihrauch lattice with composition.

References

1 Vasco Brattka and Arno Pauly. On the algebraic structure of Weihrauch degrees. *Logical Methods in Computer Science* 14(4) (2018). doi:10.23638/LMCS-14(4:4)2018.

2 Kojiro Higuchi and Arno Pauly. The degree structure of Weihrauch-reducibility. *Logical Methods in Computer Science* 9(2) (2011). doi:10.2168/LMCS-9(2:2)2013.

3 Cécilia Pradic. The equational theory of the Weihrauch lattice with (iterated) composition. Preprint (2024). arXiv: 2408.14999.

3.19 Weihrauch problems are containers
The equational theory of slightly extended Weihrauch degrees with composition

Cécilia Pradic (Swansea University, GB)

Joint work of Cécilia Pradic, Ian Price
Main reference Cécilia Pradic, Ian Price: "Weihrauch problems as containers", CoRR, Vol. abs/2501.17250, 2025.
URL https://doi.org/10.48550/ARXIV.2501.17250

I'll explain that Weihrauch problems can be regarded as containers over the category of subspaces of Baire spaces and computable maps and that Weihrauch reductions correspond exactly to container morphisms. Up to restricting to those containers that do not allow a problem not to answer a question, we get a clean equivalence. We can make similar observations and elaborations regarding extended/generalized/strong Weihrauch reducibility.

In the second part of the talk, I will discuss the equational theory of the Weihrauch lattice equipped with (iterated) composition. Terms in this theory can be translated to alternating automata, and reductions regarded as a somewhat weird kind of simulation. This leads to decidability and a complete axiomatization that includes a generalization of a result of Linda Westrick.

References

1 Cécilia Pradic. The equational theory of the Weihrauch lattice with (iterated) composition. Preprint (2025). arXiv: 2408.14999.

2 Cécilia Pradic and Ian Price. Weihrauch problems as containers. Preprint (2025). arXiv: 2501.17250.

3 Linda Westrick. A note on the diamond operator. *Computability* 10 (2023).

3.20 Principal Spaces

Matthias Schröder (TU Darmstadt, DE)

We introduce the class of principal topological spaces. Principal spaces have some bizarre properties which might be useful in Computability Theory. For example, they admit some automatic continuity properties.

Under the Axiom of Choice, principal spaces are very rare: no infinite Hausdorff space is principal under AC. By contrast, in Shelah's model of set theory and thus under the Axiom of Determinancy a big class of topological spaces relevant to Computable Analysis turn out to be principal, including all computable metric spaces and, more generally, all functionally Hausdorff qcb-spaces.

References

1. Eric Schechter. *Handbook of Analysis and Its Foundations*. Academic Press, 1997,
2. Matthias Schröder. Admissibly represented spaces and qcb-spaces. In Vasco Brattka and Peter Hertling, editors, *Handbook of Computability and Complexity in Analysis*, 305-346. Cham: Springer, 2021. doi:10.1007/978-3-030-59234-9_9.

3.21 Old directions in degree theory

Mariya I. Soskova (University of Wisconsin – Madison, US)

I was asked to present a brief overview of aspects of degree theory that have been studied throughout the years. The intention was that researchers interested in the Weihrauch degrees may use this as a source for questions that they may pursue. I focused on the following aspects:

I discussed the complexity of the theory of Turing degrees and its fragments when restricted to statements of limited quantifier complexity. I proposed the following questions about the Weihrauch lattice: How complicated are the fragments of the theory of $\mathcal{D}_{\mathrm{W}}$? At what quantifier level does decidability break down? Are there upper cones of Weihrauch degrees with a decidable/less complicated theory? Specifically, what about the cone above the degree of id?

I discussed the larger structure of the enumeration degrees and ways in which studying the Turing degrees within this larger context has been illuminating. I introduced the enumeration-Weihrauch degrees and suggested the following questions: Can enumeration Weihrauch reducibility be defined entirely in terms of Weihrauch reducibility à la Selman's theorem? How do other operators on the Weihrauch degrees live inside the $\leq_{\mathrm{eW}}$-degrees? Are the Weihrauch degrees definable in the $\leq_{\mathrm{eW}}$-degrees? What is the relationship between problems represented in the $\leq_{\mathrm{eW}}$-degrees and their total counterparts coming from the Weihrauch degrees?

I discussed local substructures such as the c.e. Turing degrees and ways in which working with them has expanded our toolbox (the priority method). I asked what local structures of the Weihrauch degrees arise naturally or determine the global structure.

Finally I discussed ways in which effective mathematics influences our view of degree structures and helps solve purely structural problems within and asked whether a similar phenomenon can be observed in the Weihrauch lattice.

3.22 Weihrauch degrees without roots

Patrick Uftring (Universität Würzburg, DE)

Main reference Patrick Uftring: "Weihrauch degrees without roots", CoRR, Vol. abs/2308.01422, 2023.
URL https://doi.org/10.48550/ARXIV.2308.01422

We answer the following question by Arno Pauly ([1, Open Question 12]): "Is there a square-root operator on the Weihrauch degrees?" In fact, we show that there are uncountably many pairwise incomparable Weihrauch degrees without any roots. We also prove that the omniscience principles LPO and LLPO do not have roots.

References

1 Arno Pauly. An update on Weihrauch complexity, and some open questions. Preprint (2020). arXiv:2008.11168.

3.23 An overview on the structure of the Weihrauch degrees

Manlio Valenti (Swansea University, GB)

In this talk, I will provide an overview of what is currently known about the structural properties of the Weihrauch degrees, including some of the more recent results about the existence and properties of chains, antichains, intervals and minimal covers, strong minimal covers, minimal pairs, and embeddings. I will also highlight some open questions and research directions.

References

1 Uri Andrews, Steffen Lempp, Alberto Marcone, Joseph S. Miller, and Manlio Valenti. A jump operator on the Weihrauch degrees. To appear in *Computability*. arXiv: 2402.13163.
2 Vasco Brattka and Guido Gherardi. Weihrauch degrees, omniscience principles and weak computability. *The Journal of Symbolic Logic* 76(1) (2011), 143–176. doi:10.2178/jsl/1294170993. arXiv: 0905.4679.
3 Vasco Brattka, Guido Gherardi, and Arno Pauly. Weihrauch complexity in computable analysis. In Vasco Brattka and Peter Hertling (editors), *Handbook of Computability and Complexity in Analysis*, 367 – 417. Springer International Publishing, 2021. doi:10.1007/978-3-030-59234-9_11. arXiv: 1707.03202.
4 Elena Z. Dyment. On some properties of the Medvedev lattice. *Mathematics of the USSR-Sbornik* 30(3) (1976), 321 – 340. doi:10.1070/SM1976v030n03ABEH002277.
5 Kojiro Higuchi and Arno Pauly. The degree structure of Weihrauch reducibility. *Logical Methods in Computer Science* 9(2:02) (2013), 1 – 17. doi:10.2168/LMCS-9(2:02)2013. arXiv: 1101.0112.

6 Steffen Lempp, Alberto Marcone, and Manlio Valenti. Chains and antichains in the Weihrauch lattice. Preprint (2024). arXiv: 2411.07792.
7 Steffen Lempp, Joseph S. Miller, Arno Pauly, Mariya I. Soskova, and Manlio Valenti. Minimal covers in the Weihrauch degrees. *Proceedings of the American Mathematical Society* 152(11) (2024), 4893 – 4901. doi:10.1090/proc/16952. arXiv: 2311.12676.
8 Arno Pauly. On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly* 56(5) (2010), 488 – 502. doi:10.1002/malq.200910104. arXiv: 10.1002/malq.200910104.
9 Paul Shafer. *On the complexity of mathematical problems: Medvedev degrees and reverse mathematics*. Ph.D. thesis. Cornell University, 2011.
10 Andrea Sorbi. The Medvedev lattice of degrees of difficulty. In *Computability, Enumerability, Unsolvability*, London Math. Soc. Lecture Note Series vol. 224, 289 – 312. New York: Cambridge University Press, 1996. doi:10.1017/CBO9780511629167.015.
11 Sebastiaan A. Terwijn. On the Structure of the Medvedev Lattice. *The Journal of Symbolic Logic* 73(2) (2008), 543 – 558. doi:10.2178/jsl/1208359059. Available at `http://www.jstor.org/stable/27588647`.

3.24 On the hierarchy above ATR in Weihrauch degrees and reverse mathematics

Keita Yokoyama (Tohoku University, JP)

Joint work of Keita Yokoyama, Yudai Suzuki

In the study of reverse mathematics, the gap between ATR_0 and Π^1_1-CA_0 is rather large, with many mathematical theorems falling in between. We focus on those theorems which are described by Π^1_2-sentences and examine the hierarchy above arithmetical transfinite recursion in the context of Weihrauch degrees and reverse mathematics. This is joint work with Yudai Suzuki.

References

1 Yudai Suzuki and Keita Yokoyama. Searching problems above arithmetical transfinite recursion. *Annals of Pure and Applied Logic* 175 (2024), 31pp. doi:10.1016/j.apal.2024.103488.
2 Yudai Suzuki and Keita Yokoyama. On the Π^1_2 consequences of Π^1_1-CA_0. Preprint (2024). arXiv: 2402.07136.

4 Open problems

4.1 What to do about all the other Weihrauch lattices?

Andrej Bauer (University of Ljubljana & Institute for Mathematics, Physics, and Mechanics – Ljubljana, SI)

The lattice of (generalized) Weihrauch degrees arises as the lattice of instance degrees [1], interpreted in the Kleene-Vesley topos $\mathsf{RT}(\mathbb{N}^{\mathbb{N}}, (\mathbb{N}^{\mathbb{N}})_{\mathsf{eff}})$, based on the function realizability model [2]. However, the instance degrees may be calculated in any topos to give many new variants of Weihrauch reduction. For example, in the relative realizability topos $\mathsf{RT}(\mathcal{P}\omega, (\mathcal{P}\omega)_{\mathsf{eff}})$ based on Scott's graph model, we obtain the so-called *enumeration* Weihrauch lattice.

More generally, any partial combinatory algebra $\mathbb{A}$ with an elementary subalgebra $\mathbb{A}'$ begets a relative realizability topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}')$, see [5], and thereby a Weihrauch-style reducibility lattice $\mathcal{W}_{\mathbb{A},\mathbb{A}'}$. Of particular interest are examples of pcas $\mathbb{A}$ that are also topological spaces, with $\mathbb{A}'$ their effective parts. Among these are van Oosten's pca of sequential functionals, universal Scott domain $\mathbb{U}$, Plotkin's universal coherent domain $\mathtt{T}^{\omega}$, and others.

We propose a new direction of research that studies the alternative Weihrauch lattices. We expect that John Longley's notion of *simulation* [3], also known as as *applicative morphism*, and his analysis of topological pcas [4] will be of some help in establishing basic results, and in particular in relating the variants of Weihrauch lattices.

References

1 Andrej Bauer. Instance reducibility and Weihrauch degrees. *Logical Methods in Computer Science*, 18(3), 2022.
2 Stephen Cole Kleene and Richard Eugène Vesley. *The Foundations of Intuitionistic Mathematics, especially in relation to recursive functions.* North-Holland Publishing Company, 1965.
3 J. Longley. *Realizability Toposes and Language Semantics.* PhD thesis, Edinburgh University, 1995.
4 John Longley. On the ubiquity of certain total type structures. *Mathematical Structures in Computer Science*, 17(5):841 – 953, 2007.
5 Jaap van Oosten. *Realizability: An Introduction To Its Categorical Side*, volume 152 of *Studies in logic and the foundations of mathematics.* Elsevier, 2008.

4.2 Interior operators in the Weihrauch lattice

Jun Le Goh (National University of Singapore, SG)

Joint work of Jun Le Goh, Vasco Brattka, Damir Dzhafarov, Reed Solomon, Keita Yokoyama, Vittorio Cipriani, Arno Pauly

1. Brattka defined an interior operator on the Weihrauch degrees called the upper Turing cone version of a problem. This problem is induced by the closure operator on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ given by upward closure under Turing reducibility.
   Question: Which other interior operators can we form by considering closure operators on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$?

2. The first-order part of a problem f is the maximum Weihrauch degree of a problem g with codomain $\mathbb{N}$ which reduces to f (Dzhafarov, Solomon, Yokoyama). The k-finitary part of a problem f is the maximum Weihrauch degree of a problem g with codomain $\mathbf{k}$ which reduces to f (Cipriani, Pauly). Pauly observed during this Dagstuhl meeting that for each represented space $\mathbf{X}$ and each problem f, the maximum Weihrauch degree among all problems with codomain $\mathbf{X}$ which reduce to f exists.
   Question: For which other represented spaces is this maximum useful? How about Sierpinski space?

4.3 Question on the strength of the infinite loop closure

Takayuki Kihara (Nagoya University, JP)

Recently, Brattka introduced the notion of infinite loop operation on Weihrauch problems. Applying Yoshimura's unpublished theorem, one can see that the Weihrauch problem F is closed under the infinite loop operation (the inverse limit) if and only if F-relative realizability validates the axiom of dependent choice. Therefore, it is an important problem to investigate which Weihrauch problems are closed under the infinite loop operation. Here, we ask about the strength of the infinite loop closure of LLPO_k (all-or-counique choice on k).

Question: Is $\mathsf{LLPO}_{k+1}^{\infty\infty\infty\cdots} <_{\mathrm{W}} \mathsf{LLPO}_{k}^{\infty\infty\infty\cdots}$?

This problem was solved by myself during the conference. That is, $\mathsf{LLPO}_{k}^{\infty\infty\infty\cdots}$ is equivalent to DNR_k, and thus the problem is positively resolved.

4.4 Strong Weihrauch compositional product

Alberto Marcone (University of Udine, IT)

Joint work of Alberto Marcone, Gian Marco Osso
Main reference Alberto Marcone, Gian Marco Osso: "The Galvin-Prikry Theorem in the Weihrauch lattice", CoRR, Vol. abs/2410/06928 2024.
URL https://arxiv.org/abs/2410.06928

In the Weihrauch degrees we have an explicit definition of a multi-valued function $f \star g$ such that

$$f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{h \circ k : h \leq_{\mathrm{W}} f \wedge k \leq_{\mathrm{W}} g\}.$$

In the paper with Gian Marco Osso we define a multi-valued function $f \tilde{\star} g$ such that if g is a cylinder then

$$f \tilde{\star} g \equiv_{\mathrm{sW}} \max_{\leq_{\mathrm{sW}}}\{h \circ k : h \leq_{\mathrm{sW}} f \wedge k \leq_{\mathrm{sW}} g\}.$$

$\tilde{\star}$ has some nice properties:

- $(f \tilde{\star} g) \tilde{\star} h \equiv_{\mathrm{W}} f \tilde{\star} (g \tilde{\star} h)$;
- $(\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times f) \tilde{\star} g \equiv_{\mathrm{W}} f \star g$;
- if $g_0 \leq_{\mathrm{W}} g_1$, then $f \tilde{\star} g_0 \leq_{\mathrm{W}} f \tilde{\star} g_1$;
- if $f_0 \leq_{\mathrm{sW}} f_1$ and $g_0 \leq_{\mathrm{sW}} g_1$, then $f_0 \tilde{\star} g_0 \leq_{\mathrm{sW}} f_1 \tilde{\star} g_1$.

However, we have examples where g is not a cylinder and $\max_{\leq_{\mathrm{sW}}}\{h \circ k : h \leq_{\mathrm{sW}} f \wedge k \leq_{\mathrm{sW}} g\}$ either exists but is not represented by $f \tilde{\star} g$ or does not exist.

It would be interesting to characterize when $\max_{\leq_{\mathrm{sW}}}\{h \circ k : h \leq_{\mathrm{sW}} f \wedge k \leq_{\mathrm{sW}} g\}$ exists and in those cases provide an explicit realizer of this strong Weihrauch degree.

4.5 A question about the the uniform content of index sets

Daniel Mourad (Nanjing University, CN)

Myhill showed that a set X is productive if and only if the complement of the halting set K is 1-reducible to X. It follows that the index sets Ind_n for partial computable functions are productive. Let IndRed be the problem which takes n and produces the graph of a 1-reduction from $\overline{K}$ to Ind_n. Let IndProd be the problem which takes n and produces a graph of a productive set for Ind_n. Myhill's proofs are uniform in the index: $\mathrm{G} \circ \mathrm{IndRed}$ is Weihrauch equivalent to $\mathrm{G} \circ \mathrm{IndProd}$, where G is the Gödel function which takes a computable set to one of its indices. It turns out that one does not need the index to produce a graph in one of the directions: IndRed is Weihrauch reducible to IndProd.

Questions: Is IndRed Weihrauch equivalent to IndProd? How about to $\mathrm{G} \circ \mathrm{IndRed}$?

4.6 On residual operators

Manlio Valenti (Swansea University, GB)

A residual lattice is a lattice equipped with a monoidal operator $*$ such that for every f and g there are maximum h and k such that $h * f \leq g$ and $f * k \leq g$. Given the large number of operators in the Weihrauch lattice, it is natural to ask what are the operators that make the Weihrauch lattice or its dual a residual lattice. Some of these questions have been already answered, but we still miss a complete picture. In particular, it is open whether there always exists a maximum h such that the compositional product $f * h$ is Weihrauch reducible to g.

4.7 Preservation results for well-quasiorders in the Weihrauch lattice

Arno Pauly (Swansea University, GB)

Results of the form "If X is well-quasiordered, then so is $F(X)$" for various constructions of quasi-orders F have been a fruitful subject of study in reverse mathematics. Kruskal's and Higman's theorems are probably the most famous example, but already "If α is an ordinal, so is 2^α" has non-trivial strength. At first glance, such results don't seem to have computational content per se. However, we can look at their contrapositives. The algorithmic task then becomes "Given a quasi-order X and a bad sequence in $F(X)$, find a bad sequence in X".

The task of finding a bad sequence in a quasi-ordered merely promised to be non-well was studied by Goh, Valenti and the author [1, 2]. By investigating how much the Weihrauch degree decreases if a bad sequence in $F(X)$ is provided as part of the input, we gain insight on how tightly the non-wqo-ness of $F(X)$ and X are linked in an effective way.

References

1 Jun Le Goh, Arno Pauly & Manlio Valenti. Finding descencing sequences through ill-founded linear orders. *Journal of Symbolic Logic* 86(2) (2021). doi:10.1017/jsl.2021.15.
2 Jun Le Goh, Arno Pauly & Manlio Valenti. The weakness of finding descending sequences in ill-founded linear orders. Preprint (2024). arXiv: 2401.11807.

4.8 A problem on the preservation of well-foundedness

Keita Yokoyama (Tohoku University, JP)

Consider the following condition for a real $X \in 2^\omega$: (†) if L is a computable linear order on ω with no computable infinite decreasing sequence, then X doesn't compute any infinite decreasing sequence for L.

Freund and Uftring [1] showed that if X is hyperimmune-free then X satisfies (†). Then, is the condition (†) equivalent to being hyperimmune-free?

Joseph Miller answered this question. If X is 1-generic, then X satisfies (†), and thus (†) is a strictly weaker notion than being hyperimmune-free.

References

1 Anton Freund and Patrick Uftring. More conservativity for weak Kőnig's lemma. Preprint (2024). arXiv: 2410.20591.

5 Bibliography on Weihrauch Complexity

For an always up-to-date version of this bibliography, see

`http://cca-net.de/publications/weibib.php`.

References

1 Nathanael Ackerman, Julian Asilis, Jieqi Di, Cameron Freer, and Jean-Baptiste Tristan. Computable PAC learning of continuous features. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '22, New York, NY, USA, 2022. Association for Computing Machinery.

2 Nathanael L. Ackerman, Cameron E. Freer, and Daniel M. Roy. On computability and disintegration. *Mathematical Structures in Computer Science*, 27(8):1287–1314, 2017.

3 Djamel Eddine Amir. *Computability of Topological Spaces.* Ph.D. thesis, Université de Lorraine, 2023.

4 Djamel Eddine Amir and Mathieu Hoyrup. Comparing computability in two topologies. hal-03702999, 2022.

5 Djamel Eddine Amir and Mathieu Hoyrup. Strong computable type. arXiv 2210.08309, 2022.

6 Djamel Eddine Amir and Mathieu Hoyrup. Strong computable type. *Computability*, 12(3):227–269, 2023.

7 Djamel Eddine Amir and Mathieu Hoyrup. Comparing computability in two topologies. *Journal of Symbolic Logic*, 89(3):1232–1250, 2024.

8 Paul-Elliot Anglès d'Auriac. *Infinite Computations in Algorithmic Randomness and Reverse Mathematics*. Ph.D. thesis, Université Paris-Est, 2020.

9 Paul-Elliot Anglès d'Auriac and Takayuki Kihara. A comparison of various analytic choice principles. *The Journal of Symbolic Logic*, 86(4):1452–1485, 2021.

10 Eric P. Astor, Damir D. Dzhafarov, Reed Solomon, and Jacob Suggs. The uniform content of partial and linear orders. *Annals of Pure and Applied Logic*, 168(6):1153 – 1171, 2017.

11 Andrej Bauer. Instance reducibility and Weihrauch degrees. arXiv 2106.01734, 2021.

12 Andrej Bauer. Instance reducibility and Weihrauch degrees. *Logical Methods in Computer Science*, 18(3):20:1–20:18, August 2022.

13 Nikolay Bazhenov, Marta Fiori-Carones, Lu Liu, and Alexander Melnikov. Primitive recursive reverse mathematics. *Annals of Pure and Applied Logic*, 175(1, Part A):103354, 2024.

14 Zach BeMent, Jeffry Hirst, and Asuka Wallace. Reverse mathematics and Weihrauch analysis motivated by finite complexity theory. *Computability*, 10(4):343–354, 2021.

15 Zack BeMent, Jeffry Hirst, and Asuka Wallace. Reverse mathematics and Weihrauch analysis motivated by finite complexity theory. arXiv 2105.01719, 2021.

16 Laurent Bienvenu and Rutger Kuyper. Parallel and serial jumps of Weak Weak König's Lemma. In Adam Day, Michael Fellows, Noam Greenberg, Bakhadyr Khoussainov, Alexander Melnikov, and Frances Rosamond, editors, *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, volume 10010 of *Lecture Notes in Computer Science*, pages 201–217. Springer, Cham, 2017.

17 H. Boche and V. Pohl. The solvability complexity index of sampling-based Hilbert transform approximations. In *2019 13th International conference on Sampling Theory and Applications (SampTA)*, pages 1–4, 2019.

18 Vasco Brattka. Computable invariance. In Tao Jiang and D.T. Lee, editors, *Computing and Combinatorics*, volume 1276 of *Lecture Notes in Computer Science*, pages 146–155, Berlin, 1997. Springer. Third Annual Conference, COCOON'97, Shanghai, China, August 1997.

19 Vasco Brattka. Computable invariance. *Theoretical Computer Science*, 210:3–20, 1999.

20 Vasco Brattka. Effective Borel measurability and reducibility of functions. *Mathematical Logic Quarterly*, 51(1):19–44, 2005.

21 Vasco Brattka. Computability and analysis, a historical approach. In Arnold Beckmann, Laurent Bienvenu, and Nataša Jonoska, editors, *Pursuit of the Universal*, volume 9709 of *Lecture Notes in Computer Science*, pages 45–57, Switzerland, 2016. Springer. 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016.

22 Vasco Brattka. The discontinuity problem. arXiv 2012.02143, 2020.

23 Vasco Brattka. Stashing-parallelization pentagons. *Logical Methods in Computer Science*, 17(4):20:1–20:29, 2021.

24 Vasco Brattka. Weihrauch complexity and the Hagen school of computable analysis. In Benedikt Löwe and Deniz Sarikaya, editors, *60 Jahre DVMLG*, volume 48 of *Tributes*, pages 13–44. College Publications, London, 2022.

25 Vasco Brattka. The discontinuity problem. *Journal of Symbolic Logic*, 88(3):1191–1212, 2023.

26 Vasco Brattka. On the complexity of computing Gödel numbers. arXiv 2302.04213, 2023.

27 Vasco Brattka. On the complexity of learning programs. In Gianluca Della Vedova, Besik Dundua, Steffen Lempp, and Florin Manea, editors, *Unity of Logic and Computation*, volume 13967 of *Lecture Notes in Computer Science*, pages 166–177, Cham, 2023. Springer. 19th Conference on Computability in Europe.

28 Vasco Brattka. Loops, inverse limits and non-determinism. arXiv arXiv:2501.17734, 2025.

29 Vasco Brattka, Andrea Cettolo, Guido Gherardi, Alberto Marcone, and Matthias Schröder. Addendum to: "The Bolzano-Weierstrass theorem is the jump of weak Kőnig's lemma". *Annals of Pure and Applied Logic*, 168(8):1605–1608, 2017.

30 Vasco Brattka, Matthew de Brecht, and Arno Pauly. Closed choice and a uniform low basis theorem. *Annals of Pure and Applied Logic*, 163:986–1008, 2012.

31 Vasco Brattka, Damir Dzhafarov, Alberto Marcone, and Arno Pauly, editors. *Special issue: Dagstuhl Seminar on Measuring the Complexity of Computational Content 2018*, volume 9 of *Computability - The Journal of the Association CiE*. IOS Press, 2020.

32 Vasco Brattka, Damir D. Dzhafarov, Alberto Marcone, and Arno Pauly, editors. *Measuring the Complexity of Computational Content: From Combinatorial Problems to Analysis (Dagstuhl Seminar 18361)*, volume 8 of *Dagstuhl Reports*, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

33 Vasco Brattka and Guido Gherardi. Borel complexity of topological operations on computable metric spaces. *Journal of Logic and Computation*, 19(1):45–76, 2009.

34 Vasco Brattka and Guido Gherardi. Effective choice and boundedness principles in computable analysis. In Andrej Bauer, Peter Hertling, and Ker-I Ko, editors, *CCA 2009, Proceedings of the Sixth International Conference on Computability and Complexity in Analysis*, pages 95–106, Schloss Dagstuhl, Germany, 2009. Leibniz-Zentrum für Informatik.

35 Vasco Brattka and Guido Gherardi. Weihrauch degrees, omniscience principles and weak computability. In Andrej Bauer, Peter Hertling, and Ker-I Ko, editors, *CCA 2009, Proceedings of the Sixth International Conference on Computability and Complexity in Analysis*, pages 83–94, Schloss Dagstuhl, Germany, 2009. Leibniz-Zentrum für Informatik.

36 Vasco Brattka and Guido Gherardi. Effective choice and boundedness principles in computable analysis. *The Bulletin of Symbolic Logic*, 17(1):73–117, 2011.

37 Vasco Brattka and Guido Gherardi. Weihrauch degrees, omniscience principles and weak computability. *The Journal of Symbolic Logic*, 76(1):143–176, 2011.

38 Vasco Brattka and Guido Gherardi. Weihrauch goes Brouwerian. arXiv 1809.00380, 2018.

39 Vasco Brattka and Guido Gherardi. Completion of choice. arXiv 1910.13186, 2019.

40 Vasco Brattka and Guido Gherardi. Weihrauch goes Brouwerian. *The Journal of Symbolic Logic*, 85(4):1614–1653, 2020.

41 Vasco Brattka and Guido Gherardi. Completion of choice. *Annals of Pure and Applied Logic*, 172(3):102914, 2021.

42 Vasco Brattka, Guido Gherardi, and Rupert Hölzl. Las Vegas computability and algorithmic randomness. In Ernst W. Mayr and Nicolas Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science (STACS 2015)*, volume 30 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 130–142, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

43 Vasco Brattka, Guido Gherardi, and Rupert Hölzl. Probabilistic computability and choice. *Information and Computation*, 242:249–286, 2015.

44 Vasco Brattka, Guido Gherardi, Rupert Hölzl, and Arno Pauly. The Vitali covering theorem in the Weihrauch lattice. In Adam Day, Michael Fellows, Noam Greenberg, Bakhadyr Khoussainov, Alexander Melnikov, and Frances Rosamond, editors, *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, volume 10010 of *Lecture Notes in Computer Science*, pages 188–200. Springer, Cham, 2017.

45 Vasco Brattka, Guido Gherardi, and Alberto Marcone. The Bolzano-Weierstrass theorem is the jump of weak Kőnig's lemma. *Annals of Pure and Applied Logic*, 163:623–655, 2012.

46 Vasco Brattka, Guido Gherardi, and Arno Pauly. Weihrauch complexity in computable analysis. arXiv 1707.03202, 2017.

47 Vasco Brattka, Guido Gherardi, and Arno Pauly. Weihrauch complexity in computable analysis. In Vasco Brattka and Peter Hertling, editors, *Handbook of Computability and Complexity in Analysis*, Theory and Applications of Computability, pages 367–417. Springer, Cham, 2021.

48 Vasco Brattka, Noam Greenberg, Iskander Kalimullin, and Mariya Soskova, editors. *Special issue: Oberwolfach Workshop on Computability Theory 2021*, volume 11 of *Computability - The Journal of the Association CiE*. IOS Press, 2022.

49 Vasco Brattka, Matthew Hendtlass, and Alexander P. Kreuzer. On the uniform computational content of computability theory. *Theory of Computing Systems*, 61(4):1376–1426, 2017.

50 Vasco Brattka, Matthew Hendtlass, and Alexander P. Kreuzer. On the uniform computational content of the Baire category theorem. *Notre Dame Journal of Formal Logic*, 59(4):605–636, 2018.

51 Vasco Brattka and Peter Hertling, editors. *Handbook of Computability and Complexity in Analysis*, Theory and Applications of Computability, Cham, 2021. Springer.

52 Vasco Brattka, Rupert Hölzl, and Rutger Kuyper. Monte Carlo computability. In Heribert Vollmer and Brigitte Vallée, editors, *34th Symposium on Theoretical Aspects of Computer Science (STACS 2017)*, volume 66 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 17:1–17:14, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

53 Vasco Brattka, Akitoshi Kawamura, Alberto Marcone, and Arno Pauly, editors. *Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis (Dagstuhl Seminar 15392)*, volume 5 of *Dagstuhl Reports*, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

54 Vasco Brattka, Stéphane Le Roux, Joseph S. Miller, and Arno Pauly. The Brouwer fixed point theorem revisited. In Arnold Beckmann, Laurent Bienvenu, and Nataša Jonoska, editors, *Pursuit of the Universal*, volume 9709 of *Lecture Notes in Computer Science*, pages 58–67, Switzerland, 2016. Springer. 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016.

55 Vasco Brattka, Stéphane Le Roux, Joseph S. Miller, and Arno Pauly. Connected choice and the Brouwer fixed point theorem. *Journal of Mathematical Logic*, 19(1):1–46, 2019.

56 Vasco Brattka, Stéphane Le Roux, and Arno Pauly. On the computational content of the Brouwer Fixed Point Theorem. In S. Barry Cooper, Anuj Dawar, and Benedikt Löwe, editors, *How the World Computes*, volume 7318 of *Lecture Notes in Computer Science*, pages 57–67, Berlin, 2012. Springer. Turing Centenary Conference and 8th Conference on Computability in Europe, CiE 2012, Cambridge, UK, June 2012.

57 Vasco Brattka and Arno Pauly. Computation with advice. In Xizhong Zheng and Ning Zhong, editors, *CCA 2010, Proceedings of the Seventh International Conference on Computability and Complexity in Analysis*, volume 24 of *Electronic Proceedings in Theoretical Computer Science*, pages 41–55, 2010.

58 Vasco Brattka and Arno Pauly. On the algebraic structure of Weihrauch degrees. *Logical Methods in Computer Science*, 14(4:4):1–36, 2018.

59 Vasco Brattka and Tahina Rakotoniaina. On the uniform computational content of Ramsey's theorem. *Journal of Symbolic Logic*, 82(4):1278–1316, 2017.

60 Vasco Brattka and Emmanuel Rauzy. Effective second countability in computable analysis. In Arnold Beckmann, Isabel Oitavem, and Florin Manea, editors, *Crossroads of Computability and Logic: Insights, Inspirations, and Innovations*, volume 15764 of *Lecture Notes in Computer Science*, pages 19–33, Cham, 2025. Springer. 21st Conference on Computability in Europe.

61 Vasco Brattka and Hendrik Smischliaew. Computability of initial value problems. arXiv arXiv:2501.00451, 2024.

62 Vasco Brattka and Hendrik Smischliaew. Computability of initial value problems. In Arnold Beckmann, Isabel Oitavem, and Florin Manea, editors, *Crossroads of Computability and Logic: Insights, Inspirations, and Innovations*, volume 15764 of *Lecture Notes in Computer Science*, page to appear, Cham, 2025. Springer. 21st Conference on Computability in Europe.

63 Matthew de Brecht, Arno Pauly, and Matthias Schröder. Overt choice. *Computability*, 9(3-4):169–191, 2020.

64 Merlin Carl. Generalized effective reducibility. arXiv 1601.01899, 2016.

65 Merlin Carl. Generalized effective reducibility. In Arnold Beckmann, Laurent Bienvenu, and Nataša Jonoska, editors, *Pursuit of the Universal*, volume 9709 of *Lecture Notes in Computer Science*, pages 225–233, Switzerland, 2016. Springer. 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016.

66 Merlin Carl. *Ordinal Computability, An Introduction to Infinitary Machines*, volume 9. de Gruyter, Berlin, 2019.

67 Merlin Carl. Effectivity and reducibility with ordinal Turing machines. *Computability*, 10(4):289–304, 2021.

68 Raphaël Carroy. *Functions of the first Baire class.* PhD thesis, University of Lausanne and University Paris 7, 2013.

69 Raphaël Carroy. A quasi-order on continuous functions. *Journal of Symbolic Logic*, 78(2):663–648, 2013.

70 Raphaël Carroy and Yann Pequignot. A well-quasi-order for continuous functions. arXiv 2410.13150v1, 2024.

71 Peter A. Cholak, Damir D. Dzhafarov, Denis R. Hirschfeldt, and Ludovic Patey. Some results concerning the SRT^2_2 vs. COH problem. *Computability*, 9(3-4):193–217, 2020.

72 Vittorio Cipriani. *Many Problems, Different Frameworks, Classification of Problems in Computable Analysis and Algorithmic Learning Theory.* Ph.D. thesis, Universitá degli Studi di Udine, 2023.

73 Vittorio Cipriani, Alberto Marcone, and Manlio Valenti. The Weihrauch lattice at the level of $\mathbf{\Pi}^1_1{-}\mathsf{CA}_0$: the Cantor-Bendixson theorem. arXiv 2210.15556, 2022.

74 Vittorio Cipriani, Alberto Marcone, and Manlio Valenti. The Weihrauch lattice at the level of $\mathbf{\Pi}^1_1{-}\mathsf{CA}_0$: the Cantor-Bendixson theorem. *Journal of Symbolic Logic*, pages 1–39, 2025.

75 Vittorio Cipriani and Arno Pauly. The complexity of finding supergraphs. In Gianluca Della Vedova, Besik Dundua, Steffen Lempp, and Florin Manea, editors, *Unity of Logic and Computation*, volume 13967 of *Lecture Notes in Computer Science*, pages 178–189, Cham, 2023. Springer. 19th Conference on Computability in Europe.

76 Caleb Davis, Denis R. Hirschfeldt, Jeffry Hirst, Jake Pardo, Arno Pauly, and Keita Yokoyama. Combinatorial principles equivalent to weak induction. *Computability*, 9(3-4):219–229, 2020.

77 Caleb Davis, Denis R. Hirschfeldt, Jeffry L. Hirst, Jake Pardo, Arno Pauly, and Keita Yokoyama. Combinatorial principles equivalent to weak induction. arXiv 1812.09943, 2018.

78 Adam R. Day, Rod Downey, and Linda Westrick. Three topological reducibilities for discontinuous functions. *Transactions of the American Mathematical Society. Series B*, 9:859–895, 2022.

79 Matthew de Brecht. Levels of discontinuity, limit-computability, and jump operators. In Vasco Brattka, Hannes Diener, and Dieter Spreen, editors, *Logic, Computation, Hierarchies*, Ontos Mathematical Logic, pages 93–122. Walter de Gruyter, Boston, 2014.

80 François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer. On uniform relationships between combinatorial problems. *Transactions of the American Mathematical Society*, 368(2):1321–1359, 2016.

81 Rod Downey, Alexander Melnikov, and Keng Meng Ng. Foundations of online structure theory II: The operator approach. arXiv 2007.07401, 2020.

82 Damir Dzhafarov, Stephen Flood, Reed Solomon, and Linda Brown Westrick. Effectiveness for the dual Ramsey theorem. arXiv 1710.00070, 2017.

83 Damir Dzhafarov, Reed Solomon, and Manlio Valenti. The tree pigeonhole principle in the Weihrauch degrees. arXiv 2312.10535, 2023.

84 Damir Dzhafarov, Reed Solomon, and Manlio Valenti. The tree pigeonhole principle in the Weihrauch degrees. *Journal of Symbolic Logic*, 2025.

85 Damir Dzhafarov, Reed Solomon, and Keita Yokoyama. On the first-order parts of problems in the Weihrauch degrees. *Computability*, 13(3–4):363–375, 2024.

86 Damir D. Dzhafarov. Cohesive avoidance and strong reductions. *Proceedings of the American Mathematical Society*, 143(2):869–876, 2015.

87 Damir D. Dzhafarov. Strong reductions between combinatorial principles. *Journal of Symbolic Logic*, 81(4):1405–1431, 2016.

88 Damir D. Dzhafarov. Joins in the strong Weihrauch degrees. *Mathematical Research Letters*, 26(3):749–767, 2019.

89 Damir D. Dzhafarov, Jun Le Goh, Denis R. Hirschfeldt, Ludovic Patey, and Arno Pauly. Ramsey's theorem and products in the Weihrauch degrees. arXiv 1804.10968, 2018.

90 Damir D. Dzhafarov, Jun Le Goh, Denis R. Hirschfeldt, Ludovic Patey, and Arno Pauly. Ramsey's theorem and products in the Weihrauch degrees. *Computability*, 9(2):85–110, 2020.

91 Damir D. Dzhafarov, Denis R. Hirschfeldt, and Sarah C. Reitzes. Reduction games, provability, and compactness. arXiv 2008.00907, 2020.

92 Damir D. Dzhafarov, Denis R. Hirschfeldt, and Sarah C. Reitzes. Reduction games, provability, and compactness. *Journal of Mathematical Logic*, 22(3):2250009, 2022.

93 Damir D. Dzhafarov and Carl Mummert. *Reverse Mathematics*. Theory and Applications of Computability. Springer, 2022.

94 Damir D. Dzhafarov and Ludovic Patey. COH, SRT22, and multiple functionals. *Computability*, 10(2):111–121, 2021.

95 Damir D. Dzhafarov, Ludovic Patey, Reed Solomon, and Linda Brown Westrick. Ramsey's theorem for singletons and strong computable reducibility. *Proceedings of the American Mathematical Society*, 145, 2017.

96 Damir D. Dzhafarov, Reed Solomon, and Keita Yokoyama. On the first-order parts of problems in the Weihrauch degrees. arXiv 2301.12733, 2023.

97 Marta Fiori-Carones and Alberto Marcone. To reorient is easier than to orient: an on-line algorithm for reorientation of graphs. *Computability*, 10(3):215–233, 2021.

98 Marta Fiori-Carones, Paul Shafer, and Giovanni Soldà. An inside/outside Ramsey theorem and recursion theory. arXiv 2006.16969, 2020.

99 Marta Fiori-Carones, Paul Shafer, and Giovanni Soldà. An inside/outside Ramsey theorem and recursion theory. *Transactions of the American Mathematical Society*, 375(3):1977–2024, 2022.

100 Stephen Flood, Matthew Jura, Oscar Levin, and Tyler Markkanen. The computational strength of matchings in countable graphs. *Annals of Pure and Applied Logic*, 173(8):Paper No. 103133, 26, 2022.

101 Makoto Fujiwara. Parallelizations in Weihrauch reducibility and constructive reverse mathematics. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability*, pages 38–49, Cham, 2020. Springer International Publishing.

102 Makoto Fujiwara. Weihrauch and constructive reducibility between existence statements. *Computability*, 10(1), 2021.

103 Makoto Fujiwara, Kojiro Higuchi, and Takayuki Kihara. On the strength of marriage theorems and uniformity. *Mathematical Logic Quarterly*, 60(3):136–153, 2014.

104 Lorenzo Galeotti and Hugo Nobrega. Towards computable analysis on the generalized real line. In Jarkko Kari, Florin Manea, and Ion Petre, editors, *Unveiling Dynamics and Complexity*, volume 10307 of *Lecture Notes in Computer Science*, pages 246–257, Cham, 2017. Springer. 13th Conference on Computability in Europe, CiE 2017, Turku, Finland, June 12-16, 2017.

105 Guido Gherardi. An analysis of the lemmas of Urysohn and Urysohn-Tietze according to effective Borel measurability. In A. Beckmann, U. Berger, B. Löwe, and J.V. Tucker, editors, *Logical Approaches to Computational Barriers*, volume 3988 of *Lecture Notes in Computer Science*, pages 199–208, Berlin, 2006. Springer. Second Conference on Computability in Europe, CiE 2006, Swansea, UK, June 30-July 5, 2006.

106 Guido Gherardi. Effective Borel degrees of some topological functions. *Mathematical Logic Quarterly*, 52(6):625–642, 2006.

107 Guido Gherardi. *Some Results in Computable Analysis and Effective Borel Measurability*. PhD thesis, University of Siena, Department of Mathematics and Computer Science, Siena, 2006.

108 Guido Gherardi and Alberto Marcone. How incomputable is the separable Hahn-Banach theorem? In Vasco Brattka, Ruth Dillhage, Tanja Grubba, and Angela Klutsch, editors, *CCA 2008, Fifth International Conference on Computability and Complexity in Analysis*, volume 221 of *Electronic Notes in Theoretical Computer Science*, pages 85–102. Elsevier, 2008. CCA 2008, Fifth International Conference, Hagen, Germany, August 21–24, 2008.

109 Guido Gherardi and Alberto Marcone. How incomputable is the separable Hahn-Banach theorem? *Notre Dame Journal of Formal Logic*, 50(4):393–425, 2009.

110 Guido Gherardi, Alberto Marcone, and Arno Pauly. Projection operators in the Weihrauch lattice. arXiv 1805.12026, 2018.

111 Guido Gherardi, Alberto Marcone, and Arno Pauly. Projection operators in the Weihrauch lattice. *Computability*, 8(3, 4):281–304, 2019.

112 Kenneth Gill. Indivisibility and uniform computational strength. arXiv 2312.03919, 2023.

113 Kenneth Gill. Indivisibility and uniform computational strength. *Logical Methods in Computer Science*, 21(2):22:1–22:23, June 2025.

114 Jun Le Goh. *Measuring the Relative Complexity of Mathematical Constructions and Theorems*. Ph.D. thesis, Cornell University, August 2019.

115 Jun Le Goh. Some computability-theoretic reductions between principles around ATR_0. arXiv 1905.06868, 2019.

116 Jun Le Goh. Compositions of multivalued functions. *Computability*, 9(3-4):231–247, 2020.

117 Jun Le Goh. Embeddings between well-orderings: Computability-theoretic reductions. *Annals of Pure and Applied Logic*, 171(6):102789, 2020.

118 Jun Le Goh, Arno Pauly, and Manlio Valenti. Finding descending sequences through ill-founded linear orders. arXiv 2010.03840, 2020.

119 Jun Le Goh, Arno Pauly, and Manlio Valenti. Finding descending sequences through ill-founded linear orders. *The Journal of Symbolic Logic*, 86(2):817–854, 2021.

120 Jun Le Goh, Arno Pauly, and Manlio Valenti. The weakness of finding descending sequences in ill-founded linear orders. arXiv 2401.11807, 2024.

121 Jun Le Goh, Arno Pauly, and Manlio Valenti. The weakness of finding descending sequences in ill-founded linear orders. In Ludovic Levy Patey, Elaine Pimentel, Lorenzo Galeotti, and Florin Manea, editors, *Twenty Years of Theoretical and Practical Synergies*, pages 339–350, Cham, 2024. Springer Nature Switzerland.

122 Noam Greenberg, Rutger Kuyper, and Dan Turetsky. Cardinal invariants, non-lowness classes, and Weihrauch reducibility. *Computability*, 8(3, 4):305–346, 2019.

123 Noam Greenberg, Joseph S. Miller, and An Nies. Highness properties close to PA completeness. *Israel Journal of Mathematics*, 244:419–465, 2021.

124 Kirill Gura, Jeffry L. Hirst, and Carl Mummert. On the existence of a connected component of a graph. *Computability*, 4(2):103–117, 2015.

125 Peter Hertling. Stetige Reduzierbarkeit auf Σ^ω von Funktionen mit zweielementigem Bild und von zweistetigen Funktionen mit diskretem Bild. Informatik Berichte 153, FernUniversität Hagen, Hagen, December 1993.

126 Peter Hertling. A topological complexity hierarchy of functions with finite range. Technical Report 223, Centre de recerca matematica, Institut d'estudis catalans, Barcelona, Barcelona, October 1993. Workshop on Continuous Algorithms and Complexity, Barcelona, October, 1993.

127 Peter Hertling. Topologische Komplexitätsgrade von Funktionen mit endlichem Bild. Informatik Berichte 152, FernUniversität Hagen, Hagen, December 1993.

128 Peter Hertling. *Unstetigkeitsgrade von Funktionen in der effektiven Analysis*. PhD thesis, Fachbereich Informatik, FernUniversität Hagen, 1996. Dissertation.

129 Peter Hertling. Forests describing Wadge degrees and topological Weihrauch degrees of certain classes of functions and relations. *Computability*, 9(3-4):249–307, 2020.

130 Peter Hertling and Victor Selivanov. Complexity issues for preorders on finite labeled forests. In Benedikt Löwe, Dag Normann, Ivan Soskov, and Alexandra Soskova, editors, *Models of computation in context*, volume 6735 of *Lecture Notes in Computer Science*, pages 112–121, Heidelberg, 2011. Springer. 7th Conference on Computability in Europe, CiE 2011, Sofia, Bulgaria, June 27–July 2, 2011.

131 Peter Hertling and Victor Selivanov. Complexity issues for preorders on finite labeled forests. In Vasco Brattka, Hannes Diener, and Dieter Spreen, editors, *Logic, Computation, Hierarchies*, Ontos Mathematical Logic, pages 165–190. Walter de Gruyter, Boston, 2014.

132 Peter Hertling and Klaus Weihrauch. Levels of degeneracy and exact lower complexity bounds for geometric algorithms. In *Proceedings of the Sixth Canadian Conference on Computational Geometry*, pages 237–242, 1994. Saskatoon, Saskatchewan, August 2–6, 1994.

133 Peter Hertling and Klaus Weihrauch. On the topological classification of degeneracies. Informatik Berichte 154, FernUniversität Hagen, Hagen, February 1994.

134 Kojiro Higuchi. *Degree Structures of Mass Problems and Choice Functions.* PhD thesis, Mathematical Institute, Tohoku University, Sendai, Japan, January 2012.

135 Kojiro Higuchi and Takayuki Kihara. Inside the Muchnik degrees I: Discontinuity, learnability and constructivism. *Annals of Pure and Applied Logic*, 165(5):1058–1114, 2014.

136 Kojiro Higuchi and Takayuki Kihara. Inside the Muchnik degrees II: The degree structures induced by the arithmetical hierarchy of countably continuous functions. *Annals of Pure and Applied Logic*, 165(6):1201–1241, 2014.

137 Kojiro Higuchi and Arno Pauly. The degree structure of Weihrauch reducibility. *Log. Methods Comput. Sci.*, 9(2):2:02, 17, 2013.

138 Denis R. Hirschfeldt. *Slicing the Truth: On the Computable and Reverse Mathematics of Combinatorial Principles*, volume 28 of *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore.* World Scientific, Singapore, 2015.

139 Denis R. Hirschfeldt. Some questions in computable mathematics. In Adam Day, Michael Fellows, Noam Greenberg, Bakhadyr Khoussainov, Alexander Melnikov, and Frances Rosamond, editors, *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, volume 10010 of *Lecture Notes in Computer Science*, pages 22–55. Springer, Cham, 2017.

140 Denis R. Hirschfeldt and Carl G. Jockusch. On notions of computability-theoretic reduction between Π^1_2 principles. *Journal of Mathematical Logic*, 16(1):1650002, 59, 2016.

141 Jeffry L. Hirst. Leaf management. arXiv 1812.09762, 2018.

142 Jeffry L. Hirst. Leaf management. *Computability*, 9(3-4):309–314, 2020.

143 Jeffry L. Hirst and Carl Mummert. Reverse mathematics of matroids. In Adam Day, Michael Fellows, Noam Greenberg, Bakhadyr Khoussainov, Alexander Melnikov, and Frances Rosamond, editors, *Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, volume 10010 of *Lecture Notes in Computer Science*, pages 143–159. Springer, Cham, 2017.

144 Jeffry L. Hirst and Carl Mummert. Using Ramsey's theorem once. *Archive for Mathematical Logic*, 58(7-8):857–866, 2019.

145 Jeffry L. Hirst and Carl Mummert. Banach's theorem in higher-order reverse mathematics. *Computability*, 12(3):203–225, 2023.

146 Rupert Hölzl and Paul Shafer. Universality, optimality, and randomness deficiency. *Annals of Pure and Applied Logic*, 166(10):1049–1069, 2015.

147 Mathieu Hoyrup. Genericity of weakly computable objects. *Theory of Computing Systems*, 60(3):396–420, 2017.

148 Mathieu Hoyrup. Notes on overt choice. *Computability*, 12(4):351–369, 2023.

149 Mathieu Hoyrup. *Topological Aspects of Representations in Computable Analysis.* PhD thesis, Laboratoire Lorrain de Recherche en Informatique et ses Applications, Nancy, France, 2023. Habilitation Thesis.

150 Mathieu Hoyrup, Cristóbal Rojas, and Klaus Weihrauch. Computability of the Radon-Nikodym derivative. In Benedikt Löwe, Dag Normann, Ivan Soskov, and Alexandra Soskova, editors, *Models of Computation in Context*, volume 6735 of *Lecture Notes in Computer Science*, pages 132–141, Heidelberg, 2011. Springer.

151 Mathieu Hoyrup, Cristóbal Rojas, and Klaus Weihrauch. Computability of the Radon-Nikodym derivative. *Computability*, 1(1):3–13, 2012.

152 Noah A. Hughes. *Applications of Computability Theory to Infinitary Combinatorics.* Ph.D. thesis, University of Connecticut, 2021.

153 Akitoshi Kawamura. Lipschitz continuous ordinary differential equations are polynomial-space complete. In *24th Annual IEEE Conference on Computational Complexity*, pages 149–160. IEEE Computer Soc., Los Alamitos, CA, 2009.

154 Akitoshi Kawamura. Lipschitz continuous ordinary differential equations are polynomial-space complete. *Computational Complexity*, 19(2):305–332, 2010.

155 Akitoshi Kawamura and Stephen Cook. Complexity theory for operators in analysis. In *Proceedings of the 42nd ACM Symposium on Theory of Computing*, STOC '10, pages 495–502, New York, 2010. ACM.

156 Akitoshi Kawamura and Hiroyuki Ota. Small complexity classes for computable analysis. In *Mathematical foundations of computer science 2014. Part II*, volume 8635 of *Lecture Notes in Comput. Sci.*, pages 432–444. Springer, Heidelberg, 2014.

157 Akitoshi Kawamura, Hiroyuki Ota, Carsten Rösnick, and Martin Ziegler. Computational complexity of smooth differential equations. In *Mathematical foundations of computer science 2012*, volume 7464 of *Lecture Notes in Comput. Sci.*, pages 578–589. Springer, Heidelberg, 2012.

158 Akitoshi Kawamura, Hiroyuki Ota, Carsten Rösnick, and Martin Ziegler. Computational complexity of smooth differential equations. *Logical Methods in Computer Science*, 10:1:6,15, 2014.

159 Akitoshi Kawamura and Arno Pauly. Function spaces for second-order polynomial time. In *Language, life, limits*, volume 8493 of *Lecture Notes in Comput. Sci.*, pages 245–254. Springer, Cham, 2014.

160 Takayuki Kihara. Borel-piecewise continuous reducibility for uniformization problems. *Logical Methods in Computer Science*, 12(4), October 2016.

161 Takayuki Kihara. Degrees of incomputability, realizability and constructive reverse mathematics. arXiv 2002.10712, 2020.

162 Takayuki Kihara. Lawvere-Tierney topologies for computability theorists. arXiv 2106.03061, 2021.

163 Takayuki Kihara. Topological reducibilities for discontinuous functions and their structures. *Israel Journal of Mathematics*, 2022.

164 Takayuki Kihara. Lawvere-Tierney topologies for computability theorists. *Transactions of the American Mathematical Society, Series B*, 10(2):48–85, 2023.

165 Takayuki Kihara, Alberto Marcone, and Arno Pauly. Searching for an analogue of ATR in the Weihrauch lattice. arXiv 1812.01549, 2018.

166 Takayuki Kihara, Alberto Marcone, and Arno Pauly. Searching for an analogue of ATR_0 in the Weihrauch lattice. *Journal of Symbolic Logic*, 85(3):1006–1043, 2020.

167 Takayuki Kihara and Arno Pauly. Dividing by zero - how bad is it, really? In Piotr Faliszewski, Anca Muscholl, and Rolf Niedermeier, editors, *41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016)*, volume 58 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 58:1–58:14, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

168 Takayuki Kihara and Arno Pauly. Dividing by zero—how bad is it, really? In *41st International Symposium on Mathematical Foundations of Computer Science*, volume 58 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 58, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

169 Takayuki Kihara and Arno Pauly. Finite choice, convex choice and sorting. In T.V. Gopal and Junzo Watada, editors, *Theory and applications of models of computation*, volume 11436 of *Lecture Notes in Computer Science*, pages 378–393. Springer, Cham, 2019. 15th Annual Conference, TAMC 2019, Kitakyushu, Japan, April 13–16, 2019.

170 Ulrich Kohlenbach. On the reverse mathematics and Weihrauch complexity of moduli of regularity and uniqueness. *Computability*, 8(3, 4):377–387, 2019.

171 Alexander P. Kreuzer. On the strength of weak compactness. *Computability*, 1(2):171–179, 2012.

172 Alexander P. Kreuzer. Bounded variation and the strength of Helly's selection theorem. *Logical Methods in Computer Science*, 10(4:16):1–23, 2014.

173 Alexander P. Kreuzer. From Bolzano-Weierstraß to Arzelà-Ascoli. *Mathematical Logic Quarterly*, 60(3):177–183, 2014.

174 Oleg V. Kudinov, Victor L. Selivanov, and Anton V. Zhukov. Undecidability in Weihrauch degrees. In Fernando Ferreira, Benedikt Löwe, Elvira Mayordomo, and Luís Mendes Gomes, editors, *Programs, Proofs, Processes*, volume 6158 of *Lecture Notes in Computer Science*, pages 256–265, Berlin, 2010. Springer. 6th Conference on Computability in Europe, CiE 2010, Ponta Delgada, Azores, Portugal, June/July 2010.

175 Rutger Kuyper. On Weihrauch reducibility and intuitionistic reverse mathematics. *Journal of Symbolic Logic*, 82(4):1438–1458, 2017.

176 Stéphane Le Roux and Arno Pauly. Closed choice for finite and for convex sets. In Paola Bonizzoni, Vasco Brattka, and Benedikt Löwe, editors, *The Nature of Computation. Logic, Algorithms, Applications*, volume 7921 of *Lecture Notes in Computer Science*, pages 294–305, Berlin, 2013. Springer. 9th Conference on Computability in Europe, CiE 2013, Milan, Italy, July 1-5, 2013.

177 Stéphane Le Roux and Arno Pauly. Finite choice, convex choice and finding roots. *Logical Methods in Computer Science*, 11(4):4:6, 31, 2015.

178 Stéphane Le Roux and Arno Pauly. Weihrauch degrees of finding equilibria in sequential games (extended abstract). In Arnold Beckmann, Victor Mitrana, and Mariya Soskova, editors, *Evolving Computability*, volume 9136 of *Lecture Notes in Computer Science*, pages 246–257, Cham, 2015. Springer. 11th Conference on Computability in Europe, CiE 2015, Bucharest, Romania, June 29–July 3, 2015.

179 Steffen Lempp, Alberto Marcone, and Manlio Valenti. Chains and antichains in the Weihrauch lattice. arXiv 2411.07792, 2024.

180 Steffen Lempp, Joseph S. Miller, Arno Pauly, Mariya I. Soskova, and Manlio Valenti. Minimal covers in the Weihrauch degrees. arXiv 2311.12676, 2023.

181 Steffen Lempp, Joseph S. Miller, Arno Pauly, Mariya I. Soskova, and Manlio Valenti. Minimal covers in the Weihrauch degrees. *Proceedings of the American Mathematical Society*, 152(11):4893–4901, 2024.

182 Ang Li. Countable ordered groups and Weihrauch reducibility. arXiv 2409.19229, 2024.

183 Patrick Lutz. *Results on Martin's Conjecture*. PhD thesis, University of California, Berkeley, 2021.

184 Patrick Lutz. The Solecki dichotomy and the Possner-Robinson theorem are almost equivalent. arXiv 2301.07259, 2023.

185 Patrick Lutz and Benjamin Siskind. Part 1 of Martin's conjecture for order-preserving and measure-preserving functions. arXiv 2305.19646, 2023.

186 Patrick Lutz and Benjamin Siskind. Part 1 of Martin's conjecture for order-preserving and measure-preserving functions. *Journal of the American Mathematical Society*, 2024.

187 Alberto Marcone and Gian Marco Osso. The Galvin-Prikry theorem in the Weihrauch lattice. arXiv 2410.06928, 2024.

188 Alberto Marcone and Manlio Valenti. The open and clopen Ramsey theorems in the Weihrauch lattice. arXiv 2003.04245, 2020.

189 Alberto Marcone and Manlio Valenti. Effective aspects of Hausdorff and Fourier dimension. arXiv 2108.06941, 2021.

190 Alberto Marcone and Manlio Valenti. The open and clopen Ramsey theorems in the Weihrauch lattice. *The Journal of Symbolic Logic*, 86(1):316–351, 2021.

191 Alberto Marcone and Manlio Valenti. Effective aspects of Hausdorff and Fourier dimension. *Computability*, 11(3-4):299–333, 2022.

192 Samuele Maschio and Davide Trotta. A topos for extended Weihrauch degrees. arXiv 2505.08697, 2025.

193 Benoit Monin and Ludovic Patey. Π^0_1-encodability and omniscient reductions. *Notre Dame Journal of Formal Logic*, 60(1):1–12, 2019.

194 Daniel Mourad. There is no composition in the computable reducibility degrees. arXiv 2405.15281, 2024.

195 Uwe Mylatz. *Vergleich unstetiger Funktionen: "Principle of Omniscience" und Vollständigkeit in der C–Hierarchie*. PhD thesis, Faculty for Mathematics and Computer Science, University Hagen, Hagen, Germany, 2006. Ph.D. thesis.

196 Eike Neumann. Computational problems in metric fixed point theory and their Weihrauch degrees. *Logical Methods in Computer Science*, 11:4:20,44, 2015.

197 Eike Neumann and Arno Pauly. A topological view on algebraic computation models. *Journal of Complexity*, 44(Supplement C):1–22, 2018.

198 David Nichols. Strong reductions between relatives of the stable Ramsey's theorem. arXiv 1711.06532, 2017.

199 Hugo Nobrega. *Games for functions - Baire classes, Weihrauch degrees, Transfinite Computations, and Ranks*. PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2018.

200 Hugo Nobrega and Arno Pauly. Game characterizations and lower cones in the Weihrauch degrees. In Jarkko Kari, Florin Manea, and Ion Petre, editors, *Unveiling Dynamics and Complexity*, volume 10307 of *Lecture Notes in Computer Science*, pages 327–337, Cham, 2017. Springer. 13th Conference on Computability in Europe, CiE 2017, Turku, Finland, June 12-16, 2017.

201 Hugo Nobrega and Arno Pauly. Game characterizations and lower cones in the Weihrauch degrees. *Logical Methods in Computer Science*, 15(3):Paper No. 11, 29, 2019.

202 Ludovic Patey. *The reverse mathematics of Ramsey-type theorems*. PhD thesis, Université Paris Diderot, Paris, France, 2016.

203 Ludovic Patey. The weakness of being cohesive, thin or free in reverse mathematics. *Israel Journal of Mathematics*, 216:905–955, 2016.

204 Arno Pauly. How discontinuous is computing Nash equilibria? (Extended abstract). In Andrej Bauer, Peter Hertling, and Ker-I Ko, editors, *6th International Conference on Computability and Complexity in Analysis (CCA'09)*, volume 11 of *OpenAccess Series in Informatics (OASIcs)*, Dagstuhl, Germany, 2009. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

205 Arno Pauly. How incomputable is finding Nash equilibria? *Journal of Universal Computer Science*, 16(18):2686–2710, 2010.

206 Arno Pauly. On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly*, 56(5):488–502, 2010.

207 Arno Pauly. *Computable Metamathematics and its Application to Game Theory*. PhD thesis, University of Cambridge, Computer Laboratory, Clare College, Cambridge, 2011. Ph.D. thesis.

208 Arno Pauly. Computability on the space of countable ordinals. arXiv 1501.00386, 2015.

209 Arno Pauly. Many-one reductions and the category of multivalued functions. *Mathematical Structures in Computer Science*, 27(3):376–404, 2017.

210 Arno Pauly. An update on Weihrauch complexity, and some open questions. arXiv 2008.11168, 2020.

211 Arno Pauly and Matthew de Brecht. Towards synthetic descriptive set theory: An instantiation with represented spaces. arXiv 1307.1850, 2013.

212 Arno Pauly and Matthew de Brecht. Non-deterministic computation and the Jayne-Rogers theorem. In Benedikt Löwe and Glynn Winskel, editors, *Proceedings 8th International Workshop on Developments in Computational Models, DCM 2012, Cambridge, United Kingdom, 17 June 2012.*, volume 143 of *Electronic Proceedings in Theoretical Computer Science*, pages 87–96, 2014.

213 Arno Pauly and Matthew de Brecht. Descriptive set theory in the category of represented spaces. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 438–449, 2015.

214 Arno Pauly, Willem Fouché, and George Davie. Weihrauch-completeness for layerwise computability. *Logical Methods in Computer Science*, 14(2), May 2018.

215 Arno Pauly and Willem L. Fouché. How constructive is constructing measures? *Journal of Logic & Analysis*, 9(c3):1–44, 2017.

216 Arno Pauly, Cécilia Pradic, and Giovanni Soldà. On the Weihrauch degree of the additive Ramsey theorem. *Computability*, 13(3–4):459–483, 2024.

217 Arno Pauly and Giovanni Soldà. Sequential discontinuity and first-order problems. arXiv 2401.12641, 2024.

218 Arno Pauly and Giovanni Soldà. Sequential discontinuity and first-order problems. In Ludovic Levy Patey, Elaine Pimentel, Lorenzo Galeotti, and Florin Manea, editors, *Twenty Years of Theoretical and Practical Synergies*, pages 351–365, Cham, 2024. Springer Nature Switzerland.

219 Arno Pauly and Florian Steinberg. Representations of analytic functions and Weihrauch degrees. In *Computer science—theory and applications*, volume 9691 of *Lecture Notes in Computer Science*, pages 367–381, Cham, 2016. Springer.

220 Arno Pauly and Florian Steinberg. Comparing representations for function spaces in computable analysis. *Theory of Computing Systems*, 62:557–582, 2018.

221 Arno Pauly and Hideki Tsuiki. Computable dyadic subbases and $\mathbb{T}^\omega$-representations of compact sets. arXiv 1604.0258, 2016.

222 Michelle Porter, Adam Day, and Rodney Downey. Notes on computable analysis. *Theory of Computing Systems*, 60(1):53–111, 2017.

223 Cécilia Pradic and Ian Price. Weihrauch problems as containers. arXiv 2501.17250, 2025.

224 Pierre Pradic and Giovanni Soldà. On the Weihrauch degree of the additive Ramsey theorem over the rationals. In *Revolutions and revelations in computability*, volume 13359 of *Lecture Notes in Computer Science*, pages 259–271. Springer, Cham, 2022.

225 Tahina Rakotoniaina. *On the Computational Strength of Ramsey's Theorem.* PhD thesis, Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, South Africa, 2015. Ph.D. thesis.

226 Victor Selivanov. Total representations. *Logical Methods in Computer Science*, 9:2:5, 30, 2013.

227 Victor Selivanov. Q-Wadge degrees as free structures. *Computability*, 9(3-4):327–341, 2020.

228 Giovanni Soldà and Manlio Valenti. Algebraic properties of the first-order part of a problem. arXiv 2203.16298, 2022.

229 Giovanni Soldà and Manlio Valenti. Algebraic properties of the first-order part of a problem. *Annals of Pure and Applied Logic*, 174(7):103270, 2023.

230 Reed Solomon. Computable reductions and reverse mathematics. In Arnold Beckmann, Laurent Bienvenu, and Nataša Jonoska, editors, *Pursuit of the Universal*, volume 9709 of *Lecture Notes in Computer Science*, pages 182–191, Switzerland, 2016. Springer. 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016.

231 Yudai Suzuki and Keita Yokoyama. Searching problems above arithmetical transfinite recursion. *Annals of Pure and Applied Logic*, 175(10):Paper No. 103488, 31, 2024.

232 Nazanin R. Tavana and Klaus Weihrauch. Turing machines on represented sets, a model of computation for analysis. *Logical Methods in Computer Science*, 7(2):2:19, 21, 2011.

233 Holger Thies. *Uniform computational complexity of ordinary differential equations with applications to dynamical systems and exact real arithmetic.* PhD thesis, Graduate School of Arts and Sciences, University of Tokyo, Tokyo, Japan, 2018.

234 Davide Trotta, Manlio Valenti, and Valeria de Paiva. Categorifying computable reducibilities. arXiv 2208.08656, 2022.

235 Patrick Uftring. The characterization of Weihrauch reducibility in systems containing E-PA$^\omega$ + QF-AC^{0,0}. arXiv 2003.13331, 2020.

236 Patrick Uftring. The characterization of Weihrauch reducibility in systems containing E-PA$^\omega$ + QF-AC^{0,0}. *The Journal of Symbolic Logic*, 86(1):224–261, 2021.

237 Patrick Uftring. Weihrauch degrees without roots. arXiv 2102.11832, 2023.

238 Manlio Valenti. *A journey through computability, topology and analysis.* Ph.D. thesis, Universitá degli Studi di Udine, 2021.

239 Klaus Weihrauch. The degrees of discontinuity of some translators between representations of the real numbers. Technical Report TR-92-050, International Computer Science Institute, Berkeley, July 1992.

240 Klaus Weihrauch. The degrees of discontinuity of some translators between representations of the real numbers. Informatik Berichte 129, FernUniversität Hagen, Hagen, July 1992.

241 Klaus Weihrauch. The TTE-interpretation of three hierarchies of omniscience principles. Informatik Berichte 130, FernUniversität Hagen, Hagen, September 1992.

242 Klaus Weihrauch. *Computable Analysis.* Springer, Berlin, 2000.

243 Klaus Weihrauch. Computable planar curves intersect in a computable point. *Computability*, 8(3, 4):399–415, 2019.

244 Linda Westrick. A note on the diamond operator. arXiv 2001.09372, 2020.

245 Linda Westrick. A note on the diamond operator. *Computability*, 10(2):107–110, 2021.

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