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*Editorial Office*

Schloss Dagstuhl – Leibniz-Zentrum für Informatik  
TGDK, Editorial Office

Oktavie-Allee, 66687 Wadern, Germany

[tgdk@dagstuhl.de](mailto:tgdk@dagstuhl.de)

<https://www.dagstuhl.de/tgdk>



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## ■ List of Authors


Abraham Bernstein  (1)  
University of Zurich, Switzerland

Ricardo Guimarães  (2)  
Zivid AS, Norway

Victor Lacerda  (2)  
University of Bergen, Norway

Ana Ozaki  (2)  
University of Oslo, Norway;  
University of Bergen, Norway

Luca Rossetto  (1)  
University of Zurich, Switzerland;  
Dublin City University, Ireland

Jan Willi  (1)  
University of Zurich, Switzerland





# Unified Multimedia Segmentation – A Comprehensive Model for URI-based Media Segment Representation

Jan Willi  

University of Zurich, Switzerland

Abraham Bernstein  

University of Zurich, Switzerland

Luca Rossetto<sup>1</sup>  

University of Zurich, Switzerland

Dublin City University, Ireland

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## Abstract

In multimedia annotation, referencing specific segments of a document is often desired due to its richness and multimodality, but no universal representation for such references exists. This significantly hampers the usage of multimedia content in knowledge graphs, as it is modeled as one large atomic information container. Unstructured data – such as text, audio, images, and video – can commonly be decomposed into its constituent parts, as such documents rarely contain only one semantic concept. Hence, it is reasonable to assume that these advances will make it possible to decompose these previous atomic components into logical segments. To be processable by the knowledge graph stack, however, one needs to break the atomic nature of multimedia content, providing a mechanism

to address media segments.

This paper proposes a Unified Segmentation Model capable of depicting arbitrary segmentations on any media document type. The work begins with a formal analysis of multimedia and segmentation, exploring segmentation operations and how to describe them. Building on this analysis, it then develops a practical scheme for expressing segmentation in Uniform Resource Identifiers (URIs). Given that this approach makes segments of multimedia content referencable, it breaks their atomic nature and makes them first-class citizens within knowledge graphs. The proposed model is implemented as a proof of concept in the MediaGraph Store, a multimedia knowledge graph storage and querying engine.

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**Software:** <https://github.com/lucaro/MeGraS> [63]

archived at [swh:1:dir:d2dfb84bd390d27ad96190f973dfa484b9e68bc0](https://swh.1.dir:d2dfb84bd390d27ad96190f973dfa484b9e68bc0)

**Audiovisual:** <https://github.com/lucaro/Unified-Media-Segmentation-Test-Suite> [62]

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<sup>1</sup> Corresponding Author



## 1 Introduction

Be it social media platforms, streaming services, news outlets, or video games, multimedia content is ubiquitous and takes up a big part of today's devices' bandwidth. Multimedia content, in comparison to pure textual content, carries vast amounts of knowledge whose capture is far from trivial. This is due to both the multimodality of multimedia as well as the interconnectedness of the individual pieces of knowledge captured by a media object. A video clip, for instance, not only includes factual information about elements of the scene, like objects, humans, or dialog, but also contains relations between these elements and implicitly conveys sentiments and emotions. To capture these knowledge elements and their relations, two ingredients are needed: First, individual elements of a multimedia object, henceforth called segments, need to be unambiguously referenceable. Such segments can be multi-dimensional and of different modalities, such as temporal, spatial, or some combination thereof. Second, a structural representation of interconnected knowledge must be able to capture semantic annotations of segments and relations between them. For knowledge that can be easily represented in textual form, this is frequently done using knowledge graphs, which are graph structures where nodes contain facts and edges establish relations. Multimedia content, however, exceeds text in richness and is thus unsuitable to be expressed in traditional knowledge graphs.

In literature, this problem has been approached from different perspectives. The rise of multimedia content on the Web led to numerous file standards, like SVG or MPEG-7, which describe metadata of multimedia and can annotate specific regions of it. Following that, efforts were made to formalize these standards using Semantic Web technologies to improve the processing of annotations by computers. This led to a number of proposed ontologies, such as the COMM Ontology [2]. At the same time, another stream of research focused on leveraging Uniform Resource Identifiers (URIs), a core concept of the Semantic Web, to uniquely describe not only complete media objects but also segments of it. Such URIs can then be used in Resource Description Framework (RDF) triples to annotate (or other knowledge graph representations cf. [33]) and relate multimedia segments. A notable mention of this is the Media Fragments URI 1.0 specification [75], which establishes a syntax for accessing fragments of a media object. Similar concepts can also be found in other sectors, like the International Image Interoperability Framework (IIIF) [71], an initiative for standards for image delivery in cultural heritage institutions, which features a URI scheme to access full or partial images. In all of this literature, work was focused on specific use cases and applications with only a limited number of segmentation and media document types. The only exception is the General Fragment Model by [23], which attempts to generalize the notion of media fragments in an ontological way. It is, however, a primarily abstract model, and no practical implementation exists.

In this paper, we present an RDF-compatible approach for the unified representation of multimedia segmentations with the aim of making multimedia first-class citizens in knowledge graphs. The goal of this paper is to tackle segmentation and its representation equally in a more holistic way. To that end, a Unified Segmentation Model is defined, which aims to be universal enough to describe all possible segments of all possible multimedia documents and yet be practical enough to be implemented into a real-world application.

Hence, the contributions of this paper are three-fold:

1. We discuss the properties of media segmentation, put them onto a solid formal foundation, independent of specific media types, and introduce our Unified Segmentation Model based on that foundation.
2. We present an RDF-compatible URI-based approach for the unified representation of arbitrary segments of multimedia content that leverages the insights of the Unified Segmentation Model.
3. We illustrate the applicability of the approach in a prototypical implementation.

These contributions provide the foundation to break the atomicity of multimedia content in a principled manner allowing multimedia *segments* to become first-class citizens in the knowledge graph domain.

The remainder of this paper is structured as follows: Section 2 discusses how literature currently annotates the created segments to close the gap between data and knowledge. In Section 3, the segmentation model is formally established and reasoned, and resulting properties, characteristics, and classes are discussed. Sections 4 and 5 then bridge the gap to the concrete implementation in Section 6 by translating the theoretical principles into applicable URI schemes and discussing their use in knowledge graph applications, respectively. A discussion follows in Section 7, after which Section 8 concludes the paper.

## 2 Segment Annotation

In multimedia content analysis, semantic knowledge is crucial for efficient manipulation and retrieval of media [9]. In many cases, though, semantic annotations may not target the whole media document but refer to specific segments. Therefore, to effectively annotate multimedia content, annotations need to be able to refer to any possible segment of that content. To this end, literature and practice have proposed and established various file types, standards, recommendations, and frameworks. The many existing approaches can be distinguished into two classes. Media documents can be annotated with segment information in metadata documents. Later, with the rise of the Semantic Web, work was put into ensuring the interoperability of applications that produce such multimedia annotations. Segments can be captured in the Uniform Resource Identifier (URI) that specifies the resource. Apart from *segment*, the term *fragment* is often found in literature. In the following, the two are used interchangeably.

This section discusses these two approaches, followed by a foray into the more generic General Fragment Model and a discussion of the insights gathered by reviewing these related works.

### 2.1 Metadata-based Annotation

This first category consists of approaches that store segment information in a file separate from the original multimedia resource. While these typically XML-based files allow for conveying extensive segments, an indirection is required. The semantic description of a segment refers to the definition of the segment, and not the segment itself [74, 52]. In other words, segmentation is defined by a part of a description document, which in turn represents the multimedia document.

#### 2.1.1 Synchronized Multimedia Integration Language

The Synchronized Multimedia Integration Language (SMIL) is an XML-based language to create interactive multimedia presentations [34]. Users can include multimedia, like audio, video, and text, and define timings, transitions, and animations. For that, SMIL supports both temporal and spatial fragments. For the former, the user can specify the beginning and end of the desired snippet of a longer video or audio object. For the latter, SMIL allows cropping rectangles from images or videos. Additionally, transitions of the rectangle can be defined, which enables the creation of a primitive spatio-temporal fragment.

#### 2.1.2 Scalable Vector Graphics

The Scalable Vector Graphics (SVG) format, also based on XML, is used to describe two-dimensional vector and raster images [40]. Besides rectangles, arbitrary paths can be specified, allowing for any fragment shapes. Additionally, with `svgView`, fragments can be established

directly in the URI of an SVG object, although only rectangular regions are supported. Temporal segments are not naturally supported, but SVG allows adding HTML or SMIL as foreign objects, which themselves could contain content with a temporal component.

### 2.1.3 MPEG-7

The MPEG-7 standard is created by the Moving Picture Experts Group (MPEG) for describing multimedia content of many different types [37]. Unlike previous MPEG standards, which focus on storing content, MPEG-7 provides means to store information about the content. MPEG-7 defines four building blocks [3, 65, 41]:

- *Descriptors* describe syntax and semantics of a feature that characterizes the content.
- *Description schemes* describe the structure of relationships between components (descriptors or description schemes).
- The *description definition language* allows creating descriptors and description schemes as well as extension and modification of existing description schemes.
- *Syntax tools* are used for binarization, synchronization, transport, and storage of descriptions.

The **Segment** description schema can be used to annotate temporal, spatial, and spatio-temporal fragments using XML-based metadata [65]. Temporal fragments are specified using starting time and duration. Spatial regions are created from polygon coordinates, allowing for arbitrary shapes. It is also possible to construct a segment from multiple components. On top of that, time frames and regions can be combined into spatio-temporal fragments.

### 2.1.4 Ontological Segment Description

[7] proposed the Semantic Web as an extension to the World Wide Web. Its goal is to share data, resulting in a Web of Data on the web of documents of the traditional web. The Semantic Web architecture makes use of various standards. XML, for example, can be used to create structure in webpages by annotating sections with tags, but it says nothing about the meaning of those structures [10]. For that, the Resource Description Framework (RDF) is used [13]. In the form of triples (i.e., **subject**, **predicate**, **object** tuples), assertions between things and properties are made, and a knowledge graph is built. The elements of the triples are either literals or URIs, with the latter pointing to further information about a resource. This Linked Data principle enables information linking and ensures unique definitions of concepts [4]. RDF alone does not prevent different systems from using different terms for the same concept. For computers to understand and reuse information across systems, shared vocabularies and ontologies, which define common concepts and relationships, are needed. A vocabulary is usually a simple description scheme specified in RDF Schema (RDFS) [29]. More complex semantics are usually modeled in the Web Ontology Language (OWL) [17]. Lastly, the SPARQL query language [68] for RDF provides an SQL-like syntax to extract data.

At the same time, multimedia content on the Web has increased, and newly created media formats, as described above, allow metadata annotations. These are, however, not easily understandable by computers, as the different standards lack formal semantics [70]. To address this issue, Semantic Web technologies are required, and a number of ontologies for single media and multimedia as well as several different ontologies to annotate various types of multimedia documents have been proposed.

For image description, for instance, [49] proposes a vocabulary to describe regions of various shapes, such as polygons, rectangles, or circles, and link them to images together with additional information. Numerous ontologies based on MPEG-7 have been proposed to describe the semantics of general multimedia. The first attempt is made by [35], who models parts of MPEG-7 using

the ABC core ontology by [45] as a foundation. [76] builds on this and extends it to an ontology called DS-MIRF. [36] takes a slightly different approach in their audio-visual core ontology and focuses more on adding genre or themes to segments. [9] proposes a framework that uses the core ontology DOLCE [25] with two multimedia ontologies, namely Multimedia Structure Ontology and Visual Descriptor Ontology. [26] is the first to produce a full MPEG-7 ontology, called Rhizomik, by automatically mapping XML Schema to OWL. [2] proposes COMM, a core ontology for multimedia annotation. Unlike previous ontologies, it does not directly translate MPEG-7, but the authors re-engineer MPEG-7 to capture the intended semantics of the standard fully. This ontology covers the most used parts of the specification.

Beyond MPEG-7, the Ontology for Media Resources 1.0 defines a core vocabulary to describe media resources and a mapping to a set of media formats that support metadata [67]. Its goal is to unify metadata properties across commonly used multimedia formats. The Multimedia Metadata Ontology (M3O) by [64] also looks beyond MPEG-7 and provides a generic modeling framework that can be integrated with media formats like SMIL or SVG.

## 2.2 URI-based Annotation

With the rising presence of multimedia on the Web, there were efforts to encode segment information in URIs. [5], later superseded by [6], defines the generic URI syntax as a sequence of components: `<scheme>:<authority>/<path>?<query>#<fragment>`

The mandatory `scheme` defines the context of the URI, like the protocol or notation used. The `authority` indicates the place of governance of the remainder of the URI, such as a domain or an IP address. The `path` then identifies the desired resource, followed by the `query` and `fragment`, identified by question marks and number signs, respectively, to target components within that resource. While both query and fragment can be used to refer to segments of the resource, they come with different consequences. Queries in the form of key-value pairs are sent to the server and interpreted, after which the server returns a new resource, which has no relation to the original one [80]. Fragments keep the reference to the original by identifying a secondary resource, which is a subset or view of the primary one. Conversely, URI fragments are not transmitted to a server but are interpreted only client-side after the complete original resource is transmitted [30].

To formalize the description of segments as part of a URI, several standards have been proposed, which the following describes in closer detail.

### 2.2.1 Hypertext Markup Language

In the Hypertext Markup Language (HTML), URI fragments are used to reference anchor elements and allow jumping to specific content on the same or another page. Anchors are created using either the `name` or `id` attribute. Such a fragment URI, for example, looks like this: `http://example.com#section-1`. To resolve it, the fragment name is compared to all element IDs of the document tree, and the first matching element is used [32]. To solve the problem of changing IDs and broken references, text fragments can be used instead, which directly reference textual content [12]. Being a relatively recent specification, it is only implemented in Chromium-based browsers at the time of writing.

### 2.2.2 XPointer

In XML, the XPointer Framework defines ways of addressing specific components through URI fragments [28]. Besides the shorthand form for referencing specific element names, two schemes allow advanced references. The `element()` scheme can be used to traverse the XML tree and

## 1:6 Unified Multimedia Segmentation

target specific children [27] (e.g., `example.xml#element(/1/4)`). The `xpointer()` scheme extends the functionality even more and allows operations on ranges and strings, allowing highly specific addressing of segments [18] (e.g., `example.xml#xpointer(string-range(//title, 'text', 4, 5))`).

### 2.2.3 Text

[78] proposes the usage of URI fragment identifiers for plain text files. Its proposal, later partly captured in [79], includes the `char` and `line` schemes, used either as single positions or ranges. For example, the segment of lines 11 to 20 is identified as: `http://example.com/text.txt#line=10,20`.

Additionally, the `length` and `md5` schemes are provided for integrity checks. In a similar fashion, [31] defines URI fragment schemes to select specific rows, columns, and cells in comma-separated values (CSV) documents. For example, a segment of cells that starts at the second row and second column and ends at the eighth row and sixth column is targeted like this:

```
http://example.com/data.csv#cell=2,2-8,6
```

### 2.2.4 MPEG-21

MPEG-21 is another standard developed by the MPEG, focusing on creating, delivering, and consuming multimedia [38]. The key components of the MPEG-21 framework include *digital items* created by and exchanged between *users*. Digital items are virtual containers for *resources*, which include both multimedia objects as well as metadata [11]. One such example is a music album, which, besides the actual audio tracks, may include lyrics, copyright information, cover art, and metadata about the songs or the artist.

Most relevant for this paper, part 17 of the specification defines URI-based fragment identifiers [39]. With four different schemes, which can also be combined, specific parts of resources can be addressed:

- `ffp()` identifies items and tracks by their name or ID.
- `offset()` is used to indicate the start and, optionally, the length of a byte segment of the resource.
- `mp()` is applicable to audio-visual resources and can be used to specify temporal, spatial, or spatio-temporal segments in numerous ways.
- `mask()` applies a binary mask to a video resource.

The following example applies both a temporal and a spatial segmentation to a video:  
`video.mp4\#mp(~time('ntp', '0:00:05', '0:00:15')*mp(~region(rect(100,100,200,200)))`

While these schemes provide powerful means to address segments without indirections, [52] deems them over-designed and ambiguous due to their complexity. For instance, both `ffp()` and `mp()` can be used to segment a track. Additionally, only MPEG formats are supported.

### 2.2.5 Temporal URI

Temporal URIs have been proposed as part of the Continuous Media Markup Language (CMML) and the file format Annodex, which together enable hyperlinking and annotating continuous media, like audio or video [57, 58]. A user can express one or multiple temporal segments by specifying start and end times: `video.mp4?t=15/25`. The specification allows both URI fragments and URI queries to be used depending on the needs.

### 2.2.6 Media Fragments URI 1.0

All file standards described above, although providing means to specify temporal and spatial segments, have certain problems associated, like indirections, complex expressions, or too narrow scopes. Therefore, [73] concludes that a standardized way to address segments is needed, which

should follow the *cool URIs* practices published by [16]. This allows media content to become a first-class citizen on the Web. Based on this proposition, the Media Fragments URI 1.0 W3C specification was created [75, 52]. URI fragments are chosen over URI queries, as fragments keep the relation to the original. That way, a media fragment is of the same file type as the parent resource. This setup ensures that media fragments follow the Linked Data principles [30]. The specification includes four fragment dimensions, which, due to being logically independent, can be combined freely:

- Temporal fragments are set with start and end time, where different time formats are supported: `video.mp4#t=5,15`
- Spatial fragments are limited to rectangles, specified as pixel coordinates or in percentages. The rectangle is distinctly defined by `xywh`, i.e., the position of the top left corner pixel and width and height of the bounding box: `image.jpg#xywh=200,200,100,50`
- One or multiple tracks can be segmented from media objects with multiple tracks (e.g., video, audio, subtitle track) by specifying track names: `video.ogg#track=audio`
- The id dimension is used to select a named fragment, such as a chapter from an audiobook: `audio.mp3#id=chapter-1`

In major browsers, however, only temporal fragments are natively supported.<sup>2</sup> Using the other dimensions requires custom software. As far as we were able to ascertain, there are also no other applications, such as dedicated media players, that natively support media fragments. In addition to that, existing Web infrastructure can only use byte ranges to refer to parts of a resource, like a temporal or named fragment. To resolve and retrieve media fragments over HTTP, different strategies are proposed in the specification and in literature [19, 22, 21, 48, 80].

- *No mapping*: As defined in [5], URI fragment identifiers are only interpreted by the user agent after the full resource is downloaded from the server. This approach, which can be found in browser implementations, only requires the user agent to be able to interpret and render media fragments. On the downside, this wastes bandwidth, as potentially large parts of the resource are discarded upon retrieval.
- *User-side mapping*: The user agent maps fragments to byte ranges. This is possible when the user agent has already downloaded parts of the resources, like headers or an index, and knows about the mapping. Then, a partial content request for this specific range can be sent to the server.
- *Server-side mapping*: The user agent requests the fragment in a special header, and the server performs the mapping. While special extensions must be installed on the server, this solution is the most efficient for bandwidth and latency. [19] implements a prototype in NinSuna, a platform for multimedia content selection and adaptation [20].
- *Third-party mapping*: A Media Fragments Translation Service (MFTS) [48] performs the mapping to byte ranges, either as a service for the user agent or as a proxy between a user agent and server. This approach requires minimal changes to the existing infrastructure.

While proper implementation of the full specification hence requires different client and server considerations, there exist various media management and retrieval systems that rely on media fragments, like the works of [42], [56], or [66]. The last, for example, implements a prototype system that can first segment videos of parliament speakers using video and audio segmentation techniques and second present the segments together with additional annotations on parliamentary websites. Beyond that, [47] uses media fragments for adaptive streaming over HTTP.

<sup>2</sup> <https://bugs.chromium.org/p/chromium/issues/detail?id=94368>  
[https://bugzilla.mozilla.org/show\\_bug.cgi?id=648595](https://bugzilla.mozilla.org/show_bug.cgi?id=648595)  
<https://trac.webkit.org/changeset/104197/webkit>

### 2.2.7 Media Fragments URI Extensions

The media fragments URI specification exhibits a number of limitations, such as no spatio-temporal fragmentation or spatial fragments being limited to rectangles. To address these limitations, extensions to the specification were proposed by [43]. Those extensions provide more options for spatial fragments, like circles, ellipses, or polygons. They also introduce static and animated shape transformations, which can be applied to spatial fragments. These additions enable more precise fragment specifications, as well as spatio-temporal fragments. The authors also provide a CSS extension to style fragments directly. On GitHub, more similar extensions can be found.<sup>3</sup>

While media fragments can be used in Semantic Web technologies, multimedia-specific queries are not supported. Hence, [44] presents SPARQL-MM, an extension that provides spatio-temporal filter and aggregation functions to SPARQL. This allows, for example, querying a video snippet that contains two specific persons, as long as the persons are identified by a fragment URI. [55] follows a similar idea and provides support for temporal SPARQL queries with media fragments. They introduce query logic that can compare temporal fragments to, for instance, query events that happened before a certain event.

### 2.2.8 International Image Interoperability Framework

The International Image Interoperability Framework (IIIF), as described by [71], is both a set of standards for image delivery and a community of cultural heritage institutions, like museums, libraries, and archives. Its goal is to allow the sharing of digital images using Linked Open Data principles. It provides application programming interfaces (APIs) for interoperable access to images and metadata stored in different repositories. At its core, IIIF has two APIs: The *Presentation API* provides metadata in the form of a JSON manifest. It describes a IIIF item (e.g., a book, newspaper, or artwork) as a collection of references to resources (e.g., scans of individual pages) and defines its structure and layout. Additionally, it can contain annotations, where media segments are typically referenced following the Media Fragments URI specification. Various IIIF clients exist that can load and display these manifests. Images are queried through the *Image API* with standard HTTP requests. A specific image and desired characteristics are specified in a URI following the format: `<image API>/<identifier>/<region>/<size>/<rotation>/<quality>.<format>`. The `identifier` references the image, `region` defines a rectangular segment, either as pixel coordinates, percentages, or “full”. `Size` and `rotation` are used to scale and rotate the region, respectively, while `quality` allows turning the image grayscale or black and white. Lastly, `format` specifies the image file format. The client application uses the region specifier to only fetch the image section currently in view. An advantage of this is that the browser can work with standard HTTP requests and does not need to distinguish between statically served and dynamically generated images. Thus, unlike when relying on the Media Fragments URI specification, the browser requires no additional logic to retrieve partial images. This fact makes it convenient to use IIIF image URIs in the general Semantic Web context independently of the presentation API [51].

With IIIF, the image server does all the work. It prepares the requested image segment from the original according to the request URI and sends it back over HTTP. The requested images are generated “on the fly” from high-quality images and transformed into the requested format. There exist several image server implementations online<sup>4</sup> and in literature [61].

<sup>3</sup> <https://github.com/tomayac/dynamic-media-fragments>  
<https://github.com/oaubert/mediafragment-prototype>

<sup>4</sup> <https://iiif.io/get-started/image-servers/>



It has been recognized that IIIF is limited to images and should be extended to support other media formats, such as audio and video. To that end, [15] coins the acronym IxIF to represent all non-image resources and provides an interim implementation; an audio and video (A/V) working group has been set up subsequently. The initial timetable was to extend the presentation API and then move on to content APIs. For the latter, [59] looks into an IIIF-based video framework and conducts some experiments on encoding and decoding speeds of videos, which is crucial for real-time delivery. Later, the working group decided to only focus on the presentation API, as it was deemed much more important. Nonetheless, there exists one account of an IIIF-like API for A/V content [50]. This API defines the request syntax `<A/V API>/<identifier>/<region>/<size>/<rotation>/<temporal_region>/<quality>.<format>`, where the added `temporal_region` parameter determines the time interval and controls playback speed.

### 2.2.9 EPUB Canonical Fragment Identifier

The EPUB Canonical Fragment Identifier (EPUB CFI) specification defines ways to access specific content inside an electronic publication (EPUB) document [72]. Such a reference code can, for example, look like this: `book.epub#epubcfi(/6/4[chap01ref]!/4[body01]/10[para05]/3:10)`. Slashes are used to navigate the document tree, where even and odd indices describe child elements and chunks of character data, respectively (/6 therefore denotes the third child element, while /3 refers to the second chunk of data). Exclamation marks perform an indirection step by accessing another document referenced in the current element. Square brackets assert the ID of the current element, and colons are used for character offsets. While the standard is generally used to extract XML or text elements, it also features identifiers for temporal, spatial, and spatio-temporal offsets. A fragment identifier in this format tends to become complicated quickly and only applies to the EPUB format, thus further limiting its applicability as a fragment representation.

### 2.2.10 OData

The Open Data Protocol (OData)<sup>5</sup> is a REST-based protocol for defining CRUD operations. It enables the encoding of filtering operations and function calls into HTTP URLs. An example for retrieving a filtered list of documents authored in or after 2024 that contain the term “Painting” in their title could look like this:

```
/OData/Documents?$filter=AuthorDate geq 2024-01-01 and contains(Name, "Painting").
```

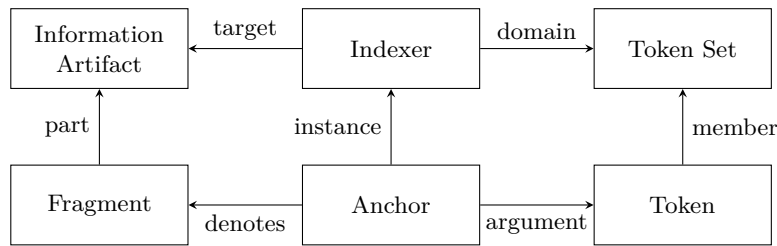
While OData is primarily used for structured data encoded in JSON, it has also found uses in hypermedia applications.

## 2.3 General Fragment Model

A more generalized approach to segment representation is presented in [23, 24]. They argue that all fragment identification schemes described so far are too narrow and basic. For this reason, they propose a General Fragment Model (GFM), an ontological model that can describe fragments of generalized media types and query them.

The model has the following three main concepts. First, *information artifacts* are objects with codified content, such as images, text, or audio files. Second, *indexers* are functions that map tokens to fragments of an information artifact. Lastly, an *anchor* is a particular token applied to an indexer and denotes a fragment of the information artifact. The set of possible tokens for a particular indexer is defined as a *token set*. Figure 1 shows the relations of these concepts.

<sup>5</sup> <https://www.odata.org/>



■ **Figure 1** Conceptual model of the General Fragment Model adapted from [23].

A simple example is an image information artifact and a `rect` indexer function that extracts rectangular sections from the image. A possible anchor is then `rect(img, [0,0,100,100])`, which defines a fragment of size  $100 \times 100$  in the top left corner of an image called `img`. Instead of `img`, another anchor can be passed, allowing for the arbitrary composition of anchors, like extracting a rectangular fragment but only the red color channel: `channel(rect(img, [0,0,100,100]), "red")`.

The authors further investigate classes of indexers and present a taxonomy of non-disjoint categories.

- *Identity indexers* map all segments to themselves.
- *Binary indexers* enumerate all bits of information of the information artifact. This indexer exists for every information artifact.
- *Tabular indexers* are further distinguished into *vector indexers* and *dictionary indexers*. Vector indexers create a vector space for the information artifact, such that the indexer domain is a totally-ordered set. This constructs a partial order of information parts. The *binary indexer* is a special type of this. Dictionary indexers map non-ordered input symbols to parts of the information artifact.
- *Spatio-temporal indexers* allow the selection of  $n$ -dimensional regions.
- *Query indexers* are formulae that reference media segments according to some criteria. The authors compare them to prepared statements in query languages like SQL.

The model is implemented in the knowledge representation framework *Hyperknowledge*, which focuses on multimedia content. It uses the construct of anchors to link fragments to graph nodes. For example, a spatial anchor on an image node represents a section of that image. By keeping the notions of indexer function and tokens very general, the GFM can be implemented in different forms supporting many use cases.

## 2.4 Interpretations and Implications

In summary, many ways exist to represent segmentation, either as a metadata annotation external to the medium or placed in a URI path, query, or fragment. One can observe some commonalities and tendencies in this multitude of proposals, further motivating this paper.

The number of proposed standards is remarkable, especially considering the relatively short period at the beginning of this century, in which most emerged. It indicates that many researchers saw a need for segment annotation at roughly the same time, but no single proposal could outdo the others. This, in turn, hints that each solution has shortcomings, be it only a very specific use case, restricted media support, or limited segment options. Conversely, universal and diverse approaches, such as MPEG-7 and later MPEG-21, quickly tend to be too complex and ambiguous. This is arguably also true for multimedia ontologies, where many either focus on specific media types or try to model MPEG-7. In many cases, the proposed standards are not widespread but occupy a rather niche space without much technological support.

■ **Table 1** Overview of segment annotation mechanisms found in the literature. ✓\* indicates that support for a functionality requires an extension beyond the original definition.

	SMIL	SVG	MPEG-7	XPointer	MPEG-21	Media Fragments URI	IIIF	EPub	GFM
External Metadata	✓	✓	✓	–	–	–	–	–	✓
Server-side	–	–	–	–	–	partial	✓	–	–
Spatial	some	✓	✓	✓	✓	✓*	some	✓	✓
Temporal	✓	–	✓	–	✓	✓	✓*	–	✓
Spatio-temporal	some	–	✓	–	✓	✓*	✓*	–	✓
Filter	–	some	–	–	–	–	some	–	✓
Text	–	–	–	✓	–	✓	–	✓	✓
Image	✓	✓	✓	–	✓	✓	✓	–	✓
Audio	✓	–	✓	–	✓	✓	✓*	–	✓
Video	✓	–	✓	–	✓	✓	✓*	–	✓
3D	–	–	–	–	–	–	–	–	✓
Type-agnostic	✓	✓	–	–	–	–	–	–	✓
Format-agnostic	✓	✓	–	✓	–	✓	✓	–	✓
Implementation Available	✓	✓	✓	✓	–	partial	✓	✓	–

The first somewhat common approach is the Media Fragments URI 1.0 specification, whose authors identified the above-mentioned issues. The result is a standard describing temporal, spatial, track, and id segments in a URI. This is, however, not without problems. Most notable for this paper is that while different segment types are supported, their flexibility is still lacking, as, for example, spatial segments can only be rectangles. Although there are proposals for extensions, none of them seem to have been picked up, which appears to be a general theme again. A considerable body of literature exists about the specification, but real applications are relatively rare, and browser support is still partially or fully lacking.

Interestingly, the standard with the most real-world relevance appears to be the IIIF image API, which is the only one coming from a different origin and motivation. Nonetheless, it offers a way to describe an image segment with a distinct URI, making it suitable for use outside of IIIF as an identifier of a resource in a knowledge context. Again, the segment flexibility is minimal, with only rectangles supported, and extensions were looked into but, so far, came to nothing.

Table 1 provides a comparison of the concrete segmentation annotation mechanisms discussed in this section. It shows that there is a broad range of capabilities, both in theory and in practice, across the various schemes that can be found in the literature. Most of the schemes focus on one specific type of media, most commonly text or images, or on a specific set of use cases.

Generalizations towards arbitrary media or segmentations are underrepresented in literature, with the General Fragment Model being the only account. It shares a similar derivation and motivation with this paper but then focuses more on an ontological structure. While it can describe virtually any segmentation, it offers few deductions of properties toward possible implementations. There appears to exist an implementation of numerous types of segmentations following the General Fragment Model, but these seem to be built case by case instead of in a general manner, as the model suggests. Unfortunately, the implementation is mostly closed-source and provides few insights into the applied mechanics.

## 1:12 Unified Multimedia Segmentation

We aim to close this gap by proposing the Unified Segmentation Model for media documents. The model combines the expressiveness of the various methods discussed above into a simple and unified representation. Furthermore, it is specified in such a way that it does not rely on the client to implement the segmentation scheme. This makes it much more easily interoperable with existing standards and technologies. The following discusses the model and how it can be used in the context of multimodal knowledge graphs.

### 3 Unified Segmentation Model

In this section, the formal part of the Unified Segmentation Model is designed and presented. It aims to abstract various media segmentations and provide a generalized segmentation operation independent of the media document type. First, the formal model is developed, and properties of different types of segmentations are discussed. The second part investigates approaches for describing segmentation rules. This sets the stage for concrete types relevant to the implementation.

#### 3.1 Multimedia Object

To arrive at a model of segmentation, the first step is to generalize multimedia documents regardless of their type. As established previously, any multimedia document consists of  $n$  dimensions, where each can be temporal or spatial. An audio track, for example, has one temporal dimension, an image has two spatial dimensions, while a video comprises one temporal and two spatial dimensions. These  $n$  dimensions span an  $n$ -dimensional space in which a concrete multimedia object can be seen as a bounded, finite subset of the space. Usually, this subspace is not continuously defined but consists of a finite amount of discrete data points. In a raster image, these are the pixels, and the subspace is bounded by the width and height of the image, whereas a video is a temporal enumeration of images additionally bounded by a duration.

The data point coordinates only determine the extent of the media object. To fully represent a specific piece of multimedia, additional description information for each data point can be required, again consisting of one or multiple dimensions. For a digital audio track, each sampling point has an amplitude. For an image, each pixel needs a color value, for instance, a mixture of red, green, and blue colors. Alternatively, the descriptions can also be empty, as is the case with a 3D model, provided that no materials or surfaces are defined.

Formally, any arbitrary multimedia object  $M$  can thus be described as

$$M \subset P \times D, \tag{1}$$

where  $P$  and  $D$  indicate the space of possible data points and descriptions, respectively. They are both defined as  $n$ -ary cartesian products of a number of dimensions

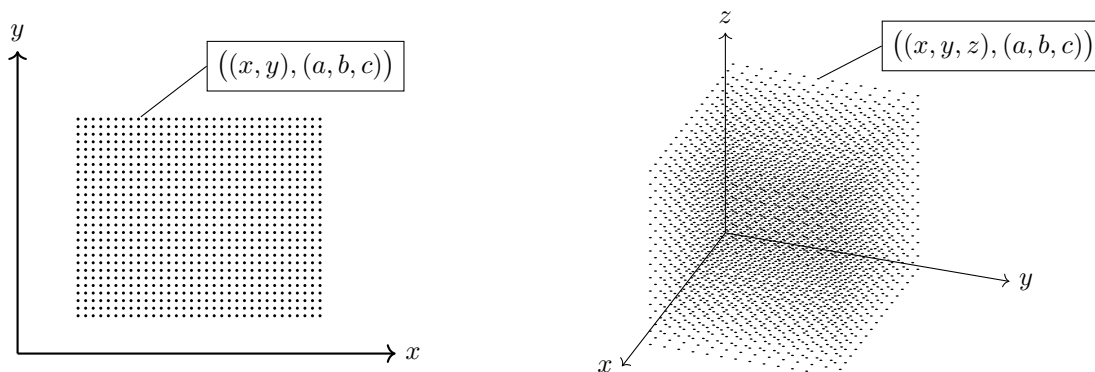
$$P = \text{dim}_1 \times \dots \times \text{dim}_n \quad \text{and} \quad D = \text{dim}_1 \times \dots \times \text{dim}_m, \tag{2}$$

where each dimension  $\text{dim}$  is an arbitrary number set, such as  $\mathbb{N}$  or  $\mathbb{R}$ . The number set depends on the application and can have constraints. An RGB color channel, for instance, is limited to 256 elements, while each ASCII character is one of 128. Figure 2 gives a visualization of this definition for two- and three-dimensional data points. Formulated differently, the media object  $M$  defines a binary relation with domain  $P$  and codomain  $D$ . Such a binary relation fulfills the functional (right-unique) and total property, i.e.

$$\forall p \in P, \forall d_1 \in D, \forall d_2 \in D : ((p, d_1) \in M \wedge (p, d_2) \in M) \Rightarrow d_1 = d_2 \tag{3}$$

$$\forall p \in P, \exists d \in D : (p, d) \in M \tag{4}$$

In other words, this means that data points are unique, while their descriptions are not, and every data point has a valid description. From this, it follows that the relation is a function  $P \rightarrow D$  but is neither injective nor surjective.



■ **Figure 2** Schematic representation of a media object consisting of ordered pairs of data points and descriptions. Each data point has  $n$  dimensions (here  $x, y$  or  $x, y, z$ ) and each description has  $m$  dimensions (here  $a, b, c$ ).

## 3.2 Segmentation Operation

Segmentation is defined as the process of extracting a part from a whole. Applying this to the formal definition of a multimedia object, a segmentation function can be defined as

$$s : M \rightarrow M', \quad M' \subseteq M. \quad (5)$$

It takes as input a media object  $M$ , which is an arbitrary subset of the space  $P \times D$ , and yields another media object  $M'$  of the same space. To uphold the definition of segmentation, the result must be a subset of the original media object. This very general function  $s$  can have all sorts of implementations depending on the segmentation and media object at hand but can be grouped into three variants – filters, reductions, and transformations – as well as post-processing operations, which we discuss in the following.

### 3.2.1 Filter Segmentation

Most segmentation tasks revolve around wanting to retain a subset of the data points of the initial media object. The filter operation is thus generally defined as

$$s_f(M) = \{m \mid m \in M \wedge f(m)\}, \quad (6)$$

where the filter function

$$f : P \times D \rightarrow \{true, false\} \quad (7)$$

is an *indicator function* that decides for each data point in the space whether it should be kept or not. It can use any properties of the data point itself or its description to make the decision. Examples of such filter segmentations are the selection of a rectangular cutout of an image, a scene from a movie, an excerpt of an audio recording, or one page from a PDF document.

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The filter segmentation is not limited to a single filter function but can consist of an arbitrary number of sub-filters  $f_1, \dots, f_k$ , which are combined using Boolean algebra operations. For only conjunctions, the filter operation is defined as

$$s_f(M) = \{m \mid m \in M \wedge f_1(m) \wedge \dots \wedge f_k(m)\}. \quad (8)$$

For arbitrary Boolean operations, data points are part of the segment if the Boolean combination of indicator functions yields 1. This allows the mixture of any filter criteria, for instance, only keeping image pixels of certain colors, which are located in a certain area of the image and do not have large illumination changes compared to neighboring pixels. From Equation 8, it follows that

$$\begin{aligned} & \{m \mid m \in M \wedge f_1(m) \wedge \dots \wedge f_k(m)\} \\ \Leftrightarrow & \{m \mid m \in M \wedge f_1(m)\} \cap \dots \cap \{m \mid m \in M \wedge f_k(m)\} \\ \Leftrightarrow & s_{f_1}(M) \cap \dots \cap s_{f_k}(M), \end{aligned} \quad (9)$$

using the definition of set intersection. Equally, by the definition of set union, it follows that

$$\{m \mid m \in M \wedge (f_1(m) \vee \dots \vee f_k(m))\} \Leftrightarrow s_{f_1}(M) \cup \dots \cup s_{f_k}(M). \quad (10)$$

This shows that the conjunction and disjunction of sub-filters are equivalent to the intersection and union of segments, respectively, where each segment is created using one particular sub-filter. The same relationship can also be shown for arbitrary combinations of operators.

As the segmentation result is another media object, multiple filter segmentations can be applied consecutively, i.e.,

$$s_{f_k}(s_{f_{k-1}}(\dots(s_{f_2}(s_{f_1}(M)))))) = (s_{f_k} \circ \dots \circ s_{f_1})(M). \quad (11)$$

Thereby, the multimedia object is increasingly filtered by every segmentation. This corresponds to applying all filter operations in the same segmentation operation, which can be written as a conjunction of filters. As derived above, this is equivalent to the intersection of the segments that result from applying the filter segmentations to the original media object. It follows that nested filter segmentations are commutative:

$$(s_{f_k} \circ \dots \circ s_{f_1})(M) = s_{f_1}(M) \cap \dots \cap s_{f_k}(M) = (s_{f_1} \circ \dots \circ s_{f_k})(M) \quad (12)$$

As an illustration, consider a temporal video segment, which is afterward cropped to a square. Its result is equivalent to first cropping the whole video and then cutting it to the desired length. The segmentations can also operate on the same dimensions, such as in an image with overlapping segment areas, where the order of segmentation does not matter for the outcome.

### 3.2.2 Reduction Segmentation

Alternatively, a user might want to keep all data points but segment the description of each point. This reduction operation

$$s_r(M) = \{(p, r(d)) \mid (p, d) \in M\} \quad (13)$$

applies a reduction function

$$r : D \rightarrow D', \quad \dim(D') \leq \dim(D) \quad (14)$$

to the description of each data point. Like the filter function, this dimensionality reduction function is not allowed to modify the individual data points but can only select which dimensions to keep or discard. A possible application could be extracting one color channel of an image.

The reduction function can be seen as a special filter function, which operates on only the codomain of the media object. However, it does not filter elements of the set but the description dimensions of each element. In other words, the scope of each data point description is changed. Hence, set theory operations are not applicable in a mathematical sense. However, one can interpret the description of a data point as an  $m$ -dimensional set instead of an  $m$ -tuple of values. Thereby, the reduction function can be described as a filter function on each set of values, enabling the combination by set operations. Then, the reduction operation is commutative. See Section 4.2 for a practical example.

Filter and reduction operations can also be combined but require caution, as commutativity is not always given. In some cases, the filter operation might use properties affected by a reduction operation, for example, a filter operation based on color and a reduction operation that drops specific color channels. In other cases, segmentations are commutative, like selecting one color channel and cropping the image. A solution to this uncertainty is adapting the filter operation to only use data point information (i.e.,  $P$ ) without changing its semantics. For instance, the filtering based on color can also be represented as filtering of certain point coordinates that happen to have said color. With this trick, filter and reduction functions operate on disjoint pieces of information, which ensures commutativity.

### 3.2.3 Transformation

Besides operations that directly affect ordered pairs of the set, utility operations need to be considered. One issue is the fact that a certain media object can have different representations, where each one is defined in a different space. To ensure that any segmentation can be applied, the media object might need to be transformed in preparation. Such a transformation function

$$t : M \rightarrow M', \quad M \subset P \times D, \quad M' \subset P' \times D' \quad (15)$$

transforms a media object  $M$  from one space to a media object  $M'$  in another space, such that  $M$  is semantically equivalent to  $M'$ . The two media objects can have different sets of dimensions. An example of that is the Fourier transformation of an audio signal from time and amplitude dimensions to frequency and magnitude dimensions. Any such transformation must be reversible, such that the inverse transformation can be applied to the segmented media object to restore the format of the original media object. Note, however, that such transformations can be lossy.

### 3.2.4 Postprocessing

Postprocessing operations can be applied to the result of a segmentation operation. Strictly speaking, these are not part of the segmentation definition, as they modify the media object instead of just selecting a fraction of it. However, for real-world applications, postprocessing can be vital for usability, as it allows speeding up a song, rotating an image, performing anti-aliasing to smoothen edges, or removing a constant dimension. The last allows turning a single frame of a video, which is so far just a very short video, into an image by removing the temporal dimension. As these operations do not fulfill the segmentation definition, they are not discussed further in this context.

## 3.3 Compound Media Object

So far, a media object is considered a subset of a single  $n$ -dimensional space. However, there are cases where the media object comprises a collection of individual media. Such a compound media object  $C$  can be defined as

$$C = \{M_i \mid i = 1, \dots, m\}, \quad (16)$$

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where each  $M_i$  corresponds to a media object as defined in Equation 1 and can potentially have different dimensionalities and structures. A prime example of a compound media object is a movie, which consists of a video track, one or multiple audio tracks, and possibly subtitles.

The segmentation operations established before are also applicable to compound media objects by applying the operation to each individual sub-object, i.e.,

$$s(C) = \{s(M_i) \mid M_i \in C\}. \quad (17)$$

In practice, this can result in any number of sub-objects being affected by the segmentation, and the others are left unchanged. If the movie is cropped, only the video track is modified, but if a scene is extracted, all sub-objects are temporally segmented. A special case of compound media object segmentation is removing all data points from one or more sub-objects. This corresponds to a subsetting operation and can, for example, be used to extract a movie's audio. While this is technically a filter operation that leaves certain media objects empty, it resembles more a reduction operation, where certain elements are dropped.

### 3.4 Segmentation Definedness

So far, the filter and reduction operations retain a subset of data points and description dimensions, respectively. An issue with this generalization is that the promise of the segmentation definition, extracting a part from a whole, is not necessarily given. With the ultimate goal of real-world applicability, some considerations need to be made.

The filter operation may select multiple non-overlapping clusters of data points. For example, two rectangles from an image with space between them. To keep the semantics of the media object and thus remain a valid segmentation, the segmented data points need to keep their relative position. Therefore, the space between the rectangles has to remain, and the two segment parts cannot be shrunk together. To this end, all data points inside the bounding box of the segmentation need to be retained. Only data points beyond the bounds can be safely discarded. To still be able to represent any segmentation, the description must be set to `null` if the data point is not part of the desired segment. For that, the notion of neutral elements  $o$  and neutral descriptions  $(o_1, \dots, o_m)$  is introduced. A neutral element is a special value of that specific dimension, which indicates the absence of that dimension. Neutral descriptions are  $m$ -tuples of neutral elements, which fully void the description part of a data point without changing its signature. In the example of the two rectangles in an image, all pixels between the rectangles are neutralized by setting them to transparent color values. Regions outside of the rectangles can be cropped safely.

The reduction operation suffers from a similar issue. If the dimensionality of the data point description is changed by discarding undesired dimensions, the media object can potentially lose its definedness. For instance, an image in RGBA color space can get rid of its alpha channel and still be a valid RGB image, but an image with only two color channels is invalid or requires special file formats for storage. As a solution, the reduction function needs to replace dimensions not part of the segmentation with a neutral element  $o_j$  of the specific dimension. For RGB, the undesired color channel would be set to zero for each data point, such that this color no longer contributes to the image.

## 4 Applied Segmentation

To ultimately be able to implement the Unified Segmentation Model in a real application, another level of abstraction needs to be discussed. First, the classes of operations outlined above need to be refined into more distinct and meaningful subclasses covering all realistically desired segmentations.



Second, thought must go into the serialization of segmentation. In other words, schemes must be developed on how the segmentation instructions can be represented textually. The goal is to fit this description into a URI to ensure both usability and compactness without sacrificing expressiveness. All segmentation URIs described in this chapter have the form

```
<reference to multimedia object>/<segmentation type>/<segmentation description>
```

First, the base of the URI has to reference the media object on which the segmentation is performed. For convenience, this part is omitted from here on. Then, the URI needs to inform about the specific segmentation type via a keyword, followed by the actual description of the segmentation of this type. The keyword is necessary to distinguish between similar segmentations, which cannot be inferred from the description alone. This allows proper segmentation processing and ensures uniqueness across types. Additionally, it needs to be discussed how to ensure that a particular segmentation has only one unambiguous URI representation, such that each segmentation is uniquely definable.

## 4.1 Filter Segmentation

For filter segmentation, we can distinguish between several different types of filter functions. The following lists their properties and presents URI schemes for the representation of each of them.

### 4.1.1 Segment Boundary

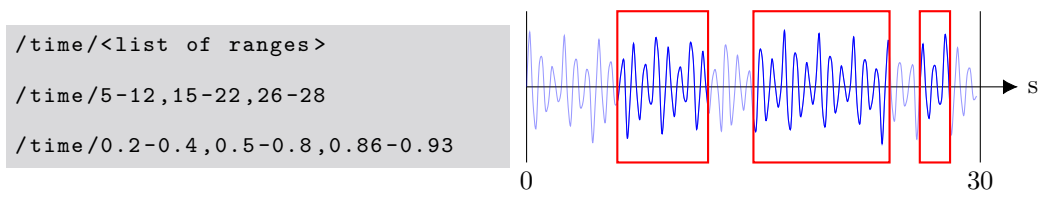
Segments can be described in terms of their boundary, as segment data points are usually coherent. Due to the limited amount of information needed, this approach is well suitable to be conveyed in a URI. This is supported by the observation that all URI-based segment annotation approaches presented in Section 2 describe segments in terms of boundaries. Depending on the dimensionality of the boundary, different considerations are necessary.

#### One-dimensional Segmentation

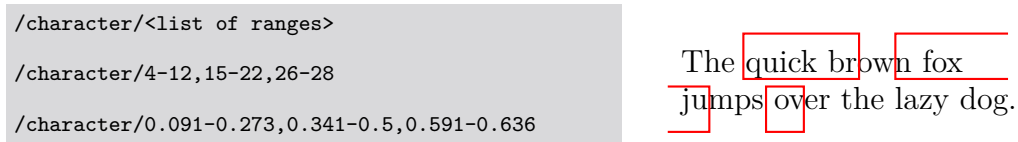
For segmentation in only one dimension, things remain relatively simple. However, flexibility is needed to handle different measures and units because many multimedia documents can be segmented along a single dimension. For instance, a video can be segmented with timestamps or frame numbers, while a PDF or text document is only segmented along integer page or character numbers, respectively. In all cases, a segment is serializable as a list of intervals in ascending order. This is showcased in Figures 3 and 4 with a 30 seconds audio clip and a 44-character text sentence, respectively. As an alternative to absolute numbers, the same segment can be equivalently defined using relative numbers in the range  $[0, 1]$ . The output can be shrunk to the bounds of the segment, cutting off the irrelevant parts at the beginning and end. Inside the bounds, parts not in the segment cannot be removed but must be nullified to adhere to the segmentation definition. In audio, this means muting non-segment parts, a video could turn those parts black or transparent, and text documents could replace the respective characters with whitespaces.

Sometimes, the media object is not represented in the correct space to apply the segmentation directly. In such a case, a transformation needs to be applied first. A prime example of this is segmenting an audio track by frequency, which requires a Fourier transformation from time and amplitude space to frequency and magnitude space. Then, one-dimensional segmentation can be performed as usual before the result is transformed back to its original space, here using an inverse Fourier transformation. It is to mention that while the formal definition required an explicit statement of this transformation and its inverse, this is no longer necessary but implicitly

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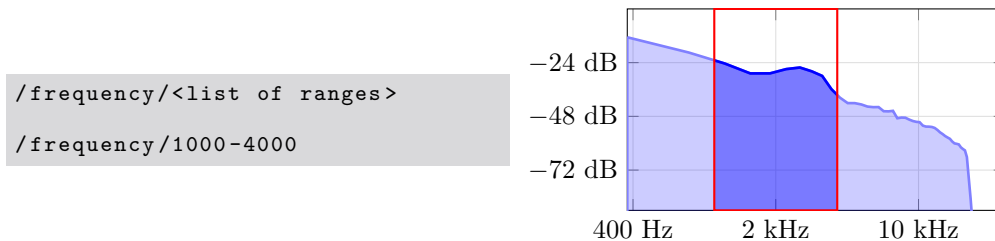


■ **Figure 3** An example segmentation of an audio snippet by time in seconds. The absolute and relative URI representations on the left correspond to the visualization on the right, with the segmented parts shown in red.



■ **Figure 4** An example segmentation of a plain text by characters. The absolute and relative URI representations on the left correspond to the visualization on the right, with the segmented parts shown in red.

derived from the specified segmentation type. An example of a frequency segmentation is shown in Figure 5, where a frequency range is selected. This particular segmentation thus corresponds to applying both a high- and low-pass filter.



■ **Figure 5** An example segmentation of an audio snippet by frequency between 1000 Hz and 4000 Hz. A Fourier transform operation precedes the segmentation to translate the time space to a frequency space.

### Two-dimensional Segmentation

Segmentations along two simultaneous dimensions need to describe areas of a plane spanned by the two dimensions. This type of segmentation can be applied to various media, such as images, videos, or PDF documents. In all cases, the media object can be viewed as a single or stack of two-dimensional planes, and the segmentation punches out a specific shape or area of each plane. A suitable analogy would be that of a cookie cutter.

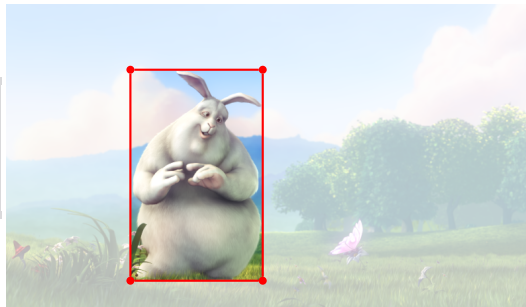
#### Rectangle

In the simplest case, a single rectangular shape is desired. Assuming that the rectangle is not tilted, it is sufficient to define the minimum and maximum  $x$  and  $y$  coordinates, either as absolute values (e.g., pixels) or relative numbers in range  $[0, 1]$ . An example is presented in Figure 6, where Big Buck Bunny is segmented from a screenshot of Blender Foundation's eponymous short film [8].

```

/rect/<xmin>,<xmax>,<ymin>,<ymax>
/rect/165,340,35,315
/rect/0.236,0.486,0.088,0.785

```



■ **Figure 6** Rectangular segmentation of a  $700 \times 400$  screenshot from Big Buck Bunny [8]. The URIs of absolute and relative point coordinates on the left correspond to the visualization on the right.

### Polygon and Spline

If the segment area is still one closed shape but more complex, polygonal or spline structures can be used when the edges are straight or rounded, respectively. Such forms are describable by an enumeration of points on the plane. If a polygon is desired, these points are connected by lines between consecutive points. In the case of closed splines, the point requirements depend on the type of spline. For instance, cubic Bézier splines require each Bézier curve to have four control points, whereas basis splines need additional care to ensure a closed shape [69]. In any case, the points can be serialized as shown in Figures 7 and 8 as absolute or relative coordinates. To ensure the unambiguity of the URI, one can establish a rule in which winding direction to specify points and with which point to start, such as starting at the left lowermost coordinate and proceeding clockwise. The output image can be cropped to the rectangular bounding box of the segmentation. Pixels inside the bounds that are not part of the segment need to be changed to transparent. The model does not explicitly define how to handle pixels on the boundary.

```

/polygon/<point1>,<point2>,<point3>,...
/polygon/(170,35),(165,180),(225,320)
, (340,275),(335,65),(300,35)
/polygon/(0.243,0.088),(0.236,0.45)
, (0.321,0.8),(0.486,0.688)
, (0.479,0.163),(0.429,0.088)

```



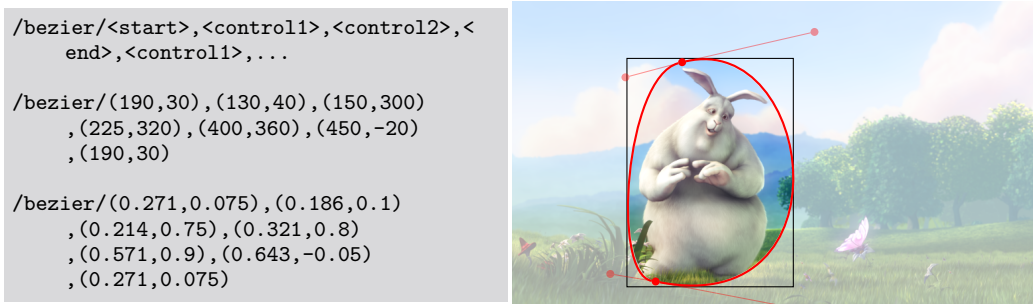
■ **Figure 7** Polygonal segmentation of a  $700 \times 400$  screenshot from Big Buck Bunny [8]. The URIs of absolute and relative point coordinates on the left correspond to the visualization on the right. The polygon is shown in red, and the resulting image is outlined in black, where pixels outside of the polygon are turned transparent.

They can either be assigned proportionally using an alpha component or completely based on the majority of their area (as is the case in our implementation, see Section 6),

### SVG Path

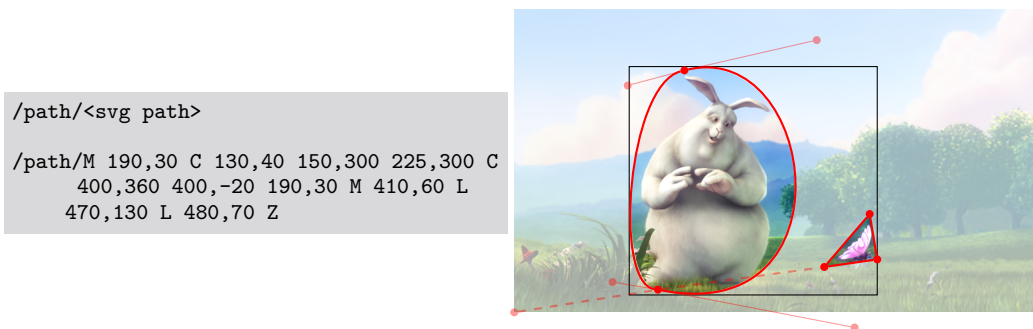
As an alternative, or if the segment comprises disjoint segments, the concept of SVG's path property can be used to outline the segment area. Its specification is made to represent a shape as a description string, serialization is thus built-in. It features instructions to define linear and

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■ **Figure 8** Bézier spline segmentation of a  $700 \times 400$  screenshot from Big Buck Bunny [8]. The URIs of absolute and relative point coordinates on the left correspond to the visualization on the right. The shape is drawn in red, with semi-transparent Bézier control points, and the resulting image is outlined in black, where pixels outside of the shape are turned transparent.

curved line segments and an operator to move the cursor to draw disconnected shapes [40]. All of them are used in the example shown in Figure 9, where the Bézier spline segment from Figure 8 is extended with a disjoint triangular segment. On the downside, unique serialization is not as straightforward, especially with composite shapes. Nonetheless, one can still come up with some ordering, especially since the move operation allows a lot of flexibility.



■ **Figure 9** SVG path segmentation of a  $700 \times 400$  screenshot from Big Buck Bunny [8]. The URI on the left corresponds to the visualization on the right. The segment is drawn in red, with the Bézier control points given as semi-transparent lines, and the cursor move command M is displayed as a dashed line. The resulting image is shown in black, where pixels outside of the boundaries are turned transparent.

### Three-dimensional Segmentation

Three-dimensional segmentations need the most care, as their dimensional complexity can lead to computational and representational limitations. The cookie-cutter analogy no longer holds here, as the third dimension is also constrained in some way. Instead, one can think of these segmentations as taking the cuboid of the media object and carving it into, possibly multiple, arbitrary shapes.

### Rotoscoping

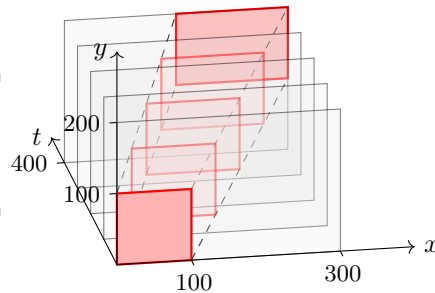
If the segmentation is simple enough, a shortcut can be made by diminishing the importance of one dimension, therefore turning the problem into a 2.5-dimensional segmentation. In what is known in the film industry as rotoscoping, a 3D segment is described as a series of keyframes.

Each keyframe defines a two-dimensional segment at a point in the third dimension. For points not directly defined, the shape can be linearly interpolated from the nearest neighbors. To, for instance, segment a car passing through a video scene at a constant speed, it might be sufficient to segment only the first and last frame of the scene in question and interpolate everything in between. This approach, therefore, allows for smooth three-dimensional segmentations with minimal instructions. This is beneficial for serialization, as seen in the example in Figure 10. Nonetheless, this approach can also be used for complex segmentations, for example, by defining an image mask for each video frame.

```

/rotoscope/<timestamp>,<type>,<description>;
...;<timestamp>,<type>,<description>

/rotoscope/0,rect,0,100,0,100;
400,rect,150,300,100,200
    
```



■ **Figure 10** Rotoscope segmentation, where a series of 2D shapes determines the 3D segment. The URI of two rectangles at different points in time on the left implicitly defines shapes for time points in between, as visualized on the right.

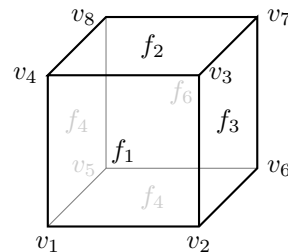
### 3D Mesh

If the segmentation becomes too complex, it can alternatively be described as a three-dimensional mesh. Applying this mesh to a media object largely depends on the media type. In the case of a video, the 3D mesh must be sliced using the time coordinate to yield a 2D segment for each video frame. If the media object is itself a 3D object, the intersection must be computed between it and the segment, for example, using Constructive Solid Geometry algorithms [60, 46].

While this approach allows virtually any three-dimensional segmentation, the URI serialization can become complex, as the complete mesh definition must be embedded into it. For that, it makes sense to draw inspiration from simple mesh file formats, such as Wavefront OBJ, Polygon File Format (PLY), or StereoLithography File Format (STL) [54, 77, 1]. In the first, for example, vertices are specified with a **v**, followed by the coordinates, and faces with **f** and a list of vertex indices. The unit cube, for example, is then defined as displayed in Figure 11.

```

1 v 0 0 0
2 v 1 0 0
3 v 1 1 0
4 v 0 1 0
5 v 0 0 1
6 v 1 0 1
7 v 1 1 1
8 v 0 1 1
1 f 1 2 3 4
2 f 5 6 7 8
3 f 1 2 6 5
4 f 2 3 7 6
5 f 3 4 8 7
6 f 1 4 8 5
    
```



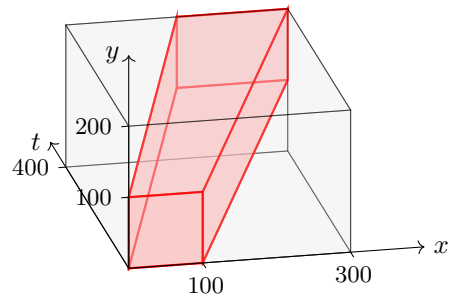
■ **Figure 11** Unit cube in the Wavefront OBJ 3D mesh format. On the left, vertices are described by their coordinates and faces by their vertex indices, and the resulting hexahedron mesh is shown on the right.

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Instead of listing the entities by line in a file, they can be encoded into a URI in a comma-separated sequence. One issue with these file formats and, hence, the URI serialization is that the order of vertices and faces can be arbitrary. The same object can thus have many different serializations. To solve this ambiguity, faces of more than three vertices can be triangulated, and faces and vertices can be sorted based on a given rule.

An applied example is shown in Figure 12, where the same three-dimensional segmentation as in Figure 10 is presented, this time as a 3D mesh in Wavefront OBJ format.

```
/mesh/<vertices>,<faces>  
  
/mesh/v 0 0 0,v 0 100 0,v 100 0 0,v 100 100 0,  
v 150 100 400,v 150 200 400,v 300 100  
400,v 300 200 400,f 1 5 6,f 1 5 7,f 1 6  
2,f 1 7 3,f 2 6 8,f 2 8 4,f 3 1 2,f 3 2  
4,f 5 7 8,f 5 8 6,f 7 3 4,f 7 4 8
```



■ **Figure 12** Three-dimensional segmentation defined using a 3D mesh. The serialized Wavefront OBJ format on the left corresponds to the red hexahedron on the right.

### 4.1.2 Cut Equation

An alternative to the straightforward boundary definition is the usage of algebraic expressions, which describe hypersurfaces and can cover a variety of smooth shapes. As these expressions require little description space, they are a good candidate for short URIs. In Figure 13, where the equation is  $y > -0.019x^2 + 9.6x - 900$ , all points above the curve are segmented as they fulfill the stated inequality. As a last step, the output can be cropped to the segment bounds, which is the whole image in this case.

```
/cut/<equation>  
  
/cut/y > -0.019x^2 + 9.6x - 900
```

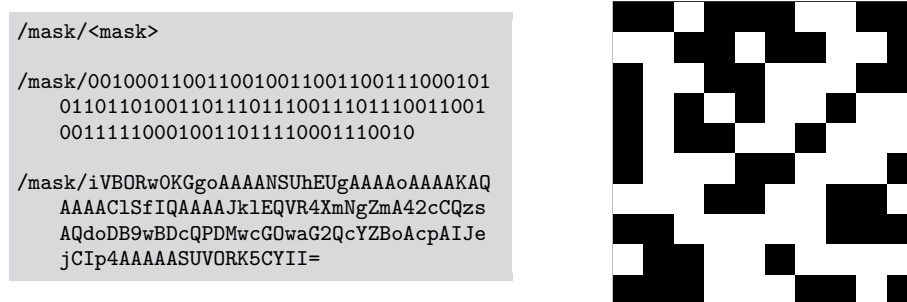


■ **Figure 13** Two-dimensional segmentation of a screenshot from Big Buck Bunny [8] using a quadratic equation to define a cut. The URI must indicate which side of the cut to keep, either as a Boolean value or as an inequality. The URIs on the left yield the segment visualized on the right.

### 4.1.3 Explicit Masking

If describing segment boundaries becomes too complex, an alternative, which allows data point-accurate segment description, is a binary mask. Such a mask can be described in a URI as an enumeration of binary values. This scheme is usable for any media object type, but an enumeration

order needs to be established for media of more than one dimension. For images, this can, for instance, be line-wise from top to bottom, as shown in the example in Figure 14. When the mask is two-dimensional, it can alternatively be represented as a serialized binary image, which can additionally be base64-encoded to shorten the length of it drastically. In that case, no ambiguities must be dealt with, provided the mask matches the media object. This approach, also presented in Figure 14, corresponds to the Data-URL scheme specified in [53]. In the example's case of only  $10 \times 10$  pixels, listing binary values is more efficient, but with larger masks, the encoded PNG becomes superior. In the worst case, when the mask is completely randomized, the base64 encoding is one sixth the size of the binary enumeration. Otherwise, the encoding can be multiple orders of magnitude smaller with more coherent masks.



■ **Figure 14** Two variants of describing a randomized two-dimensional segmentation mask, either as binary values or the base64 encoding of a binary PNG image.

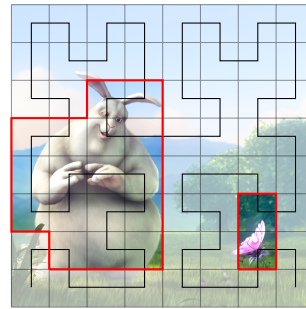
#### 4.1.4 Space-filling Curve

Clear enumeration orders of data points are provided by space-filling curves, making its application rather straightforward. Due to its locality preservation properties, the Hilbert curve is chosen for this purpose. While such a curve is defined to visit every data point, one can instead play with the curve's order. This determines the maximum index and thus controls the resolution of the segmentation. For example, a first-order Hilbert curve for two dimensions splits each dimension in half, yielding four sub-squares, each corresponding to one curve index. Therefore, the URI serialization of a segmentation needs to carry information about the order of the curve and a series of index ranges. An example of a third-order Hilbert curve applied to an image is given in Figure 15. It shows a roughly similar excerpt as the previous image segmentations, but precision could be improved by increasing the order and, hence, the number of index ranges. In the same way, Hilbert curves can also be used to segment media of higher dimensions, such as videos. In both cases, the media object can be cropped to the segment bounds, and data points not in the segment need to be neutralized.

## 4.2 Reduction Segmentation

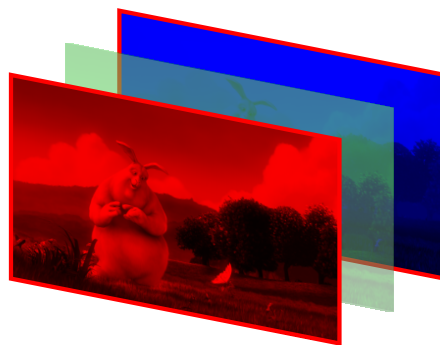
Reduction segmentations have less variability, making them simpler to serialize and able to cover all desired segmentations simultaneously. In essence, one only needs to list the description dimensions to retain. The allowed dimension names depend largely on the concrete segmentation. In Figure 16, only the red and blue color channels are selected by listing them by their name in the URI. In the output image, the green color value is set to zero for all pixels, such that the image is still storable in RGB format.

```
/hilbert/<curve order>,<index ranges>
/hilbert/3,2,6-17,28-31,50,61
```



■ **Figure 15** Hilbert curve segmentation of a square screenshot from Big Buck Bunny [8]. The Hilbert curve is of third order, which corresponds to 64 indices. The URI on the left first specifies the curve order, followed by the index ranges to segment. This segment is visualized on the right, where the curve starts with index 1 in the bottom left corner.

```
/color/<selections>
/color/red,blue
```



■ **Figure 16** Segmentation by color of a screenshot from Big Buck Bunny [8]. The URI of a two-color selection on the left is visualized on the right.

### 4.3 Compound Media Segmentation

All types described so far also translate to compound media objects. In most practical examples, one or more sub-objects are segmented while the others remain untouched. The only thing left out is the special case of selecting a subset of media objects. Although this operation is, strictly speaking, filtering, it makes sense to classify this among the reduction segmentations for practicality. The main reason is that sub-objects to segment need to be specified in the same manner with a list of keywords. In the compound media example of a movie with video, audio, and subtitles, a segmentation for only audio could then look like `/stream/audio` or alternatively `/stream/1` if the audio stream is known to have an index of 1. The result of this segmentation is then solely the audio of the movie.

### 4.4 Segmentation Combination

As established in the formal model, filter functions can be combined using Boolean operations, which correspond to set operations between the segments. The intersection, in particular, is inherently already defined through the proposed URI scheme. All other operations, such as complement, union, or set difference, exist between two segments but would require additional care and investigation. An important fact for the intersection is that the result of segmentation on a media object is another valid media object. This means that a second segmentation can be applied. As shown, this consecutive application of two filter segmentations corresponds to the intersection of the segments, where the order in which the two are listed is irrelevant. Such a URI of arbitrarily many nested segmentations then has the form



```
</segment type 1><segment description 1>/.../<segment type n><segment description n>
```

This allows two considerations. First, it creates the possibility of describing a high-dimensional segmentation in terms of a combination of lower-dimensional segmentations. For instance, a spatio-temporal segmentation of a square of a video snippet

```
/rotoscope/200,rect,0,100,0,100;600,rect,0,100,0,100
```

is equivalent to segmenting first by time and then by space or vice versa

```
/time/200-600/rect/0,100,0,100
```

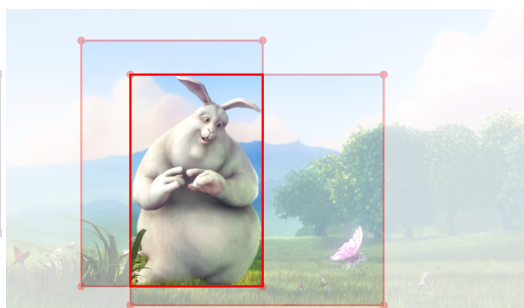
This only works if the dimensions are not disjoint, as, in this example, the rectangle is independent of time and can thus be taken apart.

Second, two filter segmentations can operate on the same dimensions. From a theoretical perspective, this combines the filter functions using logical conjunctions, as described previously. In practice, however, one must be cautious, as a segmentation can introduce a linear translation of the data point coordinates. This is due to neutral data points outside of the segment boundaries being trimmed. For illustration, consider two segmentations that extract a rectangular shape from a  $700 \times 400$  image, as shown in Figure 17. The first segmentation changes the image dimensions to  $240 \times 325$ , and the lower left corner moves from  $(100, 35)$  to  $(0, 0)$ . The second segmentation must thus be linearly translated with respect to the first one. Only then can the two URIs be concatenated to result in the correct intersection segment.

```
/rect/100,340,35,360
```

```
/rect/165,500,10,315
```

```
/rect/100,340,35,360/rect/65,400,-25,280
```



**Figure 17** Intersection of two rectangular segmentations of a  $700 \times 400$  screenshot from Big Buck Bunny [8]. On the left, combining the first two segmentations yields the third, which is adapted to the change in the image size. The corresponding visualization is on the right.

Reduction segmentation follows a similar setup. Here, the concatenation of two segment URIs is the intersection of the dimensions of the data point descriptions. Unlike filter segmentations, reduction segmentations can be combined without any transformations necessary. This is due to the elements to segment being dimensions that have no order or offset. Therefore, changing them does not change the boundaries of the media object. For instance, consider two image segments `/color/red,green` and `/color/green,blue` which, when combined, produce an image with only green-scale colors. Likewise, filter and reduction segmentations can be combined. As the two operate on fully separate layers, they are again independent, can be combined both ways and do not need any transformations.

## 5 Unified Multimedia Segmentation in Knowledge Graphs

While the segmentation mechanisms described above can be used for general-purpose media access analogously to IIF, they were primarily designed with knowledge organization tasks in mind. Specifically, it can be effectively used in knowledge graphs in order to compactly and efficiently

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refer to a concept that is better represented by a specific part of a multimedia document rather than the document as a whole. To illustrate the utility of our proposed scheme, consider the example shown in Figure 18, which shows a rectangular region of an image being annotated with a label.



■ **Figure 18** Rectangular segmentation of a larger screenshot from Big Buck Bunny [8] annotated with the label “Big Buck Bunny”.

The information depicted in Figure 18 can be represented in different ways using established graph models. Listing 1 shows an example of how a representation using traditional RDF methods could look like. In order to capture the fact that a specific image region is supposed to be annotated, the example introduces an extra graph entity `<http://example.org/bunny>` and two relation types; `<image:in>` and `<image:rectRegion>`, representing the relationship of an image region and the type of region, respectively. Since existing RDF graph stores have no notion of multimedia documents or the information contained therein, the representation requires three triples together with extra relations, the semantics of which need to be known by a client.

■ **Listing 1** Classical RDF graph example in N-Triples notation.

```
<http://example.org/bunny> <image:in> <https://peach.blender.org/wp-content/uploads/bbb-splash.png> .  
<http://example.org/bunny> <image:rectRegion> "165,340,35,315" .  
<http://example.org/bunny> <schema:label> "Big Buck Bunny" .
```

The same information could also be represented using a property graph. Listing 2 shows a possible structure using the Property Graph Exchange Format [14]. This example uses two nodes, one representing the whole image and one for the specific region. The latter has then both the label as well as the details about how the boundaries of the segmentation are defined as properties. An alternative representation would have the segmentation boundaries as properties of the relation, rather than the entity. The expressiveness would be equivalent in this case.

■ **Listing 2** Property graph example in PGEF notation.

```
101 :Image src:"https://peach.blender.org/wp-content/uploads/bbb-splash.png"  
102 :Segment type:rect coords:"165,340,35,315" label:"Big Buck Bunny"  
  
102 -> 101 :segments
```

When using a graph store that is aware of the types of media documents and supports our proposed segmentation scheme, the same information can be represented more succinctly. Listing 3 shows an RDF N-Triple example that compacts the information previously represented in three triples into just one. This example assumes that the graph store listens to `example.org` and is also directly storing the media document and implementing the segmentation. A practical implementation of such a graph store is outlined in Section 6.

■ **Listing 3** RDF graph example in N-Triples notation with support for unified multimedia segmentation scheme.

```
<http://example.org/bbb-splash/segment/rect/165,340,35,315> <schema:label> "Big Buck Bunny" .
```

RDF is especially suitable for our proposed scheme since it inherently uses URLs to represent nodes in the graph. Addressing segments of known documents, rather than the document as a whole, therefore, does not require the introduction of placeholder nodes with no inherent meaning as in Listing 1. Instead, both `<http://example.org/bbb-splash/segment/rect/165,340,35,315>` and `<http://example.org/bbb-splash>` in the context of Listing 3 would be valid URLs that directly return the expected content. An additional benefit of a graph store implementing our scheme directly is, that it would be aware that these two graph entities are related, specifically that the former is a segment of the latter. This information could then be exploited when querying the graph without the need to explicitly materialize such relations.

While our segmentation scheme is especially suited to RDF graph representations, it is by no means limited to it. Listing 4 exemplifies the use of the scheme in a property graph, which enables the representation of the entire relevant information content in a single node without the need for any relations.

■ **Listing 4** Property graph example in PGEF notation using the unified multimedia segmentation scheme.

```
101 :Image src:"http://example.org/bbb-splash/segment/rect/165,340,35,315" label:"Big Buck Bunny"
```

## 6 Implementation

Based on the theoretical considerations of the previous sections, the Unified Segmentation Model is implemented in the MediaGraph Store (MeGraS) codebase,<sup>6</sup> a storage and querying engine for multimodal data written in Kotlin. MeGraS is a graph store that follows the MediaGraph concept, which treats media components of multimodal knowledge graphs not as a collection of opaque documents external to the graph but as parts of the graph itself. The goal of the implementation is to showcase the soundness of the formal model. To that end, the schemes established in the previous section serve as the basis. The scope of the implementation is set to cover the most important document types and types of segmentation. We also implement a test suite<sup>7</sup> for all segmentations supported by MeGraS, which serves not only as validation of our implementation but can also serve to validate other implementation of our proposed scheme.

The MediaGraph Store is constructed as a web server, exposing a REST API where users can add media and metadata and run queries on it. Metadata, including references to the corresponding files on the hard drive, are stored as RDF triples in a database.

### Adding Multimedia Objects

Upon addition via an HTTP-POST request, each media object is stored in its raw format and a transcoded canonical format. The server then returns a canonical ID. With that generated canonical ID, the media object is then retrievable under the URI of the form

```
<server address>/<canonical id>
```

<sup>6</sup> <https://github.com/lucaro/MeGraS>

<sup>7</sup> <https://github.com/lucaro/Unified-Media-Segmentation-Test-Suite>

### Accessing Media Segments

The implementation provided in this paper now adds functionality to additionally request segments of a media object using a URI of the form

```
<server address>/<canonical id>/segment/<segment type>/<segment description>
```

which, after processing, generates a segment. Segment type and description largely correspond to the syntax from Section 4. Because segments remain legitimate media objects of the same type as the original, segmentations can be arbitrarily nested. One can even directly request multiple segmentations in one request URI. These are then processed one by one.

The system follows a lazy-loading approach and caches all media, meaning that a certain segmentation is performed only the first time it is requested. Repeated requests then yield the cached result directly. For that, a canonical segment ID is generated from the segmentation type and description, with which a certain segmentation is uniquely identified and requested using a URI of the form

```
<server address>/<canonical id>/c/<segment id>
```

This is the only form of URIs that will directly return the actual content of the segment. All other requests containing a segment definition, as described above, will be re-directed to the representation containing the segment ID. This mechanism also allows the combination of equivalent segment definitions into the same segment with the same ID. The equivalence of different ways of referring to the same segment is also reflected in the underlying graph structure by inserting a triple using the `sameAs` relation. An additional advantage of this representation is its compactness compared to possibly very verbose segment definitions when it is used as part of graph queries.

## 7 Discussion

The goal of the Unified Segmentation Model was to generalize the process of segmentation and describe it in mathematical terms. For that, a multimedia object and the segmentation operation on arbitrary media objects were defined. Media objects were constructed as sets of data points whose properties are separated into coordinate dimensions and description dimensions. This separation proved to be valuable, as segmentation can target either of the two. Consequently, the segmentation operation could be defined as either a filter or a reduction, where a filter selects a subset of coordinates, and a reduction retains only a part of the description dimensions. The strength of this setup is its generality, as the model can portray all conceivable types of multimedia documents and segmentations of them. Additionally, its mathematical core allows deductions using set theory and Boolean algebra. This breadth of functionality, based on a proper formal foundation, enables our model to fulfill all criteria outlined in Table 1.

In literature, only the General Fragment Model (GFM) [23, 24] tries to take a similarly general perspective. All other accounts are more of a means to an end, as they are application-oriented and focus on concrete use cases. Unlike the Unified Segmentation Model presented here, the GFM is constructed as an ontology to systemize the definition of segments. Nonetheless, parallels can be observed and are summarized in Table 2. In the GFM, an indexer function maps tokens to fragments of the information artifact. This corresponds to a segmentation function that creates segments of the multimedia object. Concrete filter or reduction functions can be seen as tokens. Applying them to media objects corresponds to the GFM's notion of anchors. The token set of the GFM defines the set of possible tokens, which is the set of valid and complete filter and reduction functions. This shows that the proposed model can be represented in terms of the GFM. While the GFM is a powerful abstract formalization of segmentation description, our model far exceeds it by providing a formal model that adds concrete instantiations that allow the investigation of properties and the extraction of characteristics.

■ **Table 2** Comparison of core concepts of the General Fragment Model with the proposed Unified Segmentation Model.

General Fragment Model	Unified Segmentation Model
information artifact	multimedia object
indexer function	segmentation function
tuple token	filter or reduction function
token set	set of all valid and complete functions
anchor	segmentation function applied to multimedia object

A vital part of the formal model is the reflection on how filter and reduction function internals can be described. Again, parallels to the GFM’s indexer taxonomy are visible. Unlike the GFM, though, it paves the way toward implementation. It can be observed that the rules by which a segmentation operates are stateable in different fashions depending on the use case and can be translated into each other. This addresses some of the model’s difficulties of its generality and creates a solid foundation for the implementation.

Before implementation was possible, the Unified Segmentation Model needed to be given a more concrete shape to prepare for implementation by closely examining the specific segmentation function types. The choice of a URI-based approach was never a question, as metadata-based segment description adds an indirection and thus defeats the purpose of the whole project: making media first-class citizens in knowledge graphs. The segmentation types and categories were created based on the investigation of how segmentation rules can be described. Surely, not all proposed segmentations are equally important for real-life applications. Space-filling curves, for instance, are a fascinating mental model to establish a locality-preserving order, but not something one would regularly create segments with. At the same time, URI serialization of the segment descriptions was investigated. The main challenge of this part is the right balance between generality and implementable concreteness. The resulting segmentation types and serializations should be capable of covering all desirable segmentations with little adaptations.

An issue introduced by this concretization is rigidity. To textually describe multi-dimensional segments and to do it uniquely requires a certain level of regulation. However, determining the best representation is not always trivial, as this can depend on the context. This factor can take away intuitiveness, as different users would represent a particular segmentation differently. To use the proposed URI schemes to describe segmentations, one has to use the right segment type keywords<sup>8</sup> and closely follow the segment description syntax.

This is comparable to URI-based segment annotation approaches from the literature, where the same necessary evil can be found many times. Most notably, existing approaches suffer at least one of two issues: weak segmentation coverage or ambiguity and complexity. For instance, the Media Fragments URI specification only supports limited segmentations but is very expressive, whereas the MPEG-21 standard is more powerful but overly ambiguous and unintuitive. This work tries to absorb the best of both worlds and combat both problems simultaneously. Against the first, the proposed schemes cover all conceivable segmentations, or at least the ones realistically used in practice. Adding more variations should be simple, though, as the same pattern can be continued. For the second, efforts were made to keep the URIs as concise and expressive as possible, such that the human eye can also understand them to a certain degree. Unlike almost all

<sup>8</sup> To standardize the kinds of segment types permissible, one could have a standard with a list of suggested/supported keywords or an “ontology”/vocabulary, which may include mathematical specifications that may/or may not be interpreted on the fly

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standards, this work fully refrains from using URI fragments because they are intended to identify secondary resources or views instead of representing a segment as its own resource in relation to its parent. Additionally, a practical reason against fragments is that they are typically interpreted by the client application, which requires client support. In a case like the Media Fragments URI specification, clients must either segment locally or translate URI fragments and send requests for partial content, neither of which is fully implemented in standard browsers. URI queries are equally not desired, as they create new resources without a relation to the original. Instead, URI paths are employed, corresponding to how knowledge graph nodes are usually represented. IIIF served as a model here, as its image processing is purely server-side, and images are fetched as path URIs using standard HTTP requests. Additionally, IIIF is the only approach that combines Semantic Web technology with multimedia on a technical level, as it defines a consistent URI scheme for accessing full and partial image content. Unfortunately, IIIF's flexibility to describe segments is minimal, and only images are supported, despite various efforts to extend to more media types.

## 8 Conclusions

This work was guided by the question of how to universally represent segments of multimedia documents for use in knowledge graphs. While the annotation of partial media content, and thus the need for a textual segment representation, is nothing new to the literature and Semantic Web applications, all theoretical and practical solutions have certain shortcomings. Metadata-based annotations use a secondary document to define segments of the main document, thus introducing an indirection. URI-based approaches are either very limited in their capabilities or are too complex to use. Further, URI fragments are most often used, which are interpreted client-side and thus require special client software. Only IIIF, through its image API, provides a standard HTTP interface for image segment retrieval. Unfortunately, available solutions are limited in the range of supported multimedia documents and segmentation types. Furthermore, only the General Fragment Model tries to generalize the conceptual segmentation model but lacks expressiveness.

To fill the gap in the literature, this paper proposes a Unified Segmentation Model, which can express an arbitrary segmentation on any multimedia document and comprises both a formal model and an applied URI scheme. The former mathematically describes multimedia objects and investigates different types of segmentation functions, which allows the inference of set-theoretical properties. It is then deduced by which rules concrete segmentation functions select partial content and how these rules can be described. In contrast to the many approaches found in literature, this theoretical foundation is defined independently of any specific existing media type. Based on these investigations, a URI scheme is presented that covers all types of segmentations and proposes serializations of segmentations typically found in research and real life. Due to its content-agnostic theoretical foundation, this scheme is not tightly bound to existing media formats and can easily be extended to future information representations. This might even include currently not widely used data types and modalities, such as, for example, holographic information or smells. Our proposed URI scheme is then implemented in the MediaGraph Store, a storage and querying engine for multimedia knowledge graphs. The implementation encompasses a representative set of segmentations and supports various media, namely text, audio, image, video, PDF, and 3D mesh.

In the future, the Unified Segmentation Model can certainly allow for more deductions and considerations. For instance, only the intersection of two segments is inherently supported, as it corresponds to the concatenation of their URIs. Here, the formal model could be leveraged further to investigate other set operations, increasing the expressive power of media segments in knowledge graphs. Additionally, future work could delve into properties that efficiently determine

equivalence, containment, and overlap, as the current deductions are too basic to be useable in practice. Some segment serializations tend to become very large, which is cumbersome, hinders performance, and ultimately can lead to HTTP errors. Therefore, more thought should go into alternative encodings of these segments to increase compaction without sacrificing segment quality. Beyond the sole representation of segments, future work should finally address the querying, discovery, and retrieval of media segments.

But even in the light of these limitations, the Unified Segmentation Model and its associated URI scheme seem to provide a crucial element to advance the state of multimedia data usage in knowledge graphs by breaking the atomicity of multimedia content in a theoretically-founded, media-type agnostic approach and on the Web and a stepping stone for extensions that can finally turn multimedia data into a first-class citizen in knowledge graphs and the Web of Data.

---

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# Strong Faithfulness for $\mathcal{ELH}$ Ontology Embeddings



Victor Lacerda  

University of Bergen, Norway

Ana Ozaki  

University of Oslo, Norway

University of Bergen, Norway

Ricardo Guimarães  

Zivid AS, Norway

## Abstract

Ontology embedding methods are powerful approaches to represent and reason over structured knowledge in various domains. One advantage of ontology embeddings over knowledge graph embeddings is their ability to capture and impose an underlying schema to which the model must conform. Despite advances, most current approaches do not guarantee that the resulting embedding respects the axioms the ontology entails. In this work, we formally prove that normalized  $\mathcal{ELH}$  has the strong faithfulness property on convex geometric models, which means that there is an embedding that precisely captures the original ontology. We present a

region-based geometric model for embedding normalized  $\mathcal{ELH}$  ontologies into a continuous vector space. To prove strong faithfulness, our construction takes advantage of the fact that normalized  $\mathcal{ELH}$  has a finite canonical model. We first prove the statement assuming (possibly) non-convex regions, allowing us to keep the required dimensions low. Then, we impose convexity on the regions and show the property still holds. Finally, we consider reasoning tasks on geometric models and analyze the complexity in the class of convex geometric models used for proving strong faithfulness.

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## 1 Introduction

Knowledge Graphs (KGs) are a popular method for representing knowledge using triples of the form (subject, predicate, object), called *facts*.

Although public KGs, such as Wikidata [25], contain a large number of facts, they are incomplete. This has sparked interest in using machine learning methods to suggest plausible facts to add to the KG based on patterns found in the data. Such methods are based on knowledge graph embedding (KGE) techniques, which aim to create representations of KGs in vector spaces. By representing individuals in a vector space, these individuals can be ranked by how similar they are to each other, based on a similarity metric.

Their proximity in a vector space may be indicative of semantic similarity, which can be leveraged to discover new facts: if two individuals are close to each other in the embedding space, it is likely that they share a pattern of relations to other individuals. These patterns of relations can indicate of assertions not explicitly stated in the source knowledge graph.



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Many attempts have been made to learn representations of knowledge graphs for use in downstream tasks [8]. These methods have traditionally focused only on embedding triples (facts), ignoring the conceptual knowledge about the domain expressed using logical operators. The former corresponds to the “*Assertion Box*” (*ABox*) of the ontology, while the latter corresponds to the “*Terminological Box*” (*TBox*) part of a knowledge base, with both being quite established notions in the fields of Description Logic and Semantic Web [2, 12]. Embeddings that consider both types of logically expressed knowledge are a more recent phenomenon (see Section 2), and we refer to them as *ontology embeddings*, where the *ontology* can have both an *ABox* and a *TBox*. Ontology embeddings offer advantages over traditional KGEs as they exploit the semantic relationships between concepts and roles. This enables ontology embeddings to better capture rich and nuanced relationships between concepts, making them good candidates for tasks requiring fine-grained reasoning, such as hierarchical reasoning and logical inference.

One question that arises in the study of ontology embeddings is the following: how similar to the source ontology are the generated embeddings? Being more strict, if we fix a semantics in order to interpret the generated embeddings, are they *guaranteed* to precisely represent the meaning of the source ontology and its entailments (of particular interest, the *TBox* entailments)? This property is called the *strong faithfulness* property [20] and, so far, no previous work for  $\mathcal{EL}$  ontology embeddings has attempted to prove the property holds for their embedding method. Moreover, the existence of embedding models satisfying this property for the  $\mathcal{ELH}$  language has not been formally proven. Given that ontologies languages in the  $\mathcal{EL}$  family have received most of the attention by the existing literature on ontology embeddings [22, 23, 1, 26, 14], this is a significant gap which we investigate in this work.

### Contribution

We investigate whether  $\mathcal{ELH}$  has the strong faithfulness property over convex geometric models. We first prove the statement for embeddings in low dimensions, considering a region-based representation for (possibly) non-convex regions (Section 4). Also, we prove that the same property does not hold when we consider convex regions and only 1 dimension. We then investigate strong faithfulness on convex geometric models with more dimensions (Section 5). This result contributes to the landscape of properties for embedding methods based on geometric models [5, Proposition 11] and it provides the foundation of the implementation of FaithEL [16]. We do so including embeddings for role inclusions, a problem that has not been well studied in the  $\mathcal{ELH}$  ontology embedding literature. We also consider model checking in convex geometric models, a topic that has not been covered in previous works (Section 6).

## 2 Ontology Embeddings

Various methods for embedding ontologies have been proposed, with ontologies in the  $\mathcal{EL}$  family being their primary targets.  $\mathcal{EL}$  is a simple yet powerful language.

These embedding methods are *region-based*, that is, they map concepts to regions and entities to vectors (in some cases, entities are transformed into nominals and also embedded as regions), and represent roles using translations or regions within the vector space.

The precise shape of the embedding regions varies depending on the method. In *EmEL* [19] and *ELem* [15], the embeddings map concepts to  $n$ -dimensional *balls*. One disadvantage of this approach is that the intersection between two balls is not itself a ball. Newer approaches addressing this issue such as *BoxEL*, *Box<sup>2</sup>EL*, and *ELBE* [26, 14, 23], starting with *BoxE* [1], represent concepts as  $n$ -dimensional *boxes*. *BoxE* introduced the use of so-called “translational bumps” to

capture relations between entities, an idea followed by *Box<sup>2</sup>EL*. Another language, *ACC*, has been studied under a *cone semantics* [20], which uses *axis-aligned cones* as its geometric interpretation. In the context of KGEs, *n*-dimensional *parallelograms* have also been used in *ExpressivE* [21].

Other approaches for accommodating TBox axioms in the embeddings have also been considered. Approaching the problem from a different direction, *OWL2Vec\** [7] targets the DL language *SR<sub>Q</sub>IQ* and does not rely on regions, but uses the NLP algorithm *word2vec* to include lexical information (such as annotations) along with the graph structure of an OWL ontology. Another framework, *TransOWL* [9], uses background knowledge injection to improve link prediction for models such as *TransE* and *TransR*. Additionally, there has been an increased interest in *querying* KGEs, with strategies utilizing query rewriting techniques being put in place to achieve better results [13].

Although expressively powerful and well performing in tasks such as subsumption checking and link prediction, the generated embeddings often lack formal guarantees with respect to the source ontology. In the KGE literature, it is a well known that, e.g., *TransE* [3] is unable to model one-to-many relations (a difficulty present even in recent ontology embedding methods such as *BoxEL*) or symmetric relations. This has spurred a quest for more expressive models, with the intention of capturing an increasing list of relation types and properties such as composition, intersection, hierarchy of relations, among others [17, 27, 24, 21].

Expressivity is a key notion in ontology embedding methods, which often also feature these relation types and potentially other forms of constraints. For example, in *Box<sup>2</sup>EL*, *ELem*, and *ELBE* [14, 15, 23], axioms of the form  $\exists r.C \sqsubseteq \perp$  are only approximated by  $\exists r.\top \sqsubseteq \perp$ . This means that strong TBox faithfulness is not respected. Moreover, only *EmEL* and *Box<sup>2</sup>EL* [19, 14] include embeddings for role inclusions. In the case of *EmEL*, the axiom  $r \sqsubseteq s$  also enforces  $s \sqsubseteq r$ , which means it is not strongly faithful, while *Box<sup>2</sup>EL* has also been shown to not be strongly faithful [5].

### 3 Basic Notions

#### 3.1 The Description Logic $\mathcal{ELH}$

Let  $N_C$ ,  $N_R$ , and  $N_I$  be countably infinite and pairwise disjoint sets of *concept names*, *role names*, and *individual names*, respectively.  $\mathcal{ELH}$  concepts  $C, D$  are built according to the syntax rule

$$C, D ::= \top \mid \perp \mid A \mid (C \sqcap D) \mid \exists r.C$$

where  $A \in N_C$  and  $r \in N_R$ .  $\mathcal{ELH}$  concept inclusions (CIs) are of the form  $C \sqsubseteq D$ , role inclusions (RIs) are of the form  $r \sqsubseteq s$ ,  $\mathcal{ELH}$  concept assertions are of the form  $A(a)$  and role assertions are of the form  $r(a, b)$ , where  $A \in N_C$ ,  $a, b \in N_I$ ,  $r, s \in N_R$ , and  $C, D$  range over  $\mathcal{ELH}$  concepts. Instance queries (IQs) are role assertions or of the form  $C(a)$ , with  $C$  being an arbitrary  $\mathcal{ELH}$  concept. An  $\mathcal{ELH}$  axiom is an  $\mathcal{ELH}$  CI, an RI, or an IQ. A *normalized  $\mathcal{ELH}$  TBox* is one that only contains CIs of the following forms:

$$A_1 \sqcap A_2 \sqsubseteq B, \exists r.A \sqsubseteq B, \text{ and } A \sqsubseteq \exists r.B$$

where  $A_1, A_2, A, B \in N_C$  and  $r \in N_R$ . We say that an  $\mathcal{ELH}$  concept is in *normal form* if it is of the form  $A$ ,  $\exists r.A$ , or  $A \sqcap B$ , with  $A, B \in N_C$  and  $r \in N_R$ . Similarly, an  $\mathcal{ELH}$  ontology is in *normal form* if its TBox part is a normalized  $\mathcal{ELH}$  TBox. An IQ is in *normal form* if it is a role assertion or of the form  $C(a)$  with  $C$  being a concept in normal form. The semantics of  $\mathcal{ELH}$  is defined classically by means of *interpretations*  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ , where  $\Delta^{\mathcal{I}}$  is a non-empty countable set called the *interpretation domain*, and  $\cdot^{\mathcal{I}}$  is an *interpretation function* mapping each concept

## 2:4 Strong Faithfulness for $\mathcal{ELH}$ Ontology Embeddings

name  $A$  in  $N_C$  to a subset  $A^{\mathcal{I}}$  of  $\Delta^{\mathcal{I}}$ , each role name  $r$  in  $N_R$  to a binary relation  $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ , and each individual name  $a$  in  $N_I$  to an element  $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ . We extend the function  $\cdot^{\mathcal{I}}$  inductively to arbitrary concepts by setting  $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}$ ,  $\perp^{\mathcal{I}} := \emptyset$ , and

$$(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}, \text{ and}$$

$$(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I}} \text{ such that } (d, e) \in r^{\mathcal{I}}\}.$$

An interpretation  $\mathcal{I}$  satisfies: (1)  $C \sqsubseteq D$  iff  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ ; (2)  $r \sqsubseteq s$  iff  $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ , (3)  $C(a)$  iff  $a^{\mathcal{I}} \in C^{\mathcal{I}}$ ; (4)  $r(a, b)$  iff  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ .

An  $\mathcal{ELH}$  TBox  $\mathcal{T}$  (Terminological Box) is a finite number of  $\mathcal{ELH}$  concept and role inclusions. An  $\mathcal{ELH}$  ABox  $\mathcal{A}$  (Assertion Box) is a finite number of  $\mathcal{ELH}$  concept and role assertions. The union of a TBox and an ABox forms an  $\mathcal{ELH}$  ontology. An  $\mathcal{ELH}$  ontology  $\mathcal{O}$  entails an  $\mathcal{ELH}$  axiom  $\alpha$ , in symbols  $\mathcal{O} \models \alpha$  if for every interpretation  $\mathcal{I}$ , we have that  $\mathcal{I} \models \mathcal{O}$  implies  $\mathcal{I} \models \alpha$  (we may write similarly for the CI and RI entailments of a TBox). We denote by  $N_C(\mathcal{O}), N_R(\mathcal{O}), N_I(\mathcal{O})$  the set of concept names, role names, and individual names occurring in an ontology  $\mathcal{O}$ . We may also write  $N_I(\mathcal{A})$  for the set of individual names occurring in an ABox  $\mathcal{A}$ . The *signature* of an ontology  $\mathcal{O}$ , denoted  $\text{sig}(\mathcal{O})$ , is the union of  $N_C(\mathcal{O}), N_R(\mathcal{O})$ , and  $N_I(\mathcal{O})$ .

### 3.2 Geometric models

We go from the traditional model-theoretic interpretation of the  $\mathcal{ELH}$  language to geometric interpretations, using definitions from previous works by [10] and [6]. Let  $m$  be a natural number and  $f: \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^{2m}$  a fixed but arbitrary linear map satisfying the following:

1. the restriction of  $f$  to  $\mathbb{R}^m \times \{0\}^m$  is injective;
2. the restriction of  $f$  to  $\{0\}^m \times \mathbb{R}^m$  is injective;
3.  $f(\mathbb{R}^m \times \{0\}^m) \cap f(\{0\}^m \times \mathbb{R}^m) = \{0^{2m}\}$ ;

where  $0^m$  denotes the vector  $(0, \dots, 0)$  with  $m$  zeros. We say that a linear map that satisfies Points 1, 2, and 3 is an *isomorphism preserving linear map*.

► **Example 1.** The concatenation function is a linear map that satisfies Points 1, 2, and 3. E.g., if we have vectors  $v_1 = (n_1, n_2, n_3)$  and  $v_2 = (m_1, m_2, m_3)$  then for  $f$  being the concatenation function we would have  $f(v_1, v_2) = (n_1, n_2, n_3, m_1, m_2, m_3)$ . Other linear maps that satisfy Points 1, 2, and 3 can be created with permutations. E.g., defining the function  $f$  such that  $f(v_1, v_2) = (n_1, m_1, n_2, m_2, n_3, m_3)$ .

► **Definition 2 (Geometric Interpretation).** Let  $f$  be an isomorphism preserving linear map and  $m$  a natural number. An  $m$ -dimensional  $f$ -geometric interpretation  $\eta$  of  $(N_C, N_R, N_I)$  assigns to each

- $A \in N_C$  a region  $\eta(A) \subseteq \mathbb{R}^m$
- $r \in N_R$  a region  $\eta(r) \subseteq \mathbb{R}^{2m}$ , and
- $a \in N_I$  a vector  $\eta(a) \in \mathbb{R}^m$ .

We now extend the definition for arbitrary  $\mathcal{ELH}$  concepts:

$$\eta(\perp) := \emptyset$$

$$\eta(\top) := \mathbb{R}^m,$$

$$\eta(C \sqcap D) := \eta(C) \cap \eta(D), \text{ and}$$

$$\eta(\exists r.C) := \{v \in \mathbb{R}^m \mid \exists u \in \eta(C) \text{ with } f(v, u) \in \eta(r)\}.$$

Intuitively, the function  $f$  combines two vectors that represent a pair of elements in a classical interpretation relation. An  $m$ -dimensional  $f$ -geometric interpretation  $\eta$  satisfies

- an  $\mathcal{ELH}$  concept assertion  $A(a)$ , if  $\eta(a) \in \eta(A)$ ,
- a role assertion  $r(a, b)$ , if  $f(\eta(a), \eta(b)) \in \eta(r)$ ,
- an  $\mathcal{ELH}$  IQ  $C(a)$ , if  $\eta(a) \in \eta(C)$ ,
- an  $\mathcal{ELH}$  CI  $C \sqsubseteq D$ , if  $\eta(C) \subseteq \eta(D)$ , and
- an RI  $r \sqsubseteq s$ , if  $\eta(r) \subseteq \eta(s)$ .

We write  $\eta \models \alpha$  if  $\eta$  satisfies an  $\mathcal{ELH}$  axiom  $\alpha$ . When speaking of  $m$ -dimensional  $f$ -geometric interpretations, we may omit  $m$ -dimensional and  $f$ -, as well as use the term “model” instead of “interpretation”. A geometric interpretation satisfies an ontology  $\mathcal{O}$ , in symbols  $\eta \models \mathcal{O}$ , if it satisfies all axioms in  $\mathcal{O}$ . We say that a geometric interpretation is finite if the regions associated with concept and role names have a finite number of vectors and we only need to consider a finite number of individual names, which is the case when considering the individual names that occur in an ontology.

Motivated by the theory of conceptual spaces and findings on cognitive science [11, 28], and by previous work on ontology embeddings for quasi-chained rules [10], we consider convexity as an interesting restriction for the regions associated with concepts and relations in a geometric model.

► **Definition 3.** A geometric interpretation  $\eta$  is convex if, for every  $E \in N_C \cup N_R$ , every  $v_1, v_2 \in \eta(E)$  and every  $\lambda \in [0, 1]$ , if  $v_1, v_2 \in \eta(E)$  then  $(1 - \lambda)v_1 + \lambda v_2 \in \eta(E)$ .

► **Definition 4.** Let  $S = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^d$ . A vector  $v$  is in the convex hull  $S^*$  of  $S$  iff there exist  $v_1, \dots, v_n \in S$  and scalars  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$  such that

$$v = \sum_{i=1}^n \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n,$$

where  $\lambda_i \geq 0$ , for  $i = 1, \dots, n$ , and  $\sum_{i=1}^n \lambda_i = 1$ .

Apropos of convexity, we highlight and prove some of its properties used later in our results.

► **Proposition 5.** For finite  $S_1, S_2 \subseteq \mathbb{R}^d$ , where  $d$  is an arbitrary dimension, we have that  $S_1 \subseteq S_2$  implies  $S_1^* \subseteq S_2^*$ .

In the following, whenever we say a vector is *binary*, we mean that its values in each dimension can only be 0 or 1.

► **Theorem 6.** Let  $S \subseteq \{0, 1\}^d$  where  $d$  is an arbitrary dimension. For any  $n \in \mathbb{N}$ , for any  $v = \sum_{i=1}^n \lambda_i v_i$ , such that  $v_i \in S$ , if  $v \in S^* \setminus S$  then  $v$  is non-binary.

► **Corollary 7.** If  $v$  is binary and  $v \in S^*$  then  $v \in S$ .

Finally, we define strong faithfulness based on the work by [20].

► **Definition 8 (Strong Faithfulness).** Let  $\mathcal{O}$  be a satisfiable ontology (or any other representation allowing the distinction between IQs and TBox axioms). Given an  $m$ -dimensional  $f$ -geometric interpretation  $\eta$ , we say that:

- $\eta$  is a strongly concept-faithful model of  $\mathcal{O}$  iff, for every concept  $C$  and individual name  $b$ , if  $\eta(b) \in \eta(C)$  then  $\mathcal{O} \models C(b)$ ;
- $\eta$  is a strongly IQ faithful model of  $\mathcal{O}$  iff it is strongly concept-faithful and for each role  $r$  and individual names  $a, b$ : if  $f(\eta(a), \eta(b)) \in \eta(r)$ , then  $\mathcal{O} \models r(a, b)$ ;
- $\eta$  is a strongly TBox-faithful model of  $\mathcal{O}$  iff for all TBox axioms  $\tau$ : if  $\eta \models \tau$ , then  $\mathcal{O} \models \tau$ .

► **Example 9.** Let  $\mathcal{O}$  be an ontology given by  $\mathcal{T} \cup \mathcal{A}$  with  $\mathcal{T} = \{A \sqsubseteq B\}$  and  $\mathcal{A} = \{A(a), B(b)\}$ . Let  $\eta_{\mathcal{I}}$  be a (non-convex) geometric interpretation of  $\mathcal{O}$  in  $\mathbb{R}$ , where  $\eta_{\mathcal{I}}(A) = \{0, 1, 2\}$ ,  $\eta_{\mathcal{I}}(B) = \{0, 1, 2, 3\}$ ,  $\eta_{\mathcal{I}}(a) = 2$ , and  $\eta_{\mathcal{I}}(b) = 3$ . Note that  $\mathcal{O} \models A(a)$  and  $\mathcal{O} \models B(b)$ , and by definition  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$ ,  $\eta_{\mathcal{I}}(b) \in \eta_{\mathcal{I}}(B)$ . Also,  $\mathcal{O} \models A \sqsubseteq B$  and  $\eta_{\mathcal{I}}(A) \subseteq \eta_{\mathcal{I}}(B)$ . So one can see that  $\eta_{\mathcal{I}}$  is both a strongly concept and TBox-faithful model of  $\mathcal{O}$ . If we let  $\eta'_{\mathcal{I}}$  be a geometric interpretation such that  $\eta'_{\mathcal{I}}(A) = \{0, 1, 2, 3\} = \eta_{\mathcal{I}}(B)$ , we now have that  $\eta'_{\mathcal{I}}(b) \in \eta'_{\mathcal{I}}(A)$ , which means  $\eta'_{\mathcal{I}}$  is not a strongly concept-faithful model of  $\mathcal{O}$  (since  $\mathcal{O} \not\models A(b)$ ), and we have that  $\eta'_{\mathcal{I}}(B) \subseteq \eta'_{\mathcal{I}}(A)$ , which means it is not a strongly TBox-faithful model of  $\mathcal{O}$  (since  $\mathcal{O} \not\models B \sqsubseteq A$ ).

We say that an ontology language has the strong faithfulness property over a class of geometric interpretations  $\mathcal{C}$  if for every satisfiable ontology  $\mathcal{O}$  in this language there is a geometric interpretation in  $\mathcal{C}$  that is both a strongly IQ faithful and a strongly TBox faithful model of  $\mathcal{O}$ .

The range of concepts, roles, and individual names in Definition 8 varies depending on the language and setting studied. We omit the notion of weak faithfulness by [20] as it does not apply for  $\mathcal{ELH}$  since ontologies in this language are always satisfiable (there is no negation). The “if-then” statements in Definition 8 become “if and only if” when  $\eta$  satisfies the ontology. Intuitively, strong faithfulness expresses how similar the generated embedding is to the original ontology.

We observe that strong faithfulness with respect to the TBox component of the ontology is extremely desirable: it guarantees that concept and role inclusions are also enforced when coupled with a geometric interpretation in the embedding space. On the other hand, strong IQ faithfulness is not a desirable property for learned embeddings. Although this might seem counter-intuitive at first, it is a reasonable statement: an embedding that is strongly IQ faithful is unsuitable for link prediction, as the only assertions that hold in the embedding are those that already hold in the original ontology. This means that no new facts are truly discovered by the model. Here we prove both strong TBox and IQ faithfulness for  $\mathcal{ELH}$  for theoretical reasons.

Finally, observe that an embedding model that is both strongly TBox and IQ faithful must have the same TBox and IQ consequences as the original ontology. This is a stronger requirement than establishing that an embedding model for an ontology  $\mathcal{O}$  (within a method) exist if and only if a classical model for  $\mathcal{O}$  exists, which is a property of sound and complete embedding methods [5].

## 4 Strong Faithfulness

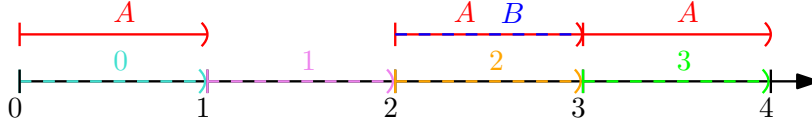
In this section we prove initial results about strong faithfulness for  $\mathcal{ELH}$ . In particular, we prove that  $\mathcal{ELH}$  has the strong faithfulness property over  $m$ -dimensional  $f$ -geometric interpretations for any  $m \geq 1$  but this is not the case if we require that regions in the geometric interpretations are convex. We first introduce a mapping from classical interpretation to (possibly) non-convex geometric interpretations and then use it with the notion of canonical model to establish strong faithfulness for  $\mathcal{ELH}$ .

► **Definition 10.** Let  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  be a classical  $\mathcal{ELH}$  interpretation, and we assume without loss of generality, since  $\Delta^{\mathcal{I}}$  is non-empty and countable, that  $\Delta^{\mathcal{I}}$  is a (possibly infinite) interval in  $\mathbb{N}$  starting on 0. Let  $\bar{\mu}: \Delta^{\mathcal{I}} \mapsto \mathbb{R}^1$  be a mapping from our classical interpretation domain to a vector space where:

$$\bar{\mu}(d) = \begin{cases} (-\infty, -d] \cup [d, \infty), & \text{if } \Delta^{\mathcal{I}} \text{ is finite and } d = \max(\Delta^{\mathcal{I}}), \\ (-d-1, -d] \cup [d, d+1), & \text{otherwise.} \end{cases}$$

where  $d \in \mathbb{N}$  and  $(-d-1, -d]$  and  $[d, d+1)$  are intervals over  $\mathbb{R}^1$ , closed on  $d$  and  $-d$ , and open on  $d+1$  and  $-d-1$ .





■ **Figure 1** A partial visualization (showing only the positive section of the real line) of a geometric interpretation  $\bar{\eta}_{\mathcal{I}}$  where elements  $d_0 \dots d_3$  are mapped to their respective intervals, and where  $\bar{\mu}(d_0), \bar{\mu}(d_2), \bar{\mu}(d_3) \in \bar{\eta}_{\mathcal{I}}(A)$  and  $\bar{\mu}(d_1) \in \bar{\eta}_{\mathcal{I}}(B)$ .

► **Remark 11.** For any interpretation  $\mathcal{I}$ ,  $\bar{\mu}$  covers the real line, that is,  $\bigcup_{d \in \Delta^{\mathcal{I}}} \bar{\mu}(d) = \mathbb{R}^1$ .

► **Definition 12.** We call  $\bar{\eta}_{\mathcal{I}}$  the geometric interpretation of  $\mathcal{I}$  and define it as follows. Let  $\mathcal{I}$  be a classical  $\mathcal{ELH}$  interpretation. The geometric interpretation of  $\mathcal{I}$ , denoted  $\bar{\eta}_{\mathcal{I}}$ , is defined as:

$$\begin{aligned} \bar{\eta}_{\mathcal{I}}(a) &:= d, \text{ such that } d = a^{\mathcal{I}}, \text{ for all } a \in N_I, \\ \bar{\eta}_{\mathcal{I}}(A) &:= \{v \in \bar{\mu}(d) \mid d \in A^{\mathcal{I}}\}, \text{ for all } A \in N_C, \text{ and} \\ \bar{\eta}_{\mathcal{I}}(r) &:= \{f(v, e) \mid v \in \bar{\mu}(d) \text{ for } (d, e) \in r^{\mathcal{I}}\}, \text{ for all } r \in N_R. \end{aligned}$$

In Figure 1, we illustrate with an example the mapping in Definition 12. We now show that for (possibly) non-convex geometric models, a classical interpretation  $\mathcal{I}$  models arbitrary IQs and arbitrary TBox axioms if and only if their geometrical interpretation  $\bar{\eta}_{\mathcal{I}}$  also models them.

► **Theorem 13.** For all  $\mathcal{ELH}$  axioms  $\alpha$ ,  $\mathcal{I} \models \alpha$  iff  $\bar{\eta}_{\mathcal{I}} \models \alpha$ .

We now provide a definition of canonical model for  $\mathcal{ELH}$  ontologies inspired by a standard chase procedure. In our definition, we use a *tree shaped interpretation*  $\mathcal{I}_D$  of an  $\mathcal{ELH}$  concept  $D$ , with the root denoted  $\rho_D$ . This is defined inductively. For  $D$  a concept name  $A \in N_C$  we define  $\mathcal{I}_A$  as the interpretation with  $\Delta^{\mathcal{I}_A} := \{\rho_A\}$ ,  $A^{\mathcal{I}_A} := \{\rho_A\}$ , and all other concept and role names interpreted as the empty set. For  $D = \exists r.C$ , we define  $\mathcal{I}_D$  as the interpretation with  $\Delta^{\mathcal{I}_D} := \{\rho_D\} \cup \Delta^{\mathcal{I}_C}$ , all concept and role name interpretations are as for  $\mathcal{I}_C$  except that we add  $(\rho_D, \rho_C)$  to  $r^{\mathcal{I}_D}$  and assume  $\rho_D$  is fresh (i.e., it is not in  $\Delta^{\mathcal{I}_C}$ ). Finally, for  $D = C_1 \sqcap C_2$  we define  $\Delta^{\mathcal{I}_D} := \Delta^{\mathcal{I}_{C_1}} \cup (\Delta^{\mathcal{I}_{C_2}} \setminus \{\rho_{C_2}\})$ , assuming  $\Delta^{\mathcal{I}_{C_1}}$  and  $\Delta^{\mathcal{I}_{C_2}}$  are disjoint, and with all concept and role name interpretations as in  $\mathcal{I}_{C_1}$  and  $\mathcal{I}_{C_2}$ , except that we connect  $\rho_{C_1}$  with the elements of  $\Delta^{\mathcal{I}_{C_2}}$  in the same way as  $\rho_{C_2}$  is connected. That is, we *identify*  $\rho_{C_1}$  with the root  $\rho_{C_2}$  of  $\mathcal{I}_{D_2}$ .

► **Definition 14.** The canonical model  $\bar{\mathcal{I}}_{\mathcal{O}}$  of a satisfiable  $\mathcal{ELH}$  ontology  $\mathcal{O}$  is defined as the union of a sequence of interpretations  $\mathcal{I}_0, \mathcal{I}_1, \dots$ , where  $\mathcal{I}_0$  is defined as:

$$\begin{aligned} \Delta^{\mathcal{I}_0} &:= \{a \mid a \in N_I(\mathcal{A})\}, \\ A^{\mathcal{I}_0} &:= \{a \mid A(a) \in \mathcal{A}\} \text{ for all } A \in N_C, \text{ and} \\ r^{\mathcal{I}_0} &:= \{(a, b) \mid r(a, b) \in \mathcal{A}\}, \text{ for all } r \in N_R. \end{aligned}$$

Suppose  $\mathcal{I}_n$  is defined. We define  $\mathcal{I}_{n+1}$  by choosing a CI or an RI in  $\mathcal{O}$  and applying one of the following rules:

- if  $C \sqsubseteq D \in \mathcal{O}$  and  $d \in C^{\mathcal{I}_n} \setminus D^{\mathcal{I}_n}$  then define  $\mathcal{I}_{n+1}$  as the result of adding to  $\mathcal{I}_n$  a copy of the tree shaped interpretation  $\mathcal{I}_D$  and identifying  $d$  with the root of  $\mathcal{I}_D$  (assume that the elements in  $\Delta^{\mathcal{I}_D}$  are fresh, that is,  $\Delta^{\mathcal{I}_D} \cap \Delta^{\mathcal{I}_n} = \emptyset$ );
- if  $r \sqsubseteq s \in \mathcal{O}$  and  $(d, e) \in r^{\mathcal{I}_n} \setminus s^{\mathcal{I}_n}$  then set  $\mathcal{I}_{n+1}$  as the result of adding  $(d, e)$  to  $s^{\mathcal{I}_n}$ .

We assume the choice of CIs and RIs and corresponding rule above to be fair, i.e., if a CI or RI applies at a certain place, it will eventually be applied there.



■ **Figure 2** An illustration of the region  $\eta_I(A) \cap \eta_I(B)$ .

► **Theorem 15.** *Let  $\mathcal{O}$  be a satisfiable  $\mathcal{ELH}$  ontology and let  $\bar{\mathcal{I}}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 14). Then,*

- *for all  $\mathcal{ELH}$  IQs and CIs  $\alpha$  over  $\text{sig}(\mathcal{O})$ ,  $\bar{\mathcal{I}}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ ; and*
- *for all RIs  $\alpha$  over  $\text{sig}(\mathcal{O})$ ,  $\bar{\mathcal{I}}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ .*

We are now ready to state our theorem combining the results of Theorems 13 and 15 and the notion of strong faithfulness for IQs and TBox axioms.

► **Theorem 16.** *Let  $\mathcal{O}$  be a satisfiable  $\mathcal{ELH}$  ontology and let  $\bar{\mathcal{I}}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (see Definition 14). The  $m$ -dimensional  $f$ -geometric interpretation of  $\bar{\mathcal{I}}_{\mathcal{O}}$  (see Definition 12) is a strongly IQ and TBox faithful model of  $\mathcal{O}$ .*

What Theorem 16 demonstrates is that the existence of canonical models for  $\mathcal{ELH}$  allows us to connect our result relating classical and geometric interpretations to faithfulness. This property of canonical models is crucial and can potentially be extended to other description logics that also have canonical models (however, many of such logics do not have polynomial size canonical models, a property we use in the next section, so we focus on  $\mathcal{ELH}$  in this work).

► **Corollary 17.** *For all  $m \geq 1$  and isomorphism preserving linear maps  $f$ ,  $\mathcal{ELH}$  has the strong faithfulness property over  $m$ -dimensional  $f$ -geometric interpretations.*

However, requiring that the regions of the geometric model are convex makes strong faithfulness more challenging. The next theorem hints that such models require more dimensions and a more principled approach to map  $\mathcal{ELH}$  ontologies in a continuous vector space.

► **Theorem 18.**  *$\mathcal{ELH}$  does not have the strong faithfulness property over convex 1-dimensional  $f$ -geometric models.*

**Proof.** We reason by cases in order to show impossibility of the strong faithfulness property for the class of *convex* 1-dimensional  $f$ -geometric model for arbitrary  $\mathcal{ELH}$  ontologies. Let  $\mathcal{O}$  be an  $\mathcal{ELH}$  ontology,  $A, B, C \in N_C$  concept names,  $a, b \in N_I$  individuals, and let  $\eta(A), \eta(B), \eta(C), \eta(a)$ , and  $\eta(b)$  be their corresponding geometric interpretations to  $\mathbb{R}^1$ . Assume  $\mathcal{O} \models A \sqcap B(a)$ . There are three initial cases on how to choose the interval placement of  $\eta(A)$  and  $\eta(B)$ :

- **Null intersection:**  $(\eta(A) \cap \eta(B)) = \emptyset$ .

If  $(\eta(A) \cap \eta(B)) = \emptyset$ , then either  $(\eta(a) \in \eta(A) \text{ and } \eta(a) \notin \eta(B))$ , or  $(\eta(a) \in \eta(B) \text{ and } \eta(a) \notin \eta(A))$ . Recall the definition of satisfiability for concept assertions. Since we assumed  $\mathcal{O} \models A \sqcap B(a)$ , we would want our geometric interpretation to be such that  $\eta(a) \in \eta(A) \cap \eta(B)$ , a contradiction.

- **Total inclusion:**  $\eta(A) \subseteq \eta(B)$  and/or  $\eta(B) \subseteq \eta(A)$ .

Consider an extension  $\mathcal{O}'$  of our ontology where  $\mathcal{O}' \models A(c)$  and  $\mathcal{O}' \not\models B(c)$ . If we let  $\eta(A) \subseteq \eta(B)$ , it is clear that our ontology cannot be faithfully modeled, since by our assumption of total inclusion, we would have that  $\eta(c) \in \eta(A)$  and  $\eta(c) \in \eta(B)$ , which goes against  $\mathcal{O}' \not\models B(c)$ . The same holds for the total inclusion in the other direction, where  $\eta(B) \subseteq \eta(A)$ . Therefore, we go to our last initial case to be considered.

- **Partial intersection:**  $(\eta(A) \cap \eta(B)) \neq \emptyset$ .

This is in fact the only way of faithfully giving a geometric interpretation to our concept assertion  $A \sqcap B(a)$ , while still leaving room for ABox axioms such that an arbitrary element could belong to one of our classes  $A$  or  $B$  without necessarily belonging to both of them. Then,  $\eta(A) \cap \eta(B)$  and  $\eta(A) \not\subseteq \eta(B)$  nor  $\eta(B) \not\subseteq \eta(A)$ .

After having forced the geometric interpretation of our two initial concepts  $A$  and  $B$  to partially intersect, we now show that by adding a third concept  $C$ , in which  $\mathcal{O} \models A \sqcap B \sqcap C(a)$ , either  $\eta(A) \subset \eta(B) \cup \eta(C)$  or  $\eta(B) \subset \eta(A) \cup \eta(C)$ , even though this interpretation is not included in our original ontology. We are unable to include a concept assertion  $A(a) \in \mathcal{O}$  without also having that  $\eta(a) \in \eta(C)$  in our geometric interpretation, or likewise for the case in which  $B(a) \in \mathcal{O}$ .

Stemming from the fact that our geometric interpretation must be convex, and it is modeled in an euclidean  $\mathbb{R}^1$  space, we can visualize our classes  $A$ ,  $B$ , and  $C$  as intervals on the real line. Assume, without loss of generality, that  $\eta(A)$  is placed to the left of  $\eta(B)$  (see Figure 2). Then,  $C$  can only be placed either to the right of  $B$  or to the left of  $A$ .

By reasoning in the same way as before, we know that  $\eta(C)$  must partially intersect with either  $\eta(A)$  or  $\eta(B)$ , so one end of the interval representing  $C$  must be placed in  $\eta(A) \cap \eta(B)$ , without us having that either  $\eta(C) \subseteq \eta(A)$ ,  $\eta(C) \subseteq \eta(B)$ ,  $\eta(C) \subseteq \eta(A) \cap \eta(B)$  or  $\eta(C) \subseteq \eta(A) \cup \eta(B)$ . This last requirement is due to the fact that we want to be able to have an ontology such that  $\mathcal{O} \models C(a)$  and where  $\mathcal{O} \not\models A(a)$ ,  $\mathcal{O} \not\models B(a)$ , or  $\mathcal{O} \not\models A(a) \sqcap B(a)$ . Assuming the intersection between  $\eta(A)$  and  $\eta(B) \neq \emptyset$  there are three more cases to be considered:

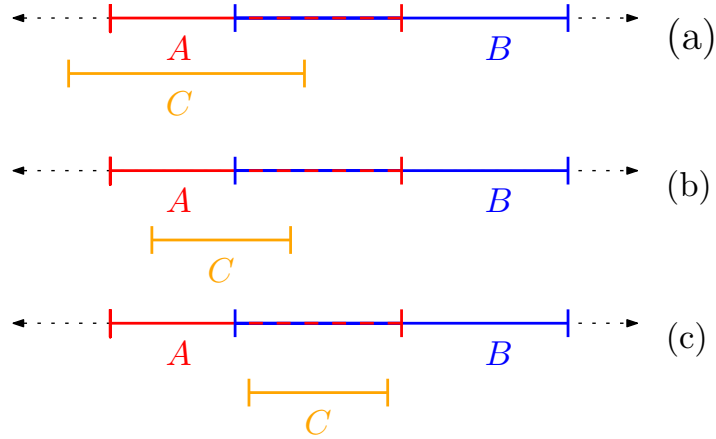
- **C is in the intersection of A and B:  $\eta(C) \subseteq \eta(A) \cap \eta(B)$  (Fig. 2 (a)).**  
If  $\eta(C) \subseteq \eta(A) \cap \eta(B)$ , it is immediately clear that by extending  $\mathcal{O}$  such that  $\mathcal{O} \models C(b)$  but  $\mathcal{O} \not\models A(b)$ , we would end up with  $\eta(b) \in \eta(C)$ . But since we assumed that  $\eta(C) \subseteq \eta(A) \cap \eta(B)$ , this means that  $\eta(b) \in \eta(A)$ , and therefore our geometric interpretation would model the concept assertion  $A(b)$ , a contradiction.
- **C goes from the intersection:  $\eta(A) \cap \eta(B)$  to  $\eta(A) \setminus \eta(B)$  (Fig. 2 (b)).**  
In this situation, we would have  $\eta(C) \subseteq \eta(A)$ , and if  $\mathcal{O} \models C(a)$ , we would necessarily have that  $\eta(a) \in \eta(C)$ , but this means we would also have  $\eta(a) \in \eta(A)$ , leading to the unwarranted consequence that  $\eta \models A(a)$ . There is one last case.
- **C is placed in a region such that:  $\eta(C) \cap (\eta(A) \cup \eta(B)) \neq \emptyset$  and  $\eta(C) \setminus (\eta(A) \cup \eta(B)) \neq \emptyset$  (Fig. 2 (c)).**  
This would mean that  $\eta(B) \subseteq \eta(A) \cup \eta(C)$ , and that any concept assertion  $B(a)$  would entail either  $C(a)$  or  $A(a)$  in our geometric interpretation, while it is not necessary that  $\mathcal{O} \models A(a)$  or  $\mathcal{O} \models B(a)$ . Since we are in  $\mathbb{R}^1$ , this desired placement can happen either to the right or to the left of the number line. By assumption that  $\eta(A)$  has been placed to the left of  $\eta(B)$  as shown in Figure 2 and following, we have just shown that placing  $\eta(C)$  to the right of  $\eta(B)$  leads to a contradiction. The same reasoning applies if we choose to place it to the left of  $\eta(A)$ .

There are no more cases to be considered. ◀

The problem illustrated in Theorem 18 arises even if the ontology language does not have roles (as it is the case, e.g., of Boolean  $\mathcal{ALC}$ , investigated by [20]). It also holds if we restrict to normalized  $\mathcal{ELH}$ . We address the problem of mapping normalized  $\mathcal{ELH}$  ontologies to convex geometric models in the next section.

## 5 Strong Faithfulness on Convex Models

We prove that normalized  $\mathcal{ELH}$  has the strong faithfulness property over a class of *convex* geometric models. We introduce a new mapping  $\mu$  from the domain of a classical interpretation  $\mathcal{I}$  to a vector space and a new geometric interpretation  $\eta_{\mathcal{I}}$  based on this mapping. Our proofs now require us to fix the isomorphism preserving linear map  $f$  used in the definition of geometric interpretations (Definition 2). We choose the concatenation function, denoted  $\oplus$ , as done in the work by [10]. The strategy for proving strong faithfulness for normalized  $\mathcal{ELH}$  requires us to (a) find a suitable non-convex geometric interpretation for concepts and roles, and (b) show that the convex hull of the region maintains the property intact.



■ **Figure 3** The three possible cases when there is an element in the intersection of  $A, B, C$ .

$$\mu(d) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & 0 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \hline a_0 & & a_m & A_0 & & A_n & r_0, d_0 & & r_0, d_p & r_o, d_o & & r_o, d_p \\ \hline \end{array}$$

■ **Figure 4** A mapping to the binary vector  $\mu(d)$  when  $d \in \Delta^{\mathcal{I}}$ , where  $d \in a_0^{\mathcal{I}}$ ,  $d \in A_0^{\mathcal{I}}$  and  $(d, d_0) \in r_0^{\mathcal{I}}$ .

► **Definition 19.** Let  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  be a classical  $\mathcal{ELH}$  interpretation, and  $\mathcal{O}$  an  $\mathcal{ELH}$  ontology. We start by defining a new map  $\mu: \Delta^{\mathcal{I}} \mapsto \mathbb{R}^d$ , where  $\mathbf{d}$  corresponds to  $|N_I(\mathcal{O})| + |N_C(\mathcal{O})| + |N_R(\mathcal{O})| \cdot |\Delta^{\mathcal{I}}|$ . We assume, without loss of generality, a fixed ordering in our indexing system for positions in vectors, where indices 0 to  $|N_I(\mathcal{O})| - 1$  correspond to the indices for individual names;  $|N_I(\mathcal{O})|$  to  $k = |N_I(\mathcal{O})| + |N_C(\mathcal{O})| - 1$  correspond to the indices for concept names; and  $k + (|N_R(\mathcal{O})| \cdot |\Delta^{\mathcal{I}}|) - 1$  correspond to the indices for role names together with an element of  $\Delta^{\mathcal{I}}$ . We adopt the notation  $v[a]$ ,  $v[A]$ , and  $v[r, d]$  to refer to the position in a vector  $v$  corresponding to  $a$ ,  $A$ , and  $r$  together with an element  $d$ , respectively (according to our indexing system). For example,  $v[a] = 0$  means that the value at the index corresponding to the individual name  $a$  is 0. A vector is binary iff  $v \in \{0, 1\}^d$ . We now define  $\mu$  using binary vectors. For all  $d \in \Delta^{\mathcal{I}}$ ,  $a \in N_I$ ,  $A \in N_C$  and  $r \in N_R$ :

- $\mu(d)[a] = 1$  if  $d = a^{\mathcal{I}}$ , otherwise  $\mu(d)[a] = 0$ ,
- $\mu(d)[A] = 1$  if  $d \in A^{\mathcal{I}}$ , otherwise  $\mu(d)[A] = 0$ , and
- $\mu(d)[r, e] = 1$  if  $(d, e) \in r^{\mathcal{I}}$ , otherwise  $\mu(d)[r, e] = 0$ .

Figure 4 illustrates a possible mapping for element  $d \in \Delta^{\mathcal{I}}$ , where  $d \in a_0^{\mathcal{I}}$ ,  $d \in A_0^{\mathcal{I}}$  and  $(d, d_0) \in r_0^{\mathcal{I}}$ .

► **Example 20.** Let  $\mathcal{O}$  be an ontology such as in Example 9, with  $\mathcal{T} = \{A \sqsubseteq B\}$ ,  $\mathcal{A}$  being extended to  $\mathcal{A}' = \{A(a), B(b), r(a, b)\}$ . Let  $\mathcal{I}$  be an interpretation such that  $\Delta^{\mathcal{I}} = \{d, e\}$ , with  $a^{\mathcal{I}} = d$ ,  $b^{\mathcal{I}} = e$ ,  $r^{\mathcal{I}} = \{(d, e)\}$ ,  $A^{\mathcal{I}} = \{d\}$ , and  $B^{\mathcal{I}} = \{d, e\}$ . In this case,  $\mu: \Delta^{\mathcal{I}} \mapsto \mathbb{R}^6$ , with  $|N_I(\mathcal{O})| = 2$  (corresponding to  $a$  and  $b$ ),  $|N_C(\mathcal{O})| = 2$  (corresponding to  $A$  and  $B$ ), and  $|N_R(\mathcal{O})| \cdot |\Delta^{\mathcal{I}}| = 2$  corresponding to  $r, d$ , and  $e$ . Assume our ordering in the definition holds, and assume further that the names in the signature of  $\mathcal{O}$  are ordered alphabetically. We have that the six dimensions correspond to, respectively:  $a, b, A, B, [r, d], [r, e]$ . By applying the mapping to the elements of  $\Delta^{\mathcal{I}}$ , we get the vectors  $\mu(d) = (1, 0, 1, 1, 0, 1)$  and  $\mu(e) = (0, 1, 0, 1, 0, 0)$ .

We now introduce a definition for (possibly) non-convex geometric interpretations, in line with the mapping  $\mu$  above.

► **Definition 21.** Let  $\mathcal{I}$  be a classical  $\mathcal{ELH}$  interpretation. The geometric interpretation of  $\mathcal{I}$ , denoted  $\eta_{\mathcal{I}}$ , is defined as:

$$\eta_{\mathcal{I}}(a) := \mu(a^{\mathcal{I}}), \text{ for all } a \in N_I,$$

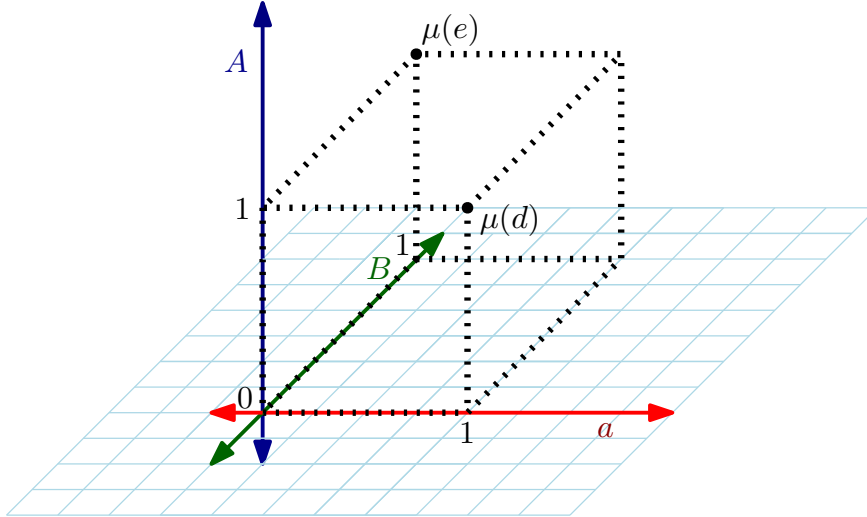
$$\eta_{\mathcal{I}}(A) := \{\mu(d) \mid \mu(d)[A] = 1, d \in \Delta^{\mathcal{I}}\}, \text{ for all } A \in N_C,$$

$$\eta_{\mathcal{I}}(r) := \{\mu(d) \oplus \mu(e) \mid \mu(d)[r, e] = 1, d, e \in \Delta^{\mathcal{I}}\}, \text{ for all } r \in N_R.$$

We provide two examples, one covering both concept and role assertions, and one (which can be represented graphically), covering only concept assertions.

► **Example 22.** Let  $\mathcal{O}, \mathcal{I}$  be as in Example 20. Then, the geometric interpretation  $\eta_{\mathcal{I}}$  of  $\mathcal{I}$  is as:  $\eta_{\mathcal{I}}(a) = \mu(d)$ ,  $\eta_{\mathcal{I}}(b) = \mu(e)$ ,  $\eta_{\mathcal{I}}(A) = \{\mu(d)\}$ ,  $\eta_{\mathcal{I}}(B) = \{\mu(d), \mu(e)\}$ ,  $\eta_{\mathcal{I}}(r) = \{\mu(d) \oplus \mu(e)\}$ . We remark that this is a strongly faithful TBox embedding.

An intuitive way of thinking about our definition  $\mu$  is that it maps domain elements to a subset of the vertex set of the  $d$ -dimensional unit hypercube (see Example 23).



■ **Figure 5** A mapping of  $\mu(d)$  and  $\mu(e)$  according to interpretation  $\mathcal{I}$ . The axes colored in red, blue, and green correspond to the dimensions associated with  $a$ ,  $A$ , and  $B$ , respectively.

► **Example 23.** Consider  $A, B \in N_C$  and  $a \in N_I$ . Let  $\mathcal{I}$  be an interpretation with  $d, e \in \Delta^{\mathcal{I}}$  such that  $d = a^{\mathcal{I}}$ ,  $d \in A^{\mathcal{I}}$ , and  $e \in A^{\mathcal{I}} \cap B^{\mathcal{I}}$ . We illustrate  $\mu(d)$  and  $\mu(e)$  in Figure 5. In symbols,  $\mu(d)[a] = 1$ ,  $\mu(d)[A] = 1$ , and  $\mu(d)[B] = 0$ , while  $\mu(e)[a] = 0$ ,  $\mu(e)[A] = 1$ , and  $\mu(e)[B] = 1$ .

Before proving strong faithfulness with convex geometric models, we show that  $\eta_{\mathcal{I}}$  preserves the axioms that hold in the original interpretation  $\mathcal{I}$ . It is possible for two elements  $d, e \in \Delta^{\mathcal{I}}$  to be mapped to the same vector  $v$  as a result of our mapping  $\mu$ . This may happen when  $d, e \notin \{a^{\mathcal{I}} \mid a \in N_I\}$  but it does hinder our results.

► **Proposition 24.** If  $\mu(d) = \mu(e)$ , then  $d \in C^{\mathcal{I}}$  iff  $e \in C^{\mathcal{I}}$ .

We use a similar strategy as before to prove our result.

► **Theorem 25.** For all  $\mathcal{ELH}$  axioms  $\alpha$ ,  $\mathcal{I} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$ .

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Since the definition of  $\eta_{\mathcal{I}}$  uses vectors in a dimensional space that depends on the size of  $\Delta^{\mathcal{I}}$  and  $\mathcal{O}$ , we need the canonical models to be finite. Therefore, we employ *finite* canonical models for normalized  $\mathcal{ELH}$  because canonical models for arbitrary  $\mathcal{ELH}$  CIs are not guaranteed to be finite. Our definition of canonical model is a non-trivial adaptation of other definitions found in the literature (e.g., [4, 18]).

Let  $\mathcal{A}$  be an  $\mathcal{ELH}$  ABox,  $\mathcal{T}$  a normalized  $\mathcal{ELH}$  TBox, and  $\mathcal{O} := \mathcal{A} \cup \mathcal{T}$ . We first define:

$$\begin{aligned} \Delta_{u^{\circ}}^{\mathcal{I}_{\mathcal{O}}} &:= \{c_A \mid A \in N_C(\mathcal{O}) \cup \{\top\}\} \text{ and} \\ \Delta_{u^+}^{\mathcal{I}_{\mathcal{O}}} &:= \Delta_{u^{\circ}}^{\mathcal{I}_{\mathcal{O}}} \cup \{c_{A \sqcap B} \mid A, B \in N_C(\mathcal{O})\} \cup \{c_{\exists r.B} \mid r \in N_R(\mathcal{O}), B \in N_C(\mathcal{O}) \cup \{\top\}\}. \end{aligned}$$

► **Definition 26.** *The canonical model  $\mathcal{I}_{\mathcal{O}}$  of  $\mathcal{O}$  is defined as*

$$\begin{aligned} \Delta^{\mathcal{I}_{\mathcal{O}}} &:= N_I(\mathcal{A}) \cup \Delta_{u^+}^{\mathcal{I}_{\mathcal{O}}}, \quad a^{\mathcal{I}_{\mathcal{O}}} := a, \\ A^{\mathcal{I}_{\mathcal{O}}} &:= \{a \in N_I(\mathcal{A}) \mid \mathcal{O} \models A(a)\} \cup \{c_D \in \Delta_{u^+}^{\mathcal{I}_{\mathcal{O}}} \mid \mathcal{T} \models D \sqsubseteq A\}, \text{ and} \\ r^{\mathcal{I}_{\mathcal{O}}} &:= \{(a, b) \in N_I(\mathcal{A}) \times N_I(\mathcal{A}) \mid \mathcal{O} \models r(a, b)\} \cup \\ &\quad \{(a, c_B) \in N_I(\mathcal{A}) \times \Delta_{u^{\circ}}^{\mathcal{I}_{\mathcal{O}}} \mid \mathcal{O} \models \exists r.B(a)\} \cup \{(c_{\exists s.B}, c_B) \in \Delta_{u^+}^{\mathcal{I}_{\mathcal{O}}} \times \Delta_{u^{\circ}}^{\mathcal{I}_{\mathcal{O}}} \mid \mathcal{T} \models s \sqsubseteq r\} \\ &\quad \cup \{(c_D, c_B) \in \Delta_{u^+}^{\mathcal{I}_{\mathcal{O}}} \times \Delta_{u^{\circ}}^{\mathcal{I}_{\mathcal{O}}} \mid \mathcal{T} \models D \sqsubseteq A, \mathcal{T} \models A \sqsubseteq \exists r.B, \text{ for some } A \in N_C(\mathcal{O})\}, \end{aligned}$$

for all  $a \in N_I$ ,  $A \in N_C$ , and  $r \in N_R$ .

The following holds for the canonical model just defined.

- **Theorem 27.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology. The following holds*
- for all  $\mathcal{ELH}$  IQs and CIs  $\alpha$  in normal form over  $\text{sig}(\mathcal{O})$ ,  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ ; and
  - for all RIs  $\alpha$  over  $\text{sig}(\mathcal{O})$ ,  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ .

The main difference between our definition and other canonical model definitions in the literature is related to our purposes of proving strong faithfulness, as we discuss in Section 5. We require the CIs and RIs (in normal form and in  $\text{sig}(\mathcal{O})$ ) that are entailed by the ontology are exactly those that hold in the canonical model.

► **Theorem 28.** *Let  $\mathcal{O}$  be an  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional (possibly non-convex)  $\oplus$ -geometric interpretation  $\eta_{\mathcal{I}_{\mathcal{O}}}$  of  $\mathcal{I}_{\mathcal{O}}$  is a strongly and IQ and TBox faithful model of  $\mathcal{O}$ .*

We now proceed with the main theorems of this section. Note that the dimensionality of the image domain of  $\mu$  can be much higher than the one for  $\bar{\mu}$  in Section 4 (which can be as low as just 1, see Corollary 17). We use the results until now as intermediate steps to bridge the gap between classical and convex geometric interpretations. In our construction of convex geometric interpretations, the vectors mapped by  $\mu$  and the regions given by the non-convex geometric interpretation  $\eta_{\mathcal{I}}$  are the anchor points for the convex closure of these sets. We introduce the notion of the *convex hull* of a geometric interpretation  $\eta_{\mathcal{I}}$  using Definition 4.

► **Definition 29.** *We denote by  $\eta_{\mathcal{I}}^*$  the convex hull of the geometric interpretation  $\eta_{\mathcal{I}}$  and define  $\eta_{\mathcal{I}}^*$  as follows:*

$$\begin{aligned} \eta_{\mathcal{I}}^*(a) &:= \mu(a^{\mathcal{I}}), \text{ for all } a \in N_I; \\ \eta_{\mathcal{I}}^*(A) &:= \{\mu(d) \mid d \in A^{\mathcal{I}}\}^*, \text{ for all } A \in N_C; \text{ and} \\ \eta_{\mathcal{I}}^*(r) &:= \{\mu(d) \oplus \mu(e) \mid (d, e) \in r^{\mathcal{I}}\}^*, \text{ for all } r \in N_R. \end{aligned}$$

► **Remark 30.** In Definition 29,  $\eta_{\mathcal{I}}^*(a) = \eta_{\mathcal{I}}(a)$  for all  $a \in N_{\mathcal{I}}$ . We include the star symbol in the notation to make it clear that we are referring to the geometric interpretation of individual names in the context of convex regions for concepts and roles.

► **Theorem 31.** *Let  $\eta_{\mathcal{I}}$  be a geometric interpretation as in Definition 21. If  $\alpha$  is an  $\mathcal{ELH}$  CI, an  $\mathcal{ELH}$  RI, or an  $\mathcal{ELH}$  IQ in normal form then  $\eta_{\mathcal{I}} \models \alpha$  iff  $\eta_{\mathcal{I}}^* \models \alpha$ .*

We are now ready to consider strong IQ and TBox faithfulness for convex regions.

► **Theorem 32.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional convex  $\oplus$ -geometric interpretation of  $\mathcal{I}_{\mathcal{O}}$  (Definition 29) is a strongly IQ and TBox faithful model of  $\mathcal{O}$ .*

We now state a corollary analogous to Corollary 17, though here we cannot state it for all classes of  $m$ -dimensional  $f$ -geometric interpretations (we know by Theorem 18 that this is impossible for any class of 1-dimensional geometric interpretations). We omit “ $m$ -dimensional” in Corollary 33 to indicate that this holds for the larger class containing geometric interpretations with an arbitrary number of dimensions (necessary to cover the whole language).

► **Corollary 33.** *Normalized  $\mathcal{ELH}$  has the strong faithfulness property over  $\oplus$ -geometric interpretations.*

► **Remark 34 (Number of parameters).** The final number of parameters for the convex geometric interpretation  $\eta_{\mathcal{I}_{\mathcal{O}}}$  of the canonical model  $\mathcal{I}_{\mathcal{O}}$  built on ontology  $\mathcal{O}$  is, thus:  $O(d \cdot n)$  where  $d$  is the embedding dimension given by map  $\mu$  (Definition 19), and  $n = |\Delta^{\mathcal{I}_{\mathcal{O}}}|$ .

## 6 Model Checking on Geometric Models

Here we study upper bounds for the complexity of model checking problems using convex geometric models as those defined in Definition 29 and normalized  $\mathcal{ELH}$  axioms. The results and algorithms in this section are underpinned by Theorem 31, which allow us to use  $\eta_{\mathcal{I}}$  instead of  $\eta_{\mathcal{I}}^*$  for model checking purposes. The advantage of using  $\eta_{\mathcal{I}}$  instead of  $\eta_{\mathcal{I}}^*$  is that the algorithms need to inspect only finitely many elements in the extension of each concept and each role, as long as the original interpretation  $\mathcal{I}$  has finite domain (and we only need to consider a finite number of concept, role, and individual names). For example, let  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  with  $\Delta^{\mathcal{I}}$  finite. If  $A \in \mathbf{N}_{\mathcal{C}}$  then  $\eta_{\mathcal{I}}^*(A)$  can have infinitely many elements, while  $\eta_{\mathcal{I}}(A)$  will have at most  $|\Delta^{\mathcal{I}}|$  elements (by Definition 21). Before presenting the algorithms, we discuss some assumptions that facilitate our analysis:

1. indexing vectors and comparing primitive types use constant time;
2. accessing the extension of an individual, concept, or role name in  $\eta_{\mathcal{I}}$  takes constant time;
3. iterating over  $\eta_{\mathcal{I}}(A)$  (and also  $\eta_{\mathcal{I}}(r)$ ) consumes time  $O(|\Delta^{\mathcal{I}}|)$  ( $O(|\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$ ) for all  $A \in \mathbf{N}_{\mathcal{C}}$  ( $r \in \mathbf{N}_{\mathcal{R}}$ ); and
4. if  $A \in \mathbf{N}_{\mathcal{C}}$  ( $r \in \mathbf{N}_{\mathcal{R}}$ ), testing if  $v \in \eta_{\mathcal{I}}(A)$  ( $v \in \eta_{\mathcal{I}}(r)$ ) consumes time  $O(d \cdot |\Delta^{\mathcal{I}}|)$  ( $O(d \cdot |\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$ ).

Assumption (1) is standard when analysing worst-case complexity. The others are pessimistic assumptions on the implementation of  $\eta_{\mathcal{I}}$  (and  $\eta_{\mathcal{I}}^*$ ). E.g., encoding the binary vectors as integers and implementing bit wise operations could reduce the complexity of membership access and iteration. Also, using a hash map with a perfect hash function would decrease the membership check to constant time.

We are now ready to present our upper bounds. For normalised  $\mathcal{ELH}$  CIs, we provide Algorithm 1 to decide if a concept inclusion holds in a convex geometric model built as in Definition 29. Theorem 31 guarantees that  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$  iff  $\eta_{\mathcal{I}} \models C \sqsubseteq D$  for any CI in normalised

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$\mathcal{ELH}$ . Thus, as long as  $\Delta^{\mathcal{I}}$  is finite, Algorithm 1 terminates and outputs whether  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$ . Theorem 35 establishes that Algorithm 1 runs in polynomial time in the size of  $\Delta^{\mathcal{I}}$  and the dimension of vectors in  $\eta_{\mathcal{I}}^*$ .

■ **Algorithm 1** Check if a convex geometric model (Definition 29) satisfies an  $\mathcal{ELH}$  CI in normal form.

**Require:** a convex geometric interpretation  $\eta_{\mathcal{I}}^*$  and an  $\mathcal{ELH}$  concept inclusion in normal form  $\alpha$

**Ensure:** returns **True** if  $\eta_{\mathcal{I}}^* \models \alpha$ , **False** otherwise

```

1: if  $\alpha = A \sqsubseteq B$  then ▷  $A, B \in \mathbf{N}_{\mathbf{C}}$ 
2:   for  $v \in \eta_{\mathcal{I}}(A)$  do
3:     if  $v[B] = 0$  then return False
4:   else if  $\alpha = A_1 \sqcap A_2 \sqsubseteq B$  then ▷  $A_1, A_2, B \in \mathbf{N}_{\mathbf{C}}$ 
5:     for  $v \in \eta_{\mathcal{I}}(A_1)$  do
6:       if  $v[A_2] = 1 \wedge v[B] = 0$  then return False
7:   else if  $\alpha = A \sqsubseteq \exists r. B$  then ▷  $A, B \in \mathbf{N}_{\mathbf{C}}, r \in \mathbf{N}_{\mathbf{R}}$ 
8:     for  $v \in \eta_{\mathcal{I}}(A)$  do  $\text{count} \leftarrow 0$ 
9:       for  $u \in \eta_{\mathcal{I}}(B)$  do
10:        if  $v \oplus u \in \eta_{\mathcal{I}}(r)$  then
11:           $\text{count} \leftarrow \text{count} + 1$ 
12:        if  $\text{count} = 0$  then return False
13:   else if  $\alpha = \exists r. A \sqsubseteq B$  then ▷  $A, B \in \mathbf{N}_{\mathbf{C}}, r \in \mathbf{N}_{\mathbf{R}}$ 
14:     for  $v \oplus u \in \eta_{\mathcal{I}}(r)$  do
15:       if  $u[A] = 1$  and  $v[B] = 0$  then return False
16: return True

```

► **Theorem 35.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  CI in normal form, Algorithm 1 runs in time in  $O(d \cdot n^4)$ , where  $d$  is as in Definition 19 and  $n = |\Delta^{\mathcal{I}}|$ .*

As  $d$  depends linearly on  $\Delta^{\mathcal{I}}$  and the size of the signature. If the latter is regarded as a constant, we can simply say that Algorithm 1 has time in  $O(n^5)$ , where  $n = |\Delta^{\mathcal{I}}|$ . Similarly as for Algorithm 1, Theorem 31 allows us to design an algorithm to determine if a convex geometric model  $\eta_{\mathcal{I}}^*$  satisfies an IQ in normal form  $\alpha$ , as we show in Algorithm 2.

■ **Algorithm 2** check if a convex geometric model (as in Definition 29) satisfies an  $\mathcal{ELH}$  IQ in normal form.

**Require:** a convex geometric interpretation  $\eta_{\mathcal{I}}^*$  and an  $\mathcal{ELH}$  IQ in normal form  $\alpha$

**Ensure:** returns **True** if  $\eta_{\mathcal{I}}^* \models \alpha$ , **False** otherwise

```

1: if  $\alpha = A(a)$  then ▷  $A \in \mathbf{N}_{\mathbf{C}}, a \in \mathbf{N}_{\mathbf{I}}$ 
2:   if  $\eta_{\mathcal{I}}(a)[A] = 1$  then return True
3:   else if  $\alpha = (A \sqcap B)(a)$  then ▷  $A, B \in \mathbf{N}_{\mathbf{C}}, a \in \mathbf{N}_{\mathbf{I}}$ 
4:     if  $(\eta_{\mathcal{I}}(a)[A] = 1) \wedge (\eta_{\mathcal{I}}(a)[B] = 1)$  then return True
5:   else if  $\alpha = (\exists r. A)(a)$  then ▷  $A \in \mathbf{N}_{\mathbf{C}}, r \in \mathbf{N}_{\mathbf{R}}, a \in \mathbf{N}_{\mathbf{I}}$ 
6:     for  $u \in \eta_{\mathcal{I}}(A)$  do
7:       if  $\eta_{\mathcal{I}}(a) \oplus u \in \eta_{\mathcal{I}}(r)$  then return True
8:   else if  $\alpha = r(a, b)$  then ▷  $r \in \mathbf{N}_{\mathbf{R}}, a, b \in \mathbf{N}_{\mathbf{I}}$ 
9:     if  $\eta_{\mathcal{I}}(a) \oplus \eta_{\mathcal{I}}(b) \in \eta_{\mathcal{I}}(r)$  then return True
   return False

```

Theorem 36 shows that Algorithm 2 runs in time polynomial in  $d \cdot |\Delta^{\mathcal{I}}|$ .



► **Theorem 36.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  IQ in normal form, Algorithm 2 runs in time  $O(d \cdot n^3)$ , with  $d$  as in Definition 19 and  $n = |\Delta^{\mathcal{I}}|$ .*

Next, we present Algorithm 3, which handles RIs. Again, as a consequence of Theorem 31, we only need to check the inclusion between two finite sets of vectors in  $\mathbb{R}^{2 \cdot d}$ . Finally, we show an upper bound using Algorithm 3.

■ **Algorithm 3** Check if a convex geometric model (as in Definition 29) satisfies an  $\mathcal{ELH}$  role inclusion.

**Require:** a convex geometric interpretation  $\eta_{\mathcal{I}}^*$  and an  $\mathcal{ELH}$  role inclusion  $r \sqsubseteq s$

**Ensure:** returns **True** if  $\eta_{\mathcal{I}}^* \models r \sqsubseteq s$ , **False** otherwise

```

1: for  $v \in \eta_{\mathcal{I}}(r)$  do
2:   if  $v \notin \eta_{\mathcal{I}}(s)$  then return False
   return True

```

► **Theorem 37.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  role inclusion, Algorithm 3 runs in time in  $O(d \cdot n^4)$ , where  $d$  is as in Definition 19 and  $n = |\Delta^{\mathcal{I}}|$ .*

The three algorithms presented in this section run in polynomial time in  $d \cdot |\Delta^{\mathcal{I}}|$ . We recall that the construction of  $\eta_{\mathcal{I}}$  (and also  $\eta_{\mathcal{I}}^*$ ) requires that both the signature and  $\Delta^{\mathcal{I}}$  are finite (which is reasonable for normalized  $\mathcal{ELH}$ ), otherwise the vectors in  $\eta_{\mathcal{I}}$  would have infinite dimension.

## 7 Conclusion and discussion

We have proven that  $\mathcal{ELH}$  has the strong faithfulness property over (possibly) non-convex geometric models, and that normalized  $\mathcal{ELH}$  has the strong faithfulness property over convex geometric models. Furthermore, we give upper bounds for the complexity of checking satisfaction for  $\mathcal{ELH}$  axioms in normal form in the class of convex geometric models that we use for strong faithfulness.

As future work, we would like to implement an embedding method that is formally guaranteed to generate strongly TBox faithful embeddings for normalized  $\mathcal{ELH}$  ontologies, as well as expand the language so as to cover more logical constructs present in  $\mathcal{EL}^{++}$ .

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## A

 Appendix

### A.1 Omitted proofs for Section 3

► **Proposition 5.** For finite  $S_1, S_2 \subseteq \mathbb{R}^d$ , where  $d$  is an arbitrary dimension, we have that  $S_1 \subseteq S_2$  implies  $S_1^* \subseteq S_2^*$ .

**Proof.** Let  $S_1, S_2$  be finite sets with  $S_1 \subseteq S_2$ . We first prove the statement for  $v \in S_1 \subseteq S_1^*$  and then for  $u \in S_1^* \setminus S_1$ . Let  $v \in S_1$  be an arbitrary vector. By assumption,  $v \in S_2$ , and by the definition of convex hull,  $v \in S_2^*$ . Now, by Definition 4 let  $u \in S_1^* \setminus S_1$  be defined by  $\sum_{i=1}^n \lambda_i v_i$  where  $v_1 \dots v_n \in S_1$  and  $n \leq |S_1|$ . Since  $S_1 \subseteq S_2$ ,  $v_1 \dots v_n \in S_2$  and, by Definition 4, since  $u = \sum_{i=1}^n \lambda_i v_i$ , this gives us that  $u \in S_2^*$ . Thus,  $S_1 \subseteq S_2$  implies  $S_1^* \subseteq S_2^*$ . ◀

► **Theorem 6.** Let  $S \subseteq \{0, 1\}^d$  where  $d$  is an arbitrary dimension. For any  $n \in \mathbb{N}$ , for any  $v = \sum_{i=1}^n \lambda_i v_i$ , such that  $v_i \in S$ , if  $v \in S^* \setminus S$  then  $v$  is non-binary.

**Proof.** For this proof we use a notation introduced in Definition 19. We reason by cases. We need to cover all combinations of values that  $\lambda_i$  may take for arbitrary  $n$ . We cover two cases. One where all  $\lambda$  are strictly greater than zero and strictly lesser than 1, and a case where some  $\lambda_i$  may be zero. By setting  $n = 1$ , we have  $v = \lambda_1 x_1$ . By definition,  $\lambda_1 = 1$ , giving us either  $v = 0$  or  $v = 1$ , both binary vectors, which means  $v \in S^*$  iff  $v \in S$ . Therefore, this case is not in the scope of our lemma, and we assume  $n > 1$ .

■ **Case 1 ( $0 < \lambda_i < 1$ ):** We prove the case by induction on the number of  $n$ .

**Base case:** In the base case  $n = 2$ . Let  $v_1, v_2 \in S$  with  $v_1 \neq v_2$ . Then, there is a dimension  $d$  such that  $v_1[d] \neq v_2[d]$ . Since  $v_1$  and  $v_2$  are binary, we can assume, without loss of generality,  $v_1[d] = 1$  and  $v_2[d] = 0$ . Now let  $v = \lambda_1 v_1 + \lambda_2 v_2$  be a vector, with  $\lambda_1 + \lambda_2 = 1$ . Since we assumed  $\forall \lambda_i$   $0 < \lambda_i < 1$ , this means  $v \notin \{0, 1\}^d$  because  $v[d] = \lambda_1$ , which is strictly between 0 and 1. Therefore,  $v$  is non-binary.

**Inductive step:** Assume our hypothesis holds for  $v_1, \dots, v_{n-1}$ .

Let  $v \in S^*$ . We know that  $v = \sum_{i=1}^n \lambda_i v_i$ , with  $0 < \lambda_i < 1$ , with  $v_i \in S$ , and with  $\sum_{i=1}^n \lambda_i = 1$ . Since  $\forall_{i \neq j} v_i \neq v_j$ , there is a dimension  $d$  such that  $\exists l, m$  with  $v_l[d] \neq v_m[d]$ . Since  $S$  is a set of binary vectors, we decompose the value of a dimension  $d$  as a sum of vectors where  $v_i[d] = 1$  and  $v_j[d] = 0$ . In order to do this, we introduce an ordering and assume, without loss of generality, that  $v_i[d] = 1 \forall 1 \leq i \leq k$  where  $k < n$ , and  $v_j[d] = 0 \forall k+1 \leq j < n$ . More explicitly:

$$v[d] = \sum_{i=1}^k \lambda_i v_i[d] + \sum_{j=k+1}^n \lambda_j v_j[d].$$

However,  $\sum_{j=k+1}^n \lambda_j v_j[d] = 0$ , so we only have to look at the first sum. Clearly,  $v[d] \neq 0$ , because  $v_l[d] \neq v_m[d]$ . Since there exists at least one  $\lambda_j > 0$  and, in this case  $\forall \lambda_i$   $0 < \lambda_i < 1$ , it is impossible for the sum to be equal to 1, giving us  $v[d] \in (0, 1)$ .

■ **Case 2 ( $\exists \lambda_i = 0$  and  $\forall \lambda_{j \neq i}$  we have  $0 \leq \lambda_j < 1$ ):**

We prove the case directly. We start by noting that for this case to hold,  $n \geq 3$ , as  $n = 2$  would mean  $\lambda_1 = 0$  and  $\lambda_2 < 1$ , which goes against the criterion that  $\sum_{i=1}^n \lambda_i v_i = 1$  from the definition. Now, assume  $n \geq 3$ . We denote by  $m$  the number of  $\lambda_i$  where  $\lambda_i = 0$ . Pick  $m$  such that  $1 \leq m \leq n - 2$ . Then, there are at least  $n - m \geq 2$   $\lambda_j$  such that  $0 < \lambda_j < 1$ . Which is the situation covered by *Case 1*.

There are no more cases to be considered. ◀

► **Corollary 7.** *If  $v$  is binary and  $v \in S^*$  then  $v \in S$ .*

**Proof.** The corollary follows directly from Theorem 6. ◀

## A.2 Omitted proofs for Section 4

► **Lemma 38.** *For all  $d \in \Delta^{\mathcal{I}}$ , for all  $\mathcal{ELH}$  concepts  $C$ , it is the case that  $d \in C^{\mathcal{I}}$  iff  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$  (see Definition 12).*

**Proof.** We provide an inductive argument in order to prove the claim.

**Base case:** Assume  $C = A \in N_C$ , and assume  $d \in A^{\mathcal{I}}$ .

By the definition of  $\bar{\eta}_{\mathcal{I}}$ ,  $d \in A^{\mathcal{I}}$  iff for all  $v \in \bar{\mu}(d)$ ,  $v \in \bar{\eta}_{\mathcal{I}}(A)$ , that is, iff  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(A)$ . Now assume  $C = \top$ , and assume  $d \in C^{\mathcal{I}}$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , if  $d \in C^{\mathcal{I}}$ , then  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ . Now assume  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ . Since we assumed  $C = \top$ , we have that  $\bar{\mu}(d) \subseteq \mathbb{R}^1$ , with  $d \in \Delta^{\mathcal{I}}$ . When  $C = \perp$ , the statement is vacuously true.

**Inductive step:** Assume our hypothesis holds for  $C_1$  and  $C_2$ . There are two cases:

- **Case 1 ( $C_1 \sqcap C_2$ ):** Assume  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$  by the semantics of  $\mathcal{ELH}$ ,  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$  iff  $d \in C_1^{\mathcal{I}}$  and  $d \in C_2^{\mathcal{I}}$ . By the inductive hypothesis,  $d \in C_i^{\mathcal{I}}$  iff  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C_i)$ ,  $i \in \{1, 2\}$ . But this happens iff  $d \in \bar{\eta}_{\mathcal{I}}(C_1) \cap \bar{\eta}_{\mathcal{I}}(C_2)$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , this means that  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C_1 \sqcap C_2)$  iff  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$ .
- **Case 2 ( $\exists r.C_1$ ):** Assume  $d \in (\exists r.C_1)^{\mathcal{I}}$  by the semantics of  $\mathcal{ELH}$ ,  $d \in (\exists r.C_1)^{\mathcal{I}}$  iff  $(d, e) \in r^{\mathcal{I}}$  and  $e \in C_1^{\mathcal{I}}$ . By the inductive hypothesis,  $e \in C_1^{\mathcal{I}}$  iff  $\bar{\mu}(e) \subseteq \bar{\eta}_{\mathcal{I}}(C_1)$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ ,  $(d, e) \in r^{\mathcal{I}}$  iff  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$  where  $v \in \bar{\mu}(d)$ . By the semantics of  $\bar{\eta}_{\mathcal{I}}$ ,  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$  and  $e \in \bar{\eta}_{\mathcal{I}}(C_1)$  iff  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(\exists r.C_1)$ . ◀

► **Lemma 39.** *For all interpretations  $\mathcal{I}$ , all  $\mathcal{ELH}$  concepts  $C$ , and all  $a \in N_{\mathcal{I}}$ , it is the case that  $\mathcal{I} \models C(a)$  iff  $\bar{\eta}_{\mathcal{I}} \models C(a)$*

**Proof.** By the semantics of  $\mathcal{ELH}$ , we know  $\mathcal{I} \models C(a)$  iff  $a^{\mathcal{I}} \in C^{\mathcal{I}}$ . By Lemma 38, we know that  $a^{\mathcal{I}} \in C^{\mathcal{I}}$  iff  $\bar{\eta}_{\mathcal{I}}(a^{\mathcal{I}}) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ . By the semantics of geometric interpretation, this is the case iff  $\bar{\eta}_{\mathcal{I}} \models C(a)$ . ◀

► **Lemma 40.** *For all  $r \in N_R$ , for all  $a, b \in N_{\mathcal{I}}$ , we have  $\bar{\eta}_{\mathcal{I}} \models r(a, b)$  iff  $\mathcal{I} \models r(a, b)$ .*

**Proof.** By the semantics of  $\mathcal{ELH}$ ,  $\mathcal{I} \models r(a, b)$  iff  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , we have  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$  iff  $f(v, b^{\mathcal{I}}) \in \bar{\eta}_{\mathcal{I}}(r)$  for all  $v \in \bar{\mu}(a^{\mathcal{I}})$ . From the Definition 12,  $b^{\mathcal{I}} = \bar{\eta}_{\mathcal{I}}(b)$ , hence  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$  iff  $f(v, \bar{\eta}_{\mathcal{I}}(b)) \in \bar{\eta}_{\mathcal{I}}(r)$  for all  $v \in \bar{\mu}(a^{\mathcal{I}})$ . Since  $\bar{\eta}_{\mathcal{I}}(a) \in \bar{\mu}(a^{\mathcal{I}})$ , we get, by the semantics of  $\bar{\eta}_{\mathcal{I}}$ , that  $f(\bar{\eta}_{\mathcal{I}}(a), \bar{\eta}_{\mathcal{I}}(b)) \in \bar{\eta}_{\mathcal{I}}(r)$  iff  $\bar{\eta}_{\mathcal{I}} \models r(a, b)$ . Giving us  $\mathcal{I} \models r(a, b)$  iff  $\bar{\eta}_{\mathcal{I}} \models r(a, b)$ . ◀

► **Lemma 41.** *Let  $\mathcal{O}$  be an  $\mathcal{ELH}$  ontology and let  $\bar{\mathcal{I}}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 14). The geometrical interpretation  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}}$  of  $\bar{\mathcal{I}}_{\mathcal{O}}$  (Definition 12) is a strongly IQ faithful model of  $\mathcal{O}$ .*

**Proof.** Since  $\bar{\mathcal{I}}_{\mathcal{O}}$  is a canonical model of  $\mathcal{O}$ ,  $\bar{\mathcal{I}}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$  (Theorem 15). By Lemmas 39 and 40,  $\bar{\mathcal{I}}_{\mathcal{O}} \models \alpha$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models \alpha$ . Then, we have that  $\mathcal{O} \models \alpha$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models \alpha$ . ◀

► **Lemma 42.** *Let  $\mathcal{I}$  be an interpretation, and  $\bar{\mu}$  be a mapping derived from Definition 10. For all  $\mathcal{ELH}$  concepts  $C$ , if  $v \in \bar{\eta}_{\mathcal{I}}(C)$ , then there is  $d \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$ , and  $d \in C^{\mathcal{I}}$ .*

**Proof.** We provide an inductive argument for the claim.

**Base case:** Assume  $C = A \in N_C$  and let  $v \in \bar{\eta}_{\mathcal{I}}(A)$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , it is the case that  $v \in \bar{\eta}_{\mathcal{I}}(A)$  iff  $v \in \{v' \in \bar{\mu}(d) \mid d \in A^{\mathcal{I}}\}$ . Assume  $C = \top$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , we have  $v \in \bar{\eta}_{\mathcal{I}}(C)$  iff  $v \in \bar{\mu}(d)$  such that  $\bar{\mu}(d) \subseteq \mathbb{R}^1$ . This means  $v \in \bar{\mu}(d)$  and  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ , for some  $d \in \Delta^{\mathcal{I}}$ . When  $C = \perp$ , the statement is vacuously true.

**Inductive step:** Assume our hypothesis holds for  $C_1$  and  $C_2$ .

- **Case 1 ( $C_1 \sqcap C_2$ ):** Assume  $v \in \bar{\eta}_{\mathcal{I}}(C_1 \sqcap C_2)$ . Then, by the definition of  $\bar{\eta}_{\mathcal{I}}$ , it is the case that  $v \in \bar{\eta}_{\mathcal{I}}(C_1)$  and  $v \in \bar{\eta}_{\mathcal{I}}(C_2)$ . By the inductive hypothesis, if  $v \in \bar{\eta}_{\mathcal{I}}(C_1)$ , then  $\exists d \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$  and  $d \in C_1^{\mathcal{I}}$ , and if  $v \in \bar{\eta}_{\mathcal{I}}(C_2)$ , then  $\exists d' \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d')$  and  $d' \in C_2^{\mathcal{I}}$ . By definition of  $\bar{\mu}$ , this can only be if  $d' = d$  since  $\bar{\mu}$  maps elements of  $\Delta^{\mathcal{I}}$  to mutually disjoint subsets of  $\mathbb{R}^1$ . By the semantics of  $\mathcal{ELH}$ , if  $d \in C_1^{\mathcal{I}}$  and  $d \in C_2^{\mathcal{I}}$  then  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$ .
- **Case 2 ( $\exists r.C_1$ ):** Assume  $v \in \bar{\eta}_{\mathcal{I}}(\exists r.C_1)$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , this means  $v$  is such that  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$  where  $v \in \bar{\mu}(d)$  for  $(d, e) \in r^{\mathcal{I}}$  and  $e \in \bar{\eta}_{\mathcal{I}}(C_1)$ . By the inductive hypothesis, there is an  $e' \in \Delta^{\mathcal{I}}$  such that  $e \in \bar{\mu}(e')$  and  $e' \in C_1^{\mathcal{I}}$ . As  $e' \in \Delta^{\mathcal{I}} \subseteq \mathbb{N}$ , by the construction of  $\bar{\mu}$ , it is the case that  $e' = e$ . Therefore, we have  $e \in C_1^{\mathcal{I}}$ . By the definition of  $\bar{\mu}$  and the semantics of  $\mathcal{ELH}$ , this means  $\exists d \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$  and  $d \in (\exists r.C_1)^{\mathcal{I}}$ . ◀

► **Lemma 43.** *Let  $\mathcal{I}$  be an interpretation and  $\bar{\eta}_{\mathcal{I}}$  the geometric interpretation of  $\mathcal{I}$  (Definition 12). For all  $\mathcal{ELH}$  concepts  $C$  and  $D$ ,  $\mathcal{I} \models C \sqsubseteq D$  iff  $\bar{\eta}_{\mathcal{I}} \models C \sqsubseteq D$ .*

**Proof.** Let  $C, D$  be  $\mathcal{ELH}$  concepts. Assume  $\mathcal{I} \models C \sqsubseteq D$ . By the semantics of  $\mathcal{ELH}$ , this means  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ . Let  $v \in \bar{\eta}_{\mathcal{I}}(C)$  be a vector. By Lemma 42, we know there is  $d \in \Delta^{\mathcal{I}}$  and  $d \in C^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$  and  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ . By Lemma 38, this means  $d \in C^{\mathcal{I}}$ , and, by assumption, that  $d \in D^{\mathcal{I}}$ . By Lemma 38, this means  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(D)$ . Since we have shown  $v \in \bar{\mu}(d)$  such that  $\bar{\eta}_{\mathcal{I}}(C)$  implies  $v \in \bar{\eta}_{\mathcal{I}}(D)$ , this means  $\bar{\eta}_{\mathcal{I}} \models C \sqsubseteq D$ .

Now assume  $\bar{\eta}_{\mathcal{I}} \models C \sqsubseteq D$ . By the semantics of geometric interpretation, this means  $\bar{\eta}_{\mathcal{I}}(C) \subseteq \bar{\eta}_{\mathcal{I}}(D)$ . Let  $d \in C^{\mathcal{I}}$ . We know, by Lemma 38, that  $d \in C^{\mathcal{I}}$  iff  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(C)$ . By assumption, this means  $\bar{\mu}(d) \subseteq \bar{\eta}_{\mathcal{I}}(D)$ . Again by Lemma 38, this means  $d \in D^{\mathcal{I}}$ . Since we have shown  $d \in C^{\mathcal{I}}$  implies  $d \in D^{\mathcal{I}}$ , we have  $\mathcal{I} \models C \sqsubseteq D$ . ◀

► **Lemma 44.** *Let  $\mathcal{I}$  be an interpretation,  $\bar{\mu}$  be a mapping (Definition 10), and  $\bar{\eta}_{\mathcal{I}}$  the geometric interpretation of  $\mathcal{I}$  (Definition 12) derived from  $\bar{\mu}$ . For all role names  $r \in N_R$ , if  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$ , then there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$  for  $(d, e) \in r^{\mathcal{I}}$ .*

**Proof.** Assume  $z = f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$ . By the definition of  $\bar{\eta}_{\mathcal{I}}$ , we have  $z \in \{f(v, e) \mid v \in \bar{\mu}(d) \text{ for } (d, e) \in r^{\mathcal{I}}\}$ . This means  $v \in \bar{\mu}(d)$  for  $d \in \Delta^{\mathcal{I}}$ , and, by definition,  $e \in \Delta^{\mathcal{I}}$ . ◀

► **Lemma 45.** *Let  $\mathcal{I}$  be an interpretation and  $\bar{\eta}_{\mathcal{I}}$  the geometric interpretation of  $\mathcal{I}$  (Definition 12). For all roles  $r, s \in N_R$ , it is the case that  $\mathcal{I} \models r \sqsubseteq s$  iff  $\bar{\eta}_{\mathcal{I}} \models r \sqsubseteq s$ .*

**Proof.** Assume  $\mathcal{I} \models r \sqsubseteq s$ . By the semantics of  $\mathcal{ELH}$ ,  $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ . Now let  $v \in \bar{\eta}_{\mathcal{I}}(r)$ . By Lemma 44, there is  $d \in \Delta^{\mathcal{I}}$  such that  $v \in \bar{\mu}(d)$ ,  $e \in \Delta^{\mathcal{I}}$ , and  $(d, e) \in r^{\mathcal{I}}$ . By assumption, this gives us  $(d, e) \in s^{\mathcal{I}}$ . By the construction of  $\bar{\eta}_{\mathcal{I}}$ , this means  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(s)$  for  $v \in \bar{\mu}(d)$ . Hence,  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$  implies  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(s)$  and we can conclude that  $\bar{\eta}_{\mathcal{I}} \models r \sqsubseteq s$ . Now assume  $\bar{\eta}_{\mathcal{I}} \models r \sqsubseteq s$ . By the semantics of  $\bar{\eta}_{\mathcal{I}}$ ,  $\bar{\eta}_{\mathcal{I}}(r) \subseteq \bar{\eta}_{\mathcal{I}}(s)$ . Let  $(d, e) \in r^{\mathcal{I}}$ . From the definition of  $\bar{\eta}_{\mathcal{I}}$ , we know there is  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(r)$  such that  $v \in \bar{\mu}(d)$ . By assumption, we have  $f(v, e) \in \bar{\eta}_{\mathcal{I}}(s)$  and, by the definition of  $\bar{\eta}_{\mathcal{I}}$ , this is the case iff  $(d, e) \in s^{\mathcal{I}}$ . Since  $(d, e)$  was arbitrary, we conclude  $\mathcal{I} \models r \sqsubseteq s$ . ◀

► **Theorem 13.** *For all  $\mathcal{ELH}$  axioms  $\alpha$ ,  $\mathcal{I} \models \alpha$  iff  $\bar{\eta}_{\mathcal{I}} \models \alpha$ .*

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**Proof.** For the case where  $\alpha$  is a concept inclusion, the result comes from Lemma 43. For the case where  $\alpha$  is a role inclusion, the result comes from Lemma 45. For the case where  $\alpha$  is an IQ, the result comes from Lemma 39 and from Lemma 40.  $\blacktriangleleft$

► **Lemma 46.** *Let  $\mathcal{O}$  be an  $\mathcal{ELH}$  ontology and let  $\bar{\mathcal{I}}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (see Definition 14). The  $m$ -dimensional  $f$ -geometric interpretation of  $\bar{\mathcal{I}}_{\mathcal{O}}$  (see Definition 12) is a strongly TBox faithful model of  $\mathcal{O}$ . That is,  $\mathcal{O} \models \tau$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models \tau$ , where  $\tau$  is either an  $\mathcal{ELH}_{\perp}$  concept inclusion or an  $\mathcal{ELH}$  role inclusion.*

**Proof.** Since we know  $\bar{\mathcal{I}}_{\mathcal{O}}$  is canonical,  $\mathcal{O} \models \alpha$  iff  $\bar{\mathcal{I}}_{\mathcal{O}} \models \alpha$ . By Lemma 43 we know  $\mathcal{I} \models C \sqsubseteq D$  iff  $\bar{\eta}_{\mathcal{I}} \models C \sqsubseteq D$ , and by Lemma 45 we know  $\mathcal{I} \models r \sqsubseteq s$  iff  $\bar{\eta}_{\mathcal{I}} \models r \sqsubseteq s$ . This means that  $\bar{\mathcal{I}}_{\mathcal{O}} \models C \sqsubseteq D$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models C \sqsubseteq D$  and  $\bar{\mathcal{I}}_{\mathcal{O}} \models r \sqsubseteq s$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models r \sqsubseteq s$ , giving us  $\mathcal{O} \models \tau$  iff  $\bar{\eta}_{\bar{\mathcal{I}}_{\mathcal{O}}} \models \tau$ .  $\blacktriangleleft$

► **Theorem 16.** *Let  $\mathcal{O}$  be a satisfiable  $\mathcal{ELH}$  ontology and let  $\bar{\mathcal{I}}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (see Definition 14). The  $m$ -dimensional  $f$ -geometric interpretation of  $\bar{\mathcal{I}}_{\mathcal{O}}$  (see Definition 12) is a strongly IQ and TBox faithful model of  $\mathcal{O}$ .*

**Proof.** The theorem follows by Lemma 41 and by Lemma 46.  $\blacktriangleleft$

### A.3 Omitted proofs for Section 5

► **Proposition 24.** *If  $\mu(d) = \mu(e)$ , then  $d \in C^{\mathcal{I}}$  iff  $e \in C^{\mathcal{I}}$ .*

**Proof.** We provide an inductive argument for the claim.

**Base case:** Notice that if  $\mu(d) = \mu(e)$ , then  $\mu(d)[i] = n$  iff  $\mu(e)[i] = n$ , for all  $i$ . That is, the value at the  $i$ th index is  $n$  for  $\mu(d)$  and  $\mu(e)$ , otherwise they would not be the same vector. Now, assume  $C = A \in N_C$ , and  $d \in C^{\mathcal{I}}$ . By the definition of  $\mu$ ,  $\mu(d)[C] = 1$ . Since  $\mu(d) = \mu(e)$ , we have that  $\mu(d)[C] = 1$  iff  $\mu(e)[C] = 1$ . But, by the definition of  $\mu$ ,  $\mu(e)[C] = 1$  iff  $e \in C^{\mathcal{I}}$ , thus giving us our result.

**Inductive step:** Assume our hypothesis holds for  $C_1$  and  $C_2$ .

Assume  $\mu(d) = \mu(e)$ . By the semantics of  $\mathcal{ELH}$ ,  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$  iff  $d \in C_1^{\mathcal{I}}$  and  $d \in C_2^{\mathcal{I}}$ . By the induction hypothesis, this happens iff  $e \in C_1^{\mathcal{I}}$  and  $e \in C_2^{\mathcal{I}}$ . This means, of course, by the semantics of  $\mathcal{ELH}$ , that  $e \in C_1^{\mathcal{I}}$  and  $e \in C_2^{\mathcal{I}}$  iff  $e \in (C_1 \sqcap C_2)^{\mathcal{I}}$ . Finally, we get  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$  iff  $e \in (C_1 \sqcap C_2)^{\mathcal{I}}$ .

We prove the case  $(\exists r.C_1)$  directly. Assume  $\mu(d) = \mu(e)$ , and  $d \in (\exists r.C_1)^{\mathcal{I}}$ . Then, by the semantics of  $\mathcal{ELH}$ ,  $\exists d'$  such that  $d' \in C_1^{\mathcal{I}}$ , and  $r(d, d')^{\mathcal{I}}$ . By the definition of  $\mu$ , we know  $\mu(d)[r, d'] = 1$ . But from our initial observation,  $\mu(d)[r, d'] = 1$  iff  $\mu(e)[r, d'] = 1$ . By definition of  $\mu$ ,  $\mu(e)[r, d'] = 1$  iff  $(e, d') \in r^{\mathcal{I}}$ . By the semantics of  $\mathcal{ELH}$ , whenever  $d' \in C_1^{\mathcal{I}}$  and  $(e, d') \in r^{\mathcal{I}}$  we have that  $e \in (\exists r.C_1)^{\mathcal{I}}$ .  $\blacktriangleleft$

► **Lemma 47.** *Let  $\mathcal{I}$  be an interpretation, and  $\mu$  a mapping derived from Definition 19. For all normalized  $\mathcal{ELH}$  concepts  $C$ , if  $v \in \eta_{\mathcal{I}}(C)$ , then there is  $d \in \Delta^{\mathcal{I}}$  such that  $v = \mu(d)$  and  $d \in C^{\mathcal{I}}$ .*

**Proof.** We provide an inductive argument for the claim.

**Base case:** Assume  $C = A \in N_C$  and assume  $v \in \eta_{\mathcal{I}}(C)$ . By the definition of  $\eta_{\mathcal{I}}$ , it is the case that  $v \in \eta_{\mathcal{I}}(C)$  iff  $v[C] = 1$ . This is the case iff  $v = \mu(d)$ , for some  $d \in \Delta^{\mathcal{I}}$ .

**Inductive step:** Assume our hypothesis holds for  $C_1$  and  $C_2$ . We prove two cases.

- **Case 1 ( $C_1 \sqcap C_2$ ):** Assume  $v \in \eta_{\mathcal{I}}(C_1 \sqcap C_2)$ . Then, by definition of  $\eta_{\mathcal{I}}$ , it is true that  $v \in \eta_{\mathcal{I}}(C_1)$  and  $v \in \eta_{\mathcal{I}}(C_2)$ . By the inductive hypothesis, if this is the case, then  $v = \mu(d) \in C_1$  and  $v = \mu(d) \in C_2$ , for  $d \in \Delta^{\mathcal{I}}$ . This gives us  $v = \mu(d) \in \eta_{\mathcal{I}}(C_1) \cap \eta_{\mathcal{I}}(C_2)$ , which means  $v = \mu(d) \in \eta_{\mathcal{I}}(C_1 \sqcap C_2)$ , for  $d \in \Delta^{\mathcal{I}}$ .
- **Case 2 ( $\exists r.C_1$ ):** Assume  $v \in \eta_{\mathcal{I}}(\exists r.C_1)$ . Then, by the definition of  $\eta_{\mathcal{I}}$ ,  $\exists u \in \eta_{\mathcal{I}}(C_1)$  and  $v \oplus u \in \eta_{\mathcal{I}}(r)$ . By the inductive hypothesis, if  $u \in \eta_{\mathcal{I}}(C_1)$ , we get  $u = \mu(e) \in \eta_{\mathcal{I}}(C_1)$ , for  $e \in \Delta^{\mathcal{I}}$ . Now,  $v \oplus u \in \eta_{\mathcal{I}}(r)$  iff  $v \oplus u \in \{\mu(d) \oplus \mu(e) \mid \mu(d)[r, e] = 1\}$ , for  $d, e \in \Delta^{\mathcal{I}}$ . This gives us  $v = \mu(d)$  such that  $\mu(d)[r, e] = 1$ . By construction of  $\eta_{\mathcal{I}}$ , if we have  $u = \mu(e) \in \eta_{\mathcal{I}}(C_1)$ , and  $v = \mu(d)$  such that  $\mu(d)[r, e] = 1$  with  $v \oplus u \in \eta_{\mathcal{I}}(r)$ , this means  $v = \mu(d) \in \eta_{\mathcal{I}}(\exists r.C_1)$ , for some  $d \in \Delta^{\mathcal{I}}$ . ◀

► **Lemma 48.** *Let  $\mathcal{I}$  be an interpretation and let  $\mu$  be as in Definition 19. For all  $r \in N_R$ , if  $u \oplus w \in \eta_{\mathcal{I}}(r)$ , then there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $u = \mu(d)$ ,  $w = \mu(e)$ , and  $(d, e) \in r^{\mathcal{I}}$ .*

**Proof.** Assume  $v = u \oplus w \in \eta_{\mathcal{I}}(r)$ . Then, by the definition of  $\eta_{\mathcal{I}}(r)$ , it is the case that  $v \in \{\mu(d) \oplus \mu(e) \mid \mu(d)[r, e] = 1, \text{ for } d, e \in \Delta^{\mathcal{I}}\}$ . This means there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $v = \mu(d) \oplus \mu(e)$  and  $\mu(d)[r, e] = 1$ . By construction of  $\mu$ , it is true that  $\mu(d)[r, e] = 1$  iff  $(d, e) \in r^{\mathcal{I}}$ . This means there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $u = \mu(d)$ ,  $w = \mu(e)$  and  $(d, e) \in r^{\mathcal{I}}$ . ◀

► **Lemma 49.** *For all  $d \in \Delta^{\mathcal{I}}$ , for all  $\mathcal{ELH}$  concepts  $C$ ,  $d \in C^{\mathcal{I}}$  iff  $\mu(d) \in \eta_{\mathcal{I}}(C)$ .*

**Proof.** We provide an inductive argument for the claim.

For all  $d \in \Delta^{\mathcal{I}}$ , for all  $\mathcal{ELH}$  concepts  $C$ ,  $d \in C^{\mathcal{I}}$  iff  $\mu(d) \in \eta_{\mathcal{I}}(C)$ .

**Base case:** Assume  $C = A \in N_C$  and  $d \in C^{\mathcal{I}}$ . By the definition of  $\mu$ ,  $d \in C^{\mathcal{I}}$  iff  $\mu(d)[C] = 1$ . By the definition of geometric interpretation,  $\mu(d)[C] = 1$  iff  $\mu(d) \in \eta_{\mathcal{I}}(C)$ .

**Inductive step:** assume our hypothesis holds for  $C_1$  and  $C_2$ . We consider two cases:

- **Case 1 ( $C_1 \sqcap C_2$ ):** Assume  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$ . This is the case iff  $d \in C_1^{\mathcal{I}}$  and  $d \in C_2^{\mathcal{I}}$ . By the inductive hypothesis, we have that  $\mu(d) \in \eta_{\mathcal{I}}(C_1)$  and  $d \in \eta_{\mathcal{I}}(C_2)$ . But  $\mu(d) \in \eta_{\mathcal{I}}(C_1)$  and  $d \in \eta_{\mathcal{I}}(C_2)$  iff  $\mu(d) \in \eta_{\mathcal{I}}(C_1 \sqcap C_2)$ . Finally, by the semantics of geometric interpretation,  $\mu(d) \in \eta_{\mathcal{I}}(C_1 \sqcap C_2)$  iff  $d \in (C_1 \sqcap C_2)^{\mathcal{I}}$ .
- **Case 2 ( $\exists r.C_1$ ):** Assume  $d \in (\exists r.C_1)^{\mathcal{I}}$ . Then, by the semantics of  $\mathcal{ELH}$ ,  $\exists e \in C_1^{\mathcal{I}}$  such that  $(d, e) \in r^{\mathcal{I}}$ . By the inductive hypothesis, we get  $\mu(e) \in \eta_{\mathcal{I}}(C_1)$ . By the definition of  $\eta_{\mathcal{I}}$ ,  $(d, e) \in r^{\mathcal{I}}$  iff  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(r)$ . But, by the semantics of our geometric interpretation,  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(r)$  and  $\mu(e) \in \eta_{\mathcal{I}}(C_1)$  iff  $\mu(d) \in \eta_{\mathcal{I}}(\exists r.C_1)$ . ◀

► **Lemma 50.** *For all interpretations  $\mathcal{I}$ , all  $\mathcal{ELH}$  concepts  $C$ , all  $a \in N_I$ ,  $\mathcal{I} \models C(a)$  iff  $\eta_{\mathcal{I}} \models C(a)$ .*

**Proof.**  $\mathcal{I} \models C(a)$  iff  $a^{\mathcal{I}} \in C^{\mathcal{I}}$ . By Lemma 49,  $a^{\mathcal{I}} \in C^{\mathcal{I}}$  iff  $\mu(a^{\mathcal{I}}) \in \eta_{\mathcal{I}}(C)$ . By the semantics of geometric interpretation,  $\mu(a^{\mathcal{I}}) \in \eta_{\mathcal{I}}(C)$  iff  $\eta_{\mathcal{I}} \models C(a)$ . ◀

► **Lemma 51.** *For all  $r \in N_R$ , all  $a, b \in N_I$ ,  $\mathcal{I} \models r(a, b)$  iff  $\eta_{\mathcal{I}} \models r(a, b)$ .*

**Proof.** Assume  $\mathcal{I} \models r(a, b)$ . By the semantics of  $\mathcal{ELH}$ , this means there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $d = a^{\mathcal{I}}$ ,  $e = b^{\mathcal{I}}$ , and  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ . By the definition of  $\mu$ , this means  $\mu(d)[a] = 1$ , that  $\mu(e)[b] = 1$ , and that  $\mu(d)[r, e] = 1$ . By the definition of geometric interpretation, this means  $\mu(d) \in \eta_{\mathcal{I}}(a)$ , that  $\mu(e) \in \eta_{\mathcal{I}}(b)$ , and that  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(r)$ , which is the case iff  $\eta_{\mathcal{I}} \models r(a, b)$ .

Now assume  $\eta_{\mathcal{I}} \models r(a, b)$ . This means that  $\eta_{\mathcal{I}}(a) \oplus \eta_{\mathcal{I}}(b) \in \eta_{\mathcal{I}}(r)$ . By Lemma 48, we have that  $\exists d, e \in \Delta^{\mathcal{I}}$  such that  $\eta_{\mathcal{I}}(a) = \mu(d)$ ,  $\eta_{\mathcal{I}}(b) = \mu(e)$ , and  $(d, e) \in r^{\mathcal{I}}$ . But, by the definition of geometric interpretation and construction of  $\mu$ , we have  $\eta_{\mathcal{I}}(a) = \mu(d)$  iff  $d = a^{\mathcal{I}}$ , and  $\eta_{\mathcal{I}}(b) = \mu(e)$  iff  $e = b^{\mathcal{I}}$ , and  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ . By the semantics of  $\mathcal{ELH}$ , this means  $\mathcal{I} \models r(a, b)$ . ◀

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► **Lemma 52.** *If  $\mathcal{I}_{\mathcal{O}}$  is the canonical model of  $\mathcal{O}$ , then the geometrical interpretation  $\eta_{\mathcal{I}_{\mathcal{O}}}$  of  $\mathcal{I}_{\mathcal{O}}$  is strongly IQ faithful with respect to  $\mathcal{O}$ . That is,  $\mathcal{O} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$ , where  $\alpha$  is an  $\mathcal{ELH}$  IQ.*

**Proof.**  $\mathcal{I}_{\mathcal{O}}$  is canonical, therefore  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ . By Lemma 50 we have that  $\mathcal{I} \models C(a)$  iff  $\eta_{\mathcal{I}} \models C(a)$ , and by Lemma 51 we have that  $\mathcal{I} \models r(a, b)$  iff  $\eta_{\mathcal{I}} \models r(a, b)$ . This just means  $\mathcal{I} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$ , giving us  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ . ◀

► **Lemma 53.** *For all  $C, D$  it is the case that  $\mathcal{I} \models C \sqsubseteq D$  iff  $\eta_{\mathcal{I}} \models C \sqsubseteq D$ .*

**Proof.** Let  $C, D$  be  $\mathcal{ELH}$  concepts. Assume  $\mathcal{I} \models C \sqsubseteq D$ . By the semantics of  $\mathcal{ELH}$ , this means  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ . Let  $v \in \eta_{\mathcal{I}}(C)$ . By Lemma 47 we have that  $v = \mu(d) \in \eta_{\mathcal{I}}(C)$ . We know, by Lemma 49, that  $\mu(d) \in \eta_{\mathcal{I}}(C)$  iff  $d \in C^{\mathcal{I}}$ . Since we have  $d \in C^{\mathcal{I}}$ , we also have, by assumption,  $d \in D^{\mathcal{I}}$ . Again by Lemma 49, this gives us  $\mu(d) \in \eta_{\mathcal{I}}(D)$ . Since  $d$  was chosen arbitrarily, this is the case iff  $\eta_{\mathcal{I}} \models C \sqsubseteq D$ .

Now assume  $\eta_{\mathcal{I}} \models C \sqsubseteq D$ . By the semantics of  $\mathcal{ELH}$ ,  $\eta_{\mathcal{I}}(C) \subseteq \eta_{\mathcal{I}}(D)$ . Now assume  $d \in C^{\mathcal{I}}$ . We know, by Lemma 49, that this is the case iff  $\mu(d) \in \eta_{\mathcal{I}}(C)$ . By assumption, we get  $\mu(d) \in \eta_{\mathcal{I}}(D)$ . Since  $v$  was arbitrary, and we showed that  $d \in C^{\mathcal{I}}$  implies  $d \in D^{\mathcal{I}}$ , this means  $\mathcal{I} \models C \sqsubseteq D$ . ◀

► **Lemma 54.** *For all  $r, s \in N_R$ , it is the case that  $\mathcal{I} \models r \sqsubseteq s$  iff  $\eta_{\mathcal{I}} \models r \sqsubseteq s$ .*

**Proof.** Assume  $\mathcal{I} \models r \sqsubseteq s$ . By the semantics of  $\mathcal{ELH}$ ,  $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ . Now let  $v = u \oplus w \in \eta_{\mathcal{I}}(r)$ . This means  $v \in \{\mu(d) \oplus \mu(e) \mid (d, e) \in r^{\mathcal{I}}\}$ , and, by Lemma 48 there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $u = \mu(d)$ ,  $w = \mu(e)$  and  $(d, e) \in r^{\mathcal{I}}$ . By assumption,  $(d, e) \in s^{\mathcal{I}}$ . By construction of  $\mu$ , this means  $\mu(d)[s, e] = 1$ . Since we know  $v = \mu(d) \oplus \mu(e)$  and  $\mu(d)[s, e] = 1$ , by the definition of  $\eta_{\mathcal{I}}$  we have that  $v \in \eta_{\mathcal{I}}(s)$ , and, therefore  $\eta_{\mathcal{I}} \models r \sqsubseteq s$ .

Now assume  $\eta_{\mathcal{I}} \models r \sqsubseteq s$ . By the semantics of  $\mathcal{ELH}$ , this means  $\eta_{\mathcal{I}}(r) \subseteq \eta_{\mathcal{I}}(s)$ . Let  $(d, e) \in r^{\mathcal{I}}$ . By the construction of  $\mu$ , this means  $\mu(d)[r, e] = 1$ . By the definition of  $\eta_{\mathcal{I}}$ , there is  $v = \mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(r)$ . By assumption,  $v \in \eta_{\mathcal{I}}(s)$ . But, by Lemma 48, there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $u = \mu(d)$ ,  $w = \mu(e)$ , and  $(d, e) \in s^{\mathcal{I}}$ . Since we have proven  $(d, e) \in r^{\mathcal{I}}$  implies  $(d, e) \in s^{\mathcal{I}}$ , this means  $\mathcal{I} \models r \sqsubseteq s$ . ◀

► **Theorem 25.** *For all  $\mathcal{ELH}$  axioms  $\alpha$ ,  $\mathcal{I} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$ .*

**Proof.** When  $\alpha$  is a concept inclusion, the result comes from Lemma 53. When  $\alpha$  is a role inclusion, the result comes from Lemma 54. When  $\alpha$  is an IQ, the result comes from Lemma 50 and from Lemma 51. ◀

► **Theorem 27.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology. The following holds*

- *for all  $\mathcal{ELH}$  IQs and CIs  $\alpha$  in normal form over  $\text{sig}(\mathcal{O})$ ,  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ ; and*
- *for all RIs  $\alpha$  over  $\text{sig}(\mathcal{O})$ ,  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ .*

**Proof.** We divide the proof into claims, first for assertions and then for concept and role inclusions. In the following, let  $\mathcal{O} = \mathcal{T} \cup \mathcal{A}$  be an  $\mathcal{ELH}$  ontology in normal form, with  $\mathcal{T}$  being the set of  $\mathcal{ELH}$  concept and role inclusions in  $\mathcal{O}$  and  $\mathcal{A}$  being the set of  $\mathcal{ELH}$  assertions in  $\mathcal{O}$ . As mentioned before,  $N_C(\mathcal{O})$ ,  $N_R(\mathcal{O})$ , and  $N_I(\mathcal{A})$  denote the set of concept, role, and individual names occurring in  $\mathcal{O}$ , respectively. In the following, let  $A, A_1, A_2, B, B'$  be arbitrary concept names in  $N_C(\mathcal{O})$ , let  $a, b$  be arbitrary individual names in  $N_I(\mathcal{A})$ , and let  $r, s, s'$  be arbitrary role names in  $N_R(\mathcal{O})$ .

▷ **Claim 55.**  $\mathcal{I}_{\mathcal{O}} \models A(a)$  iff  $\mathcal{O} \models A(a)$ .

**Proof.** **Assume  $\mathcal{O} \models A(a)$ .** Now, by the definition of  $\mathcal{I}_{\mathcal{O}}$  (Definition 26), it is the case that  $A^{\mathcal{I}_{\mathcal{O}}} \supseteq \{a \in N_I(\mathcal{A}) \mid \mathcal{O} \models A(a)\}$ . By assumption, we have that  $a \in A^{\mathcal{I}_{\mathcal{O}}}$ . But since  $a \in N_I(\mathcal{A})$ , by the definition of  $\mathcal{I}_{\mathcal{O}}$ , we have  $a^{\mathcal{I}_{\mathcal{O}}} = a$  and, therefore,  $a^{\mathcal{I}_{\mathcal{O}}} \in A^{\mathcal{I}_{\mathcal{O}}}$ , which means  $\mathcal{I}_{\mathcal{O}} \models A(a)$ .



**Now assume  $\mathcal{I}_O \models A(a)$ .** This means  $a^{\mathcal{I}_O} \in A^{\mathcal{I}_O}$ . We know, by the definition of  $\mathcal{I}_O$ , that  $a^{\mathcal{I}_O} = a$ . Also by the definition of  $\mathcal{I}_O$ , we know  $A^{\mathcal{I}_O} = \{a \in N_I(\mathcal{A}) \mid \mathcal{O} \models A(a)\} \cup \{c_D \in \Delta_{u+}^{\mathcal{I}_O} \mid \mathcal{O} \models D \sqsubseteq A\}$ . Since  $a \in N_I(\mathcal{A})$ , we have that  $a \notin \Delta_{u+}^{\mathcal{I}_O}$ , and thus,  $\mathcal{O} \models A(a)$ .  $\triangleleft$

▷ Claim 56.  $\mathcal{I}_O \models r(a, b)$  iff  $\mathcal{O} \models r(a, b)$ .

**Proof. Assume  $\mathcal{O} \models r(a, b)$ .** By the definition of canonical model (Definition 26),  $r^{\mathcal{I}_O} \supseteq \{(a, b) \in N_I(\mathcal{A}) \times N_I(\mathcal{A}) \mid \mathcal{O} \models r(a, b)\}$ . Since we assumed that  $\mathcal{O} \models r(a, b)$ , we have that  $(a, b) \in r^{\mathcal{I}_O}$ . Now, again by the definition of  $\mathcal{I}_O$ , we have that  $a^{\mathcal{I}_O} = a$ , and  $b^{\mathcal{I}_O} = b$ . This means  $(a^{\mathcal{I}_O}, b^{\mathcal{I}_O}) \in r^{\mathcal{I}_O}$ , which is the case iff  $\mathcal{I}_O \models r(a, b)$ .

**Now assume  $\mathcal{I}_O \models r(a, b)$ .** Then, we know  $(a^{\mathcal{I}_O}, b^{\mathcal{I}_O}) \in r^{\mathcal{I}_O}$ . By definition of  $r^{\mathcal{I}_O}$ , we have that  $(a, b) \in r^{\mathcal{I}_O}$ . Since  $a, b \in N_I$ , by definition of  $\mathcal{I}_O$ , we have  $\mathcal{O} \models r(a, b)$ .  $\triangleleft$

▷ Claim 57.  $\mathcal{I}_O \models \exists r.A(a)$  iff  $\mathcal{O} \models \exists r.A(a)$ .

**Proof. Assume  $\mathcal{O} \models \exists r.A(a)$ .** By the definition of  $\mathcal{I}_O$  (Definition 26), we have  $r^{\mathcal{I}_O} \supseteq \{(a, c_A) \in N_I(\mathcal{A}) \times \Delta^{\mathcal{I}_O} \mid \mathcal{O} \models \exists r.A(a)\}$ . This means  $(a, c_A) \in r^{\mathcal{I}_O}$ . Also, by the definition of the canonical model,  $a^{\mathcal{I}_O} = a$  and  $c_A \in A^{\mathcal{I}_O}$ , and therefore  $a^{\mathcal{I}_O} \in (\exists r.A)^{\mathcal{I}_O}$ . This gives us  $\mathcal{I}_O \models \exists r.A(a)$ .

**Now assume  $\mathcal{I}_O \models \exists r.A(a)$ .** Then,  $a^{\mathcal{I}_O} \in (\exists r.A)^{\mathcal{I}_O}$ . By the definition of the canonical model, either (1) there is  $b \in N_I(\mathcal{A})$  such that  $(a, b) \in r^{\mathcal{I}_O}$  and  $b \in A^{\mathcal{I}_O}$  or (2) there is  $c_{A'} \in \Delta_{u+}^{\mathcal{I}_O}$  such that  $(a, c_{A'}) \in r^{\mathcal{I}_O}$  and  $c_{A'} \in A^{\mathcal{I}_O}$ . In case (1), by the definition of  $\mathcal{I}_O$ , we have that  $(a, b) \in r^{\mathcal{I}_O}$  means that  $\mathcal{O} \models r(a, b)$ . We also have that it is the case that  $b \in A^{\mathcal{I}_O}$ . By the definition of the canonical model, this means that  $b \in \{b \in N_I(\mathcal{A}) \mid \mathcal{O} \models A(b)\}$ , so  $\mathcal{O} \models A(b)$ . By the semantics of  $\mathcal{ELH}$ ,  $\mathcal{O} \models r(a, b)$  and  $\mathcal{O} \models A(b)$  implies  $\mathcal{O} \models \exists r.A(a)$ . In case (2), by the definition of  $\mathcal{I}_O$ ,  $(a, c_{A'}) \in r^{\mathcal{I}_O}$  means that  $\mathcal{O} \models \exists r.A'(a)$ . Again by the definition of  $\mathcal{I}_O$ ,  $c_{A'} \in A^{\mathcal{I}_O}$  implies  $\mathcal{T} \models A' \sqsubseteq A$ . This gives us  $\mathcal{O} \models \exists r.A(a)$ .  $\triangleleft$

▷ Claim 58.  $\mathcal{I}_O \models A_1 \sqcap A_2 \sqsubseteq B$  iff  $\mathcal{O} \models A_1 \sqcap A_2 \sqsubseteq B$ .

**Proof. Assume  $\mathcal{O} \models A_1 \sqcap A_2 \sqsubseteq B$ .** We make a case distinction based on the elements in  $\Delta^{\mathcal{I}_O} := N_I(\mathcal{A}) \cup \Delta_{u+}^{\mathcal{I}_O}$ .

- $a \in N_I(\mathcal{A})$ : Assume  $a \in (A_1 \sqcap A_2)^{\mathcal{I}_O}$ . This is the case iff  $a \in A_1^{\mathcal{I}_O}$  and  $a \in A_2^{\mathcal{I}_O}$ . By the definition of  $\mathcal{I}_O$ , this means  $\mathcal{O} \models A_1(a)$  and  $\mathcal{O} \models A_2(a)$ . By assumption, this gives us  $\mathcal{O} \models B(a)$ , which, by the definition of  $\mathcal{I}_O$ , means that  $a \in B^{\mathcal{I}_O}$ . Therefore,  $\mathcal{I}_O \models B(a)$ . Since  $a$  was an arbitrary element in  $N_I(\mathcal{A})$ , this holds for all elements of this kind.
- $c_D \in \Delta_{u+}^{\mathcal{I}_O}$ : Assume  $c_D \in (A_1 \sqcap A_2)^{\mathcal{I}_O}$ . This means  $c_D \in A_1^{\mathcal{I}_O}$  and  $c_D \in A_2^{\mathcal{I}_O}$ . By the definition of  $\mathcal{I}_O$ , this gives us that  $\mathcal{T} \models D \sqsubseteq A_1$  and  $\mathcal{T} \models D \sqsubseteq A_2$ . By assumption, this means  $\mathcal{T} \models D \sqsubseteq B$ . But, by the definition of  $\mathcal{I}_O$ , this means  $c_D \in B^{\mathcal{I}_O}$ . Since  $c_D$  was an arbitrary element in  $\Delta_{u+}^{\mathcal{I}_O}$ , this argument can be applied for all elements of this kind.

We have thus shown that, for all elements  $d$  in  $\Delta^{\mathcal{I}_O}$ , if  $d \in (A_1 \sqcap A_2)^{\mathcal{I}_O}$  then  $d \in B^{\mathcal{I}_O}$ . So  $\mathcal{I}_O \models A_1 \sqcap A_2 \sqsubseteq B$ .

**Now, assume  $\mathcal{O} \not\models A_1 \sqcap A_2 \sqsubseteq B$ .** We show that  $\mathcal{I}_O \not\models A_1 \sqcap A_2 \sqsubseteq B$  by showing that  $c_{A_1 \sqcap A_2} \in (A_1 \sqcap A_2)^{\mathcal{I}_O}$  but  $c_{A_1 \sqcap A_2} \notin B^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ ,  $c_{A_1 \sqcap A_2} \in A_i^{\mathcal{I}_O}$  since  $\mathcal{T} \models A_1 \sqcap A_2 \sqsubseteq A_i$  (trivially), where  $i \in \{1, 2\}$ . Then, by the semantics of  $\mathcal{ELH}$ ,  $c_{A_1 \sqcap A_2} \in (A_1 \sqcap A_2)^{\mathcal{I}_O}$ . We now argue that  $c_{A_1 \sqcap A_2} \notin B^{\mathcal{I}_O}$ . This follows again by the definition of  $\mathcal{I}_O$  and the assumption that  $\mathcal{O} \not\models A_1 \sqcap A_2 \sqsubseteq B$ , since the definition means that  $c_D \notin B^{\mathcal{I}_O}$  iff  $\mathcal{O} \models D \sqsubseteq B$  and we can take  $D = A_1 \sqcap A_2$ .  $\triangleleft$

▷ Claim 59.  $\mathcal{I}_O \models \exists r.B \sqsubseteq A$  iff  $\mathcal{O} \models \exists r.B \sqsubseteq A$ .

Proof. **Assume**  $\mathcal{O} \models \exists r.B \sqsubseteq A$ . We make a case distinction based on the elements in  $\Delta^{\mathcal{I}_O} := N_I(\mathcal{A}) \cup \Delta_{u+}^{\mathcal{I}_O}$ .

- $a \in N_I(\mathcal{A})$ : Assume  $a \in (\exists r.B)^{\mathcal{I}_O}$ . In this case, by definition of  $\mathcal{I}_O$ , either (1) there is  $b \in N_I(\mathcal{A})$  such that  $(a, b) \in r^{\mathcal{I}_O}$  and  $b \in B^{\mathcal{I}_O}$  or (2) there is  $c_{B'} \in \Delta_u^{\mathcal{I}_O}$  such that  $(a, c_{B'}) \in r^{\mathcal{I}_O}$  and  $c_{B'} \in B^{\mathcal{I}_O}$ . In case (1), by definition of  $\mathcal{I}_O$ ,  $(a, b) \in r^{\mathcal{I}_O}$  implies that  $\mathcal{O} \models r(a, b)$ . Also,  $b \in B^{\mathcal{I}_O}$  implies that  $\mathcal{O} \models B(b)$ . Together with the assumption that  $\mathcal{O} \models \exists r.B \sqsubseteq A$ , this means that  $\mathcal{O} \models A(a)$ . Again by definition of  $\mathcal{I}_O$ , we have that  $a \in A^{\mathcal{I}_O}$ . In case (2), by definition of  $\mathcal{I}_O$ ,  $(a, c_{B'}) \in r^{\mathcal{I}_O}$  implies that  $\mathcal{O} \models \exists r.B'(a)$ . Also, by definition of  $\mathcal{I}_O$ ,  $c_{B'} \in B^{\mathcal{I}_O}$  implies that  $\mathcal{T} \models B' \sqsubseteq B$ . Then,  $\mathcal{O} \models \exists r.B(a)$ . By assumption  $\mathcal{O} \models \exists r.B \sqsubseteq A$ , which means that  $\mathcal{O} \models A(a)$ . Again by definition of  $\mathcal{I}_O$ , we have that  $a \in A^{\mathcal{I}_O}$ . Since  $a$  was an arbitrary element in  $N_I(\mathcal{A})$ , this argument can be applied for all elements of this kind.
- $c_D \in \Delta_{u+}^{\mathcal{I}_O}$ : Assume  $c_D \in (\exists r.B)^{\mathcal{I}_O}$ . In this case, by definition of  $\mathcal{I}_O$ , either (1) there is  $c_{B'} \in \Delta_u^{\mathcal{I}_O}$  such that  $(c_D, c_{B'}) \in r^{\mathcal{I}_O}$  and  $c_{B'} \in B^{\mathcal{I}_O}$  or (2)  $D$  is of the form  $\exists s.B'$ ,  $(c_D, c_{B'}) \in r^{\mathcal{I}_O}$ ,  $c_{B'} \in B^{\mathcal{I}_O}$ , and  $\mathcal{T} \models s \sqsubseteq r$ . In case (1), by definition of  $\mathcal{I}_O$ ,  $\mathcal{T} \models D \sqsubseteq A$  and  $\mathcal{T} \models A \sqsubseteq \exists r.B'$ . Again by definition of  $\mathcal{I}_O$ ,  $c_{B'} \in B^{\mathcal{I}_O}$  implies  $\mathcal{T} \models B' \sqsubseteq B$ . This means that  $\mathcal{T} \models D \sqsubseteq \exists r.B$ . By assumption  $\mathcal{O} \models \exists r.B \sqsubseteq A$ , which means  $\mathcal{T} \models \exists r.B \sqsubseteq A$ . Then,  $\mathcal{T} \models D \sqsubseteq A$ . By definition of  $\mathcal{I}_O$ , we have that  $c_D \in A^{\mathcal{I}_O}$ . In case (2), we have that  $\mathcal{T} \models D \sqsubseteq \exists r.B'$  since  $D$  is of the form  $\exists s.B'$  and  $\mathcal{T} \models s \sqsubseteq r$ . Also, as  $c_{B'} \in B^{\mathcal{I}_O}$ , by definition of  $\mathcal{I}_O$ ,  $\mathcal{T} \models B' \sqsubseteq B$ . Then,  $\mathcal{T} \models D \sqsubseteq \exists r.B$ . By assumption,  $\mathcal{O} \models \exists r.B \sqsubseteq A$ , which then means that  $\mathcal{T} \models D \sqsubseteq A$ . By definition of  $\mathcal{I}_O$ , we have that  $c_D \in A^{\mathcal{I}_O}$ . Since  $c_D$  was an arbitrary element in  $\Delta_{u+}^{\mathcal{I}_O}$ , this argument can be applied for all elements of this kind.

We have thus shown that, for all elements  $d$  in  $\Delta^{\mathcal{I}_O}$ , if  $d \in (\exists r.B)^{\mathcal{I}_O}$  then  $d \in A^{\mathcal{I}_O}$ . So  $\mathcal{I}_O \models \exists r.B \sqsubseteq A$ .

**Now, assume**  $\mathcal{O} \not\models \exists r.B \sqsubseteq A$ . We show that  $\mathcal{I}_O \not\models \exists r.B \sqsubseteq A$  by showing that  $c_{\exists r.B} \in (\exists r.B)^{\mathcal{I}_O}$  but  $c_{\exists r.B} \notin A^{\mathcal{I}_O}$ . By the definition of  $\mathcal{I}_O$ ,  $(c_{\exists r.B}, c_B) \in r^{\mathcal{I}_O}$  if  $\mathcal{T} \models s \sqsubseteq r$ , which is trivially the case for  $s = r$ , and  $c_B \in B^{\mathcal{I}_O}$  by definition of  $\mathcal{I}_O$ . We now argue that  $c_{\exists r.B} \notin A^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , an element of the form  $c_D$  is in  $A^{\mathcal{I}_O}$  iff  $\mathcal{T} \models D \sqsubseteq A$ . By assumption  $\mathcal{O} \not\models \exists r.B \sqsubseteq A$  which means  $\mathcal{T} \not\models \exists r.B \sqsubseteq A$ . So  $c_{\exists r.B}$  is not in  $A^{\mathcal{I}_O}$ .  $\triangleleft$

$\triangleright$  Claim 60.  $\mathcal{I}_O \models A \sqsubseteq \exists r.B$  iff  $\mathcal{O} \models A \sqsubseteq \exists r.B$ .

Proof. **Assume**  $\mathcal{O} \models A \sqsubseteq \exists r.B$ . We make a case distinction based on the elements in  $\Delta^{\mathcal{I}_O} := N_I(\mathcal{A}) \cup \Delta_{u+}^{\mathcal{I}_O}$ .

- $a \in N_I(\mathcal{A})$ : Assume  $a \in A^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , we have  $\mathcal{O} \models A(a)$ . By assumption  $\mathcal{O} \models A \sqsubseteq \exists r.B$ , so  $\mathcal{O} \models \exists r.B(a)$ . Then, by definition of  $\mathcal{I}_O$ ,  $(a, c_B) \in r^{\mathcal{I}_O}$ . Again by definition of  $\mathcal{I}_O$ , we have  $c_B \in B^{\mathcal{I}_O}$ . So  $a \in (\exists r.B)^{\mathcal{I}_O}$ . Since  $a$  was an arbitrary element in  $N_I(\mathcal{A})$ , the argument holds for all similar elements.
- $c_D \in \Delta_{u+}^{\mathcal{I}_O}$ : Assume  $c_D \in A^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , we have that  $\mathcal{T} \models D \sqsubseteq A$ . By assumption,  $\mathcal{O} \models A \sqsubseteq \exists r.B$  which means  $\mathcal{T} \models A \sqsubseteq \exists r.B$ . Then, by definition of  $\mathcal{I}_O$ ,  $(c_D, c_B) \in r^{\mathcal{I}_O}$ . Again by definition of  $\mathcal{I}_O$ , we have that  $c_B \in B^{\mathcal{I}_O}$ . So  $c_D \in (\exists r.B)^{\mathcal{I}_O}$ . Since  $c_D$  was an arbitrary element in  $\Delta_{u+}^{\mathcal{I}_O}$ , this argument holds for all similar elements.

We have thus shown that, for all elements  $d$  in  $\Delta^{\mathcal{I}_O}$ , if  $d \in A^{\mathcal{I}_O}$  then  $d \in (\exists r.B)^{\mathcal{I}_O}$ . This means that  $\mathcal{I}_O \models A \sqsubseteq \exists r.B$ .

**Now, assume**  $\mathcal{O} \not\models A \sqsubseteq \exists r.B$ . We show that  $\mathcal{I}_O \not\models A \sqsubseteq \exists r.B$  by showing that  $c_A \in A^{\mathcal{I}_O}$  but  $c_A \notin (\exists r.B)^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , we have that  $\{c_D \in \Delta_{u+}^{\mathcal{I}_O} \mid \mathcal{T} \models D \sqsubseteq A\} \subseteq A^{\mathcal{I}_O}$ . For  $D = A$  we trivially have that  $\mathcal{T} \models A \sqsubseteq A$ , so  $c_A \in A^{\mathcal{I}_O}$ . We now show that  $c_A \notin (\exists r.B)^{\mathcal{I}_O}$ . Suppose this is not the case and there is some element  $d \in \Delta^{\mathcal{I}_O}$  such that  $(c_A, d) \in r^{\mathcal{I}_O}$  and  $d \in B^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , this can happen iff  $d$  is of the form  $c_{B'}$  in  $\Delta_u^{\mathcal{I}_O}$  and, moreover,  $\mathcal{T} \models A \sqsubseteq A'$  and  $\mathcal{T} \models A' \sqsubseteq \exists r.B'$  for some  $A' \in N_C(\mathcal{O})$ . We now argue  $d = c_{B'} \in B^{\mathcal{I}_O}$  implies  $\mathcal{T} \models B' \sqsubseteq B$ . By definition of  $\mathcal{I}_O$ ,

$c_{B'} \in B^{\mathcal{I}_O}$  iff  $\mathcal{T} \models B' \sqsubseteq B$ . Since  $\mathcal{T} \models A \sqsubseteq A'$  and  $\mathcal{T} \models A' \sqsubseteq \exists r.B'$ , we have  $\mathcal{T} \models A \sqsubseteq \exists r.B$ , which means  $\mathcal{O} \models A \sqsubseteq \exists r.B$ . This contradicts our assumption that there is some element  $d \in \Delta^{\mathcal{I}_O}$  such that  $(c_A, d) \in r^{\mathcal{I}_O}$  and  $d \in B^{\mathcal{I}_O}$ . Thus,  $c_A \notin (\exists r.B)^{\mathcal{I}_O}$ , as required.  $\triangleleft$

▷ **Claim 61.**  $\mathcal{I}_O \models r \sqsubseteq s$  iff  $\mathcal{O} \models r \sqsubseteq s$ .

**Proof.** **Assume  $\mathcal{O} \models r \sqsubseteq s$ .** We make a case distinction based on the elements in  $\Delta^{\mathcal{I}_O}$  and how they can be related in the extension of a role name in the definition of  $\mathcal{I}_O$ .

- $(a, b) \in N_I(\mathcal{A}) \times N_I(\mathcal{A})$ : Assume  $(a, b) \in r^{\mathcal{I}_O}$ . We first argue that in this case  $\mathcal{O} \models r(a, b)$ . By definition of  $\mathcal{I}_O$ ,  $(a, b) \in r^{\mathcal{I}_O}$  iff  $\mathcal{O} \models r(a, b)$ . Since by assumption  $\mathcal{O} \models r \sqsubseteq s$  we have that  $\mathcal{O} \models s(a, b)$ , so  $(a, b) \in s^{\mathcal{I}_O}$ . Since  $(a, b)$  was an arbitrary pair in  $N_I(\mathcal{A}) \times N_I(\mathcal{A})$ , the argument can be applied for all such kinds of pairs.
- $(a, c_B) \in N_I(\mathcal{A}) \times \Delta_u^{\mathcal{I}_O}$ : Assume  $(a, c_B) \in r^{\mathcal{I}_O}$ . We first argue that in this case  $\mathcal{O} \models \exists r.B(a)$ . By definition of  $\mathcal{I}_O$ , we have that  $(a, c_B) \in r^{\mathcal{I}_O}$  iff  $\mathcal{O} \models \exists r.B(a)$ . By assumption  $\mathcal{O} \models r \sqsubseteq s$ . So  $\mathcal{O} \models \exists s.B(a)$ . Then, again by definition of  $\mathcal{I}_O$ , we have that  $(a, c_B) \in s^{\mathcal{I}_O}$ . Since  $(a, c_B)$  was an arbitrary pair in  $N_I(\mathcal{A}) \times \Delta_u^{\mathcal{I}_O}$ , this argument can be applied for all such kinds of pairs.
- $(c_D, c_B) \in \Delta_{u+}^{\mathcal{I}_O} \times \Delta_u^{\mathcal{I}_O}$ : Assume  $(c_D, c_B) \in r^{\mathcal{I}_O}$ . In this case, by definition of  $\mathcal{I}_O$ , either (1)  $\mathcal{T} \models D \sqsubseteq A$  and  $\mathcal{T} \models A \sqsubseteq \exists r.B$ , for some  $A \in N_C(\mathcal{O})$ , or (2)  $D$  is of the form  $\exists s'.B$  and  $\mathcal{T} \models s' \sqsubseteq r$ . In case (1), since by assumption  $\mathcal{O} \models r \sqsubseteq s$ , we have that  $\mathcal{T} \models D \sqsubseteq A$  and  $\mathcal{T} \models A \sqsubseteq \exists s.B$ , for some  $A \in N_C(\mathcal{O})$ . Then, by definition of  $\mathcal{I}_O$ , it follows that  $(c_D, c_B) \in s^{\mathcal{I}_O}$ . In case (2), since  $\mathcal{T} \models s' \sqsubseteq r$  and by assumption  $\mathcal{O} \models r \sqsubseteq s$  (which means  $\mathcal{T} \models r \sqsubseteq s$ ), we have that  $\mathcal{T} \models s' \sqsubseteq s$ . Then, again by definition of  $\mathcal{I}_O$ , as in this case  $D$  is of the form  $\exists s'.B$ , it follows that  $(c_D, c_B) \in s^{\mathcal{I}_O}$ . Since  $(c_D, c_B)$  was an arbitrary pair in  $\Delta_{u+}^{\mathcal{I}_O} \times \Delta_u^{\mathcal{I}_O}$ , this argument can be applied for all such kinds of pairs.

We have thus shown that  $\mathcal{I}_O \models r \sqsubseteq s$ .

**Now, assume  $\mathcal{O} \not\models r \sqsubseteq s$ .** We show that  $\mathcal{I}_O \not\models r \sqsubseteq s$ . By definition of  $\mathcal{I}_O$ , we have that  $\{(c_{\exists s.B}, c_B) \in \Delta_{u+}^{\mathcal{I}_O} \times \Delta_u^{\mathcal{I}_O} \mid \mathcal{T} \models s \sqsubseteq r\} \subseteq r^{\mathcal{I}_O}$ . By taking  $B = \top$  and  $s = r$  (and since trivially  $\mathcal{T} \models r \sqsubseteq r$ ), we have in particular that  $(c_{\exists r.\top}, c_\top) \in r^{\mathcal{I}_O}$ . We now argue that  $(c_{\exists r.\top}, c_\top) \notin s^{\mathcal{I}_O}$ . By definition of  $\mathcal{I}_O$ , a pair of the form  $(c_{\exists s'.B}, c_B)$  is in  $s^{\mathcal{I}_O}$  iff  $\mathcal{T} \models s' \sqsubseteq s$ . By assumption  $\mathcal{O} \not\models r \sqsubseteq s$ , which means  $\mathcal{T} \not\models r \sqsubseteq s$ . So the pair  $(c_{\exists r.\top}, c_\top)$  is not in  $s^{\mathcal{I}_O}$ .  $\triangleleft$

This finishes our proof.  $\blacktriangleleft$

► **Lemma 62.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_O$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional  $\oplus$ -geometric interpretation of  $\mathcal{I}_O$  (Definition 21) is a strongly TBox faithful model of  $\mathcal{O}$ .*

**Proof.** From Theorem 27, if  $\tau$  is an  $\mathcal{ELH}$  CI in normal form or an  $\mathcal{ELH}$  role inclusion over  $\text{sig}(\mathcal{O})$ , then  $\mathcal{I}_O \models \tau$  iff  $\mathcal{O} \models \tau$ . Since, by Lemma 53 it is the case that  $\mathcal{I} \models C \sqsubseteq D$  iff  $\eta_{\mathcal{I}} \models C \sqsubseteq D$  (where  $C$  and  $D$  are arbitrary  $\mathcal{ELH}$  concepts) and by Lemma 54 it is the case that  $\mathcal{I} \models r \sqsubseteq s$  iff  $\eta_{\mathcal{I}} \models r \sqsubseteq s$  (with  $r, s \in \mathbf{N}_R$ ), we have that  $\mathcal{I} \models \tau$  iff  $\eta_{\mathcal{I}_O} \models \tau$ , where  $\tau$  is a TBox axiom in normal form. This gives us  $\eta_{\mathcal{I}_O} \models \tau$  iff  $\mathcal{O} \models \tau$  for any normalized TBox axiom.  $\blacktriangleleft$

► **Theorem 28.** *Let  $\mathcal{O}$  be an  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_O$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional (possibly non-convex)  $\oplus$ -geometric interpretation  $\eta_{\mathcal{I}_O}$  of  $\mathcal{I}_O$  is a strongly and IQ and TBox faithful model of  $\mathcal{O}$ .*

**Proof.** This result follows from Lemmas 52 and 62.  $\blacktriangleleft$

► **Lemma 63.** *For all  $r \in N_R$ , all  $a, b \in N_I$ , it is the case that  $\eta_{\mathcal{I}} \models r(a, b)$  iff  $\eta_{\mathcal{I}}^* \models r(a, b)$ .*

**Proof.** We know that  $\eta_{\mathcal{I}}^* \models r(a, b)$  iff it is true that  $\eta_{\mathcal{I}}^*(a) \oplus \eta_{\mathcal{I}}^*(b) \in \eta_{\mathcal{I}}^*(r)$ . From the definition of  $\eta_{\mathcal{I}}^*$  we know  $\eta_{\mathcal{I}}^*(a) \oplus \eta_{\mathcal{I}}^*(b) = \eta_{\mathcal{I}}(a) \oplus \eta_{\mathcal{I}}(b)$ . Since  $\mu(d)$  is binary for any  $d$ , we have  $\eta_{\mathcal{I}}(a) \oplus \eta_{\mathcal{I}}(b)$  is binary. From Corollary 7, we have  $\eta_{\mathcal{I}}(a) \oplus \eta_{\mathcal{I}}(b) \in \eta_{\mathcal{I}}(r)$ , which, by the definition of satisfaction is the case iff  $\eta_{\mathcal{I}} \models r(a, b)$ .  $\blacktriangleleft$

► **Lemma 64.** *For any vector  $v$ , such that  $v$  is a result of the mapping in Definition 19, if  $v \in \eta_{\mathcal{I}}^*(A)$ , then  $v[A] = 1$ .*

**Proof.** By the definition of  $\eta_{\mathcal{I}}^*$  and that of convex hull, for all  $v$ , it holds that  $v \in \eta_{\mathcal{I}}^*(A)$  means  $\exists \lambda_i 0 \leq \lambda_i \leq 1$  such that  $v = \sum_{i=1}^n v_i \lambda_i$ , with  $v_i \in \eta_{\mathcal{I}}(A)$ . By the definition of  $\eta_{\mathcal{I}}$ , it is true that  $v_i \in \eta_{\mathcal{I}}(A)$  is the case iff  $v_i[A] = 1$ , for all  $1 \leq i \leq n$ . By the definition of convex hull, this means  $v[A] = 1$ .  $\blacktriangleleft$

► **Lemma 65.** *For all  $\mathcal{ELH}$  IQs in normal form  $\alpha$ , it is the case that  $\eta_{\mathcal{I}}^* \models \alpha$  iff  $\eta_{\mathcal{I}} \models \alpha$ .*

**Proof.** If  $\alpha$  is a role assertion the lemma follows from Lemma 63. Now, we will consider the remaining cases. Let  $A, B \in N_C$  be concept names, and  $a \in N_I$  be an individual name. We make a case distinction and divide the proof into claims for readability.

▷ **Claim 66.** Case 1:  $\eta_{\mathcal{I}}^* \models A(a)$  iff  $\eta_{\mathcal{I}} \models A(a)$ .

**Proof.** **Assume  $\eta_{\mathcal{I}}^* \models A(a)$ .** By the semantics of geometric interpretation,  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(A)$ . By the definition of  $\mu$ , it is the case that  $\eta_{\mathcal{I}}^*(a)$  is binary and, by the definition of  $\eta_{\mathcal{I}}^*$ , it is the case that  $\eta_{\mathcal{I}}^*(a) = \eta_{\mathcal{I}}(a)$ . From Corollary 7 we get that  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$ , which is the case iff  $\eta_{\mathcal{I}} \models A(a)$ .

**Now assume  $\eta_{\mathcal{I}} \models A(a)$ .** This means  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$ . By definition of  $\eta_{\mathcal{I}}^*$ , we know  $\eta_{\mathcal{I}}(a) = \eta_{\mathcal{I}}^*(a)$ , and by Proposition 5 we know  $\eta_{\mathcal{I}}(A) \subseteq \eta_{\mathcal{I}}^*(A)$ . By assumption,  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$ . By the semantics of geometric interpretation, this means  $\eta_{\mathcal{I}}^* \models A(a)$ .  $\triangleleft$

▷ **Claim 67.** Case 2:  $\eta_{\mathcal{I}}^* \models (\exists r.A(a))$  iff  $\eta_{\mathcal{I}} \models (\exists r.A(a))$ .

**Proof.** **Assume  $\eta_{\mathcal{I}}^* \models \exists r.A(a)$ .** By the semantics of  $\eta_{\mathcal{I}}^*$ , we have that  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(\exists r.A)$ . By the definition of  $\eta_{\mathcal{I}}^*$ , we know  $\eta_{\mathcal{I}}^*(a) = \eta_{\mathcal{I}}(a)$ . Also, by construction of  $\mu$ , it is the case that  $\eta_{\mathcal{I}}(a)$  is binary. If there is a binary  $v \in \eta_{\mathcal{I}}^*(A)$  such that  $\eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}^*(r)$  then we are done. In this case, by Corollary 7, we have that  $v \in \eta_{\mathcal{I}}(A)$  and  $\eta_{\mathcal{I}}(a) \oplus v \in \eta_{\mathcal{I}}(r)$ . This means, by the semantics of  $\eta_{\mathcal{I}}$ , that  $\eta_{\mathcal{I}} \models \exists r.A(a)$ .

**Otherwise,** for all  $v \in \eta_{\mathcal{I}}^*(A)$  such that  $\eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}^*(r)$  we have that  $v$  is non-binary (and, moreover, such  $v$  exists). We rename this vector to  $z$ , giving us  $z = \eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}^*(r)$ . This means that  $z = \sum_{i=1}^{n'} v'_i \lambda'_i$ , such that  $\exists \lambda'_i$  with  $0 \leq \lambda'_i \leq 1$  and  $\sum_{i=1}^{n'} \lambda'_i = 1$ , and it also means that  $v'_1, \dots, v'_{n'} \in \eta_{\mathcal{I}}(r)$ . For clarity, we call the vector on the left-hand side of the concatenation operation its *prefix*  $\text{pref}(x)$ , and the one on the right-hand side its *suffix*  $\text{suf}(x)$ . For example, regarding the vector  $z \in \mathbb{R}^{2 \cdot d}$  renamed above, we have  $\text{pref}(z) = \eta_{\mathcal{I}}^*(a) \in \mathbb{R}^d$  and  $\text{suf}(z) = v \in \mathbb{R}^d$ .

We now need to demonstrate that  $z \in \eta_{\mathcal{I}}(\exists r.A(a))$ . We show that (1)  $\text{pref}(z)[a] = 1$ , (2)  $\text{pref}(z)[r, e] = 1$ , and (3)  $\text{suf}(z)[A] = 1$ .

1. We now argue that, for any  $v'_i \in \eta_{\mathcal{I}}(r)$  such that  $\sum_{i=1}^{n'} v'_i \lambda'_i = z$ , it must be the case that  $\text{pref}(v'_i) = \eta_{\mathcal{I}}^*(a)$ . This is because  $\eta_{\mathcal{I}}^*(a)$  cannot be written as a convex combination of vectors  $w' \in (\eta_{\mathcal{I}}(r) \setminus \{\eta_{\mathcal{I}}^*(a) \oplus v \mid v \in \mathbb{R}^d\})$  such that  $\text{pref}(v'_i) = \sum_{i=1}^n w'_i \lambda_k$ . If this was the case, every  $w'$  would have  $\text{pref}(w')[a] = 0$ , which, multiplied by any  $\lambda'_i$ , would of course still result in  $\text{pref}(w')[a] = 0$ , contradicting the fact that  $z = \eta_{\mathcal{I}}^*(a) \oplus v$ . Since we know  $\text{pref}(z) = \eta_{\mathcal{I}}^*(a)$ , we have that  $\text{pref}(z)[a] = 1$ .
2. We now argue that  $\text{pref}(z)[r, e] = 1$ . By Lemma 48, we know that, for  $v'_i \in \eta_{\mathcal{I}}(r)$ , there are  $d, e \in \Delta^{\mathcal{I}}$  such that  $\text{pref}(v'_i) = \mu(d)$ ,  $\text{suf}(v'_i) = \mu(e)$ , and  $(d, e) \in r^{\mathcal{I}}$ , which, by the definition of  $\mu$ , gives us  $\text{pref}(v'_i)[r, e] = 1$ .

3. From the fact we have assumed  $v \in \eta_{\mathcal{I}}^*(A)$  and  $v = \text{suf}(z)$ , we know that  $\text{suf}(z) = \sum_{i=1}^n v_i \lambda_i$  with  $v_i \in \eta_{\mathcal{I}}(A)$ . As  $v \in \eta_{\mathcal{I}}^*(A)$ , we get from Lemma 64 that  $\text{suf}(z)[A] = 1$ .

From these facts, we have that for  $z = \sum_{i=1}^n v_i \lambda'_i$ , it is true that  $\text{pref}(z)[a] = 1$ , that  $\text{pref}(z)[r, e] = 1$ , and that  $\text{suf}(z)[A] = 1$ . By definition of  $\eta_{\mathcal{I}}$ , this means  $\text{pref}(z) = \eta_{\mathcal{I}}(a)$ , that  $z \in \eta_{\mathcal{I}}(r)$ , and that  $\text{suf}(z) = v \in \eta_{\mathcal{I}}(A)$ . Finally, by the semantics of  $\eta_{\mathcal{I}}$ , we have  $\eta_{\mathcal{I}} \models \exists r.A(a)$ .

**Now assume  $\eta_{\mathcal{I}} \models \exists r.A(a)$ .** By the semantics of  $\eta_{\mathcal{I}}$ , this means  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(\exists r.A)$ . We know, by the definition of  $\eta_{\mathcal{I}}^*$ , that  $\eta_{\mathcal{I}}(a) = \eta_{\mathcal{I}}^*(a)$ , and therefore it is binary. Now,  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}(\exists r.A)$  means  $\eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}(r)$  and  $v \in \eta_{\mathcal{I}}(A)$ . Since  $\eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}(r)$ , this means it is a binary vector, and by Proposition 5, it gives us  $\eta_{\mathcal{I}}^*(a) \oplus v \in \eta_{\mathcal{I}}^*(r)$ . Since  $v$  itself is binary and  $v \in \eta_{\mathcal{I}}(A)$ , again by Proposition 5, we have  $v \in \eta_{\mathcal{I}}^*(A)$ . This means, by the semantics of  $\eta_{\mathcal{I}}^*$ , that  $\eta_{\mathcal{I}}^* \models \exists r.A(a)$ .

▷ **Claim 68.** Case 3:  $\eta_{\mathcal{I}}^* \models A \sqcap B(a)$  iff  $\eta_{\mathcal{I}} \models A \sqcap B(a)$

**Assume  $\eta_{\mathcal{I}}^* \models A \sqcap B(a)$ .** By the semantics of geometric interpretation, this means  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(A)$  and  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(B)$ . By the definition of  $\eta_{\mathcal{I}}^*$ , it is the case that  $\eta_{\mathcal{I}}^*(a) = \eta_{\mathcal{I}}(a)$ , and it is therefore binary. But, by Corollary 7 this means  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$  and  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(B)$ . This means  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A) \cap \eta_{\mathcal{I}}(B)$ , which gives us  $\eta_{\mathcal{I}} \models A \sqcap B(a)$ .

**Now assume  $\eta_{\mathcal{I}} \models A \sqcap B(a)$ .** This means  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(A)$  and  $\eta_{\mathcal{I}}(a) \in \eta_{\mathcal{I}}(B)$ . By definition of  $\eta_{\mathcal{I}}^*$  we have  $\eta_{\mathcal{I}}(a) = \eta_{\mathcal{I}}^*(a)$ , and by Proposition 5 we have  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(A)$  and  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(B)$ . This means  $\eta_{\mathcal{I}}^*(a) \in \eta_{\mathcal{I}}^*(A) \cap \eta_{\mathcal{I}}^*(B)$ , giving us  $\eta_{\mathcal{I}}^* \models A \sqcap B(a)$ . ◁

This finishes our proof. ◀

► **Lemma 69.** Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology and  $\mathcal{I}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$ . The geometrical interpretation  $\eta_{\mathcal{I}_{\mathcal{O}}}^*$  of  $\mathcal{I}_{\mathcal{O}}$  is strongly IQ faithful with respect to  $\mathcal{O}$ . That is,  $\mathcal{O} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}}^* \models \alpha$ , where  $\alpha$  is an  $\mathcal{ELH}$  IQ in normal form.

**Proof.** Since  $\mathcal{I}_{\mathcal{O}}$  is canonical,  $\mathcal{I}_{\mathcal{O}} \models \alpha$  iff  $\mathcal{O} \models \alpha$ . By Lemma 52, we know  $\mathcal{O} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$ . By Lemma 65, we have that if  $\alpha$  is an  $\mathcal{ELH}$  IQ in normal form then  $\eta_{\mathcal{I}} \models \alpha$  iff  $\eta_{\mathcal{I}}^* \models \alpha$ . This means  $\eta_{\mathcal{I}_{\mathcal{O}}} \models \alpha$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}}^* \models \alpha$ . Hence,  $\eta_{\mathcal{I}_{\mathcal{O}}}^* \models \alpha$  iff  $\mathcal{O} \models \alpha$ . ◀

► **Lemma 70.** For all  $C, D$ , it is the case that  $\mathcal{I} \models C \sqsubseteq D$  iff  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$ , where  $C \sqsubseteq D$  is a TBox axiom.

**Proof.** Let  $C, D$  be  $\mathcal{ELH}$  concepts. We prove the statement in two directions.

Assume  $\mathcal{I} \models C \sqsubseteq D$ . By Lemma 53, we know  $\mathcal{I} \models C \sqsubseteq D$  iff  $\eta_{\mathcal{I}} \models C \sqsubseteq D$ , which means  $\eta_{\mathcal{I}}(C) \subseteq \eta_{\mathcal{I}}(D)$ . By Proposition 5, this implies  $\eta_{\mathcal{I}}^*(C) \subseteq \eta_{\mathcal{I}}^*(D)$ . Finally, by the definition of satisfaction, this is the case iff  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$ . Now assume  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$ . Then, by the semantics of geometric interpretation,  $\eta_{\mathcal{I}}^*(C) \subseteq \eta_{\mathcal{I}}^*(D)$ . This means if  $v \in \eta_{\mathcal{I}}^*(C)$ , then  $v \in \eta_{\mathcal{I}}^*(D)$ , with  $v = \sum_{i=1}^n \lambda_i v_i$  and  $v_1, \dots, v_n \in \eta_{\mathcal{I}}(C)$ . So, assume  $C^{\mathcal{I}}$  is non-empty. Then, there is  $d \in C^{\mathcal{I}}$ , which, by Lemma 49 is the case iff  $\mu(d) \in \eta_{\mathcal{I}}(C)$ . By the definition of convex hull,  $\mu(d) \in \eta_{\mathcal{I}}^*(C)$ . By assumption,  $\mu(d) \in \eta_{\mathcal{I}}^*(D)$ , and since  $\mu(d)$  is binary, Corollary 7 gives us that  $\mu(d) \in \eta_{\mathcal{I}}(D)$ . But again by Lemma 49, this is the case iff  $d \in D^{\mathcal{I}}$ . Since  $d$  was arbitrary, we have  $\mathcal{I} \models C \sqsubseteq D$ . ◀

► **Lemma 71.** For all  $r, s \in N_R$ , it is the case that  $\mathcal{I} \models r \sqsubseteq s$  iff  $\eta_{\mathcal{I}}^* \models r \sqsubseteq s$ .

**Proof.** First, assume  $\mathcal{I} \models r \sqsubseteq s$ . By Lemma 54, we know  $\mathcal{I} \models r \sqsubseteq s$  iff  $\eta_{\mathcal{I}} \models r \sqsubseteq s$ , which means  $\eta_{\mathcal{I}}(r) \subseteq \eta_{\mathcal{I}}(s)$ . By Proposition 5, this implies  $\eta_{\mathcal{I}}^*(r) \subseteq \eta_{\mathcal{I}}^*(s)$ , which, by the definition of satisfaction is the case iff  $\eta_{\mathcal{I}}^* \models r \sqsubseteq s$ .

Assume  $\eta_{\mathcal{I}}^* \models r \sqsubseteq s$ . Then, by the semantics of geometric interpretation,  $\eta_{\mathcal{I}}^*(r) \subseteq \eta_{\mathcal{I}}^*(s)$ , which means if  $v \in \eta_{\mathcal{I}}^*(r)$ , then  $v \in \eta_{\mathcal{I}}^*(s)$ , where  $v = \sum_{i=1}^n \lambda_i v_i$  for  $v_1, \dots, v_n \in \eta_{\mathcal{I}}(r)$ . Assume  $r^{\mathcal{I}}$  is non-empty. Then, there must be  $(d, e) \in r^{\mathcal{I}}$ . We must now show  $(d, e) \in s^{\mathcal{I}}$  is true. Since  $(d, e) \in r^{\mathcal{I}}$ ,

by the definition of  $\eta_{\mathcal{I}}$ , we have  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(r)$  with both  $\mu(d)$  and  $\mu(e)$  being binary vectors. By the definition of convex hull,  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}^*(r)$ . Now, by assumption,  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}^*(s)$ , but since  $\mu(d) \oplus \mu(e)$  is binary, by Corollary 7 we have that  $\mu(d) \oplus \mu(e) \in \eta_{\mathcal{I}}(s)$ . By definition of  $\eta_{\mathcal{I}}$ , we have that  $\mu(d)[s, e] = 1$ . By definition of  $\mu$ , for all  $d'$  such that  $\mu(d') = \mu(d)$  we have that  $(d', e) \in s^{\mathcal{I}}$ . In particular, this holds for  $d' = d$ . So  $(d, e) \in s^{\mathcal{I}}$ . We have shown that if  $(d, e) \in r^{\mathcal{I}}$ , then  $(d, e) \in s^{\mathcal{I}}$ , which is the case iff  $\mathcal{I} \models r \sqsubseteq s$ .  $\blacktriangleleft$

► **Theorem 31.** *Let  $\eta_{\mathcal{I}}$  be a geometric interpretation as in Definition 21. If  $\alpha$  is an  $\mathcal{ELH}$  CI, an  $\mathcal{ELH}$  RI, or an  $\mathcal{ELH}$  IQ in normal form then  $\eta_{\mathcal{I}} \models \alpha$  iff  $\eta_{\mathcal{I}}^* \models \alpha$ .*

**Proof.** The result for IQs in normal form follows from Lemma 65; the one for concept inclusions follows from Lemmas 53 and 70; and the one for role inclusion follows from Lemma 54 and from Lemma 71.  $\blacktriangleleft$

► **Lemma 72.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional convex  $\oplus$ -geometric interpretation of  $\mathcal{I}_{\mathcal{O}}$  (Definition 29) is a strongly TBox faithful model of  $\mathcal{O}$ . That is,  $\mathcal{O} \models \tau$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}}^* \models \tau$ , where  $\tau$  is either a concept inclusion in normal form or a role inclusion.*

**Proof.** Theorem 27 implies that if  $\tau$  is an  $\mathcal{ELH}$  CI in normal form or an  $\mathcal{ELH}$  RI then  $\mathcal{O} \models \tau$  iff  $\mathcal{I}_{\mathcal{O}} \models \tau$ . From Lemma 70, we know  $\eta_{\mathcal{I}}^* \models C \sqsubseteq D$  iff  $\mathcal{I} \models C \sqsubseteq D$ , and by Lemma 71 we get that  $\eta_{\mathcal{I}}^* \models r \sqsubseteq s$  iff  $\mathcal{I} \models r \sqsubseteq s$ . This means that if  $\tau$  is an  $\mathcal{ELH}$  CI in normal form or an  $\mathcal{ELH}$  RI then  $\mathcal{I}_{\mathcal{O}} \models \tau$  iff  $\eta_{\mathcal{I}_{\mathcal{O}}}^* \models \tau$ .  $\blacktriangleleft$

► **Theorem 32.** *Let  $\mathcal{O}$  be a normalized  $\mathcal{ELH}$  ontology and let  $\mathcal{I}_{\mathcal{O}}$  be the canonical model of  $\mathcal{O}$  (Definition 26). The  $d$ -dimensional convex  $\oplus$ -geometric interpretation of  $\mathcal{I}_{\mathcal{O}}$  (Definition 29) is a strongly IQ and TBox faithful model of  $\mathcal{O}$ .*

**Proof.** The theorem follows from Lemmas 69 and 72.  $\blacktriangleleft$

## A.4 Omitted proofs for Section 6

► **Theorem 35.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  CI in normal form, Algorithm 1 runs in time in  $O(d \cdot n^4)$ , where  $d$  is as in Definition 19 and  $n = |\Delta^{\mathcal{I}}|$ .*

**Proof.** Algorithm 1 has four main parts that are never executed in the same run, each corresponding to one of the normal forms that the input concept inclusion  $\alpha$  can take.

$\alpha = A \sqsubseteq B$ : In this case, the algorithm will execute lines Algorithms 1–1. From assumption 1, Algorithm 1 spends time  $O(1)$  and by assumption 3 this line is run  $O(|\Delta^{\mathcal{I}}|)$  times. Hence, in this case, the algorithm consumes time  $O(|\Delta^{\mathcal{I}}|)$ .

$\alpha = A_1 \sqcap A_2 \sqsubseteq B$ : From assumption 3, the loop from Algorithms 1–1 is executed  $O(|\Delta^{\mathcal{I}}|)$  times. Each iteration consumes time  $O(1)$  by assumption 1. Thus, Algorithm 1 runs in time  $O(|\Delta^{\mathcal{I}}|)$  in this case.

$\alpha = A \sqsubseteq \exists r.B$ : According to assumption 3, the nested loop from Algorithms 1–1 uses time  $O(|\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$ . The membership check in Algorithm 1 takes time  $O(d \cdot |\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$ , by assumption 4. Therefore, we get that Algorithm 1 requires time  $O(d \cdot n^4)$ , where  $n = |\Delta^{\mathcal{I}}|$ .

$\alpha = \exists r.A \sqsubseteq B$ : Algorithm 1 will execute from Algorithms 1–1 for CIs in this normal form. Each iteration of the for loop starting in Algorithm 1 consumes constant time according to assumption 1. Furthermore, the loop has  $O(|\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  iterations due to assumption 3. Hence, Algorithm 1 uses time  $O(|\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  for CIs in this normal form.

Therefore, Algorithm 1 consumes time  $O(d \cdot n^4)$ .  $\blacktriangleleft$

► **Theorem 36.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  IQ in normal form, Algorithm 2 runs in time  $O(\mathbf{d} \cdot \mathbf{n}^3)$ , with  $\mathbf{d}$  as in Definition 19 and  $\mathbf{n} = |\Delta^{\mathcal{I}}|$ .*

**Proof.** We consider each the four forms that an  $\mathcal{ELH}$  IQ in normal form  $\alpha$  can assume separately. In each of them  $a \in \mathbf{N}_I$ ,  $A, B \in \mathbf{N}_C$ , and  $r \in \mathbf{N}_R$ .

$\alpha = A(a)$ : Due to assumptions 1 and 2, Algorithm 2 uses time  $O(1)$ .

$\alpha = (A \sqcap B)(a)$ : As in the previous case, the assumption 1 and 2 imply that Algorithm 2 executes in time  $O(1)$ .

$\alpha = (\exists r.A)(a)$ : By assumption 3, Algorithm 2 is run  $O(|\Delta^{\mathcal{I}}|)$  times, each iteration consuming time in  $O(\mathbf{d} \cdot |\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  (from assumptions 2 and 4). Therefore, Algorithm 2 spends time  $O(\mathbf{d} \cdot \mathbf{n}^3)$  in such instance queries, where  $\mathbf{n} = |\Delta^{\mathcal{I}}|$ .

$\alpha = r(a, b)$ : Algorithm 2 runs in time  $O(\mathbf{d} \cdot |\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  due to assumptions 2 and 4.

Therefore, Algorithm 2 consumes time  $O(\mathbf{d} \cdot \mathbf{n}^3)$ . ◀

► **Theorem 37.** *Given a finite geometric interpretation  $\eta_{\mathcal{I}}$  and an  $\mathcal{ELH}$  role inclusion, Algorithm 3 runs in time in  $O(\mathbf{d} \cdot \mathbf{n}^4)$ , where  $\mathbf{d}$  is as in Definition 19 and  $\mathbf{n} = |\Delta^{\mathcal{I}}|$ .*

**Proof.** There are  $O(|\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  iterations of the for loop starting in Algorithm 3 in a single run of Algorithm 3 as a consequence of the assumption 3. Additionally, each iteration consumes time  $O(\mathbf{d} \cdot |\Delta^{\mathcal{I}}| \cdot |\Delta^{\mathcal{I}}|)$  by assumption 4. Therefore, Algorithm 3 runs in time  $O(\mathbf{d} \cdot \mathbf{n}^4)$ , where  $\mathbf{n} = |\Delta^{\mathcal{I}}|$ . ◀

