Dagstuhl-Seminar

Algorithms and Complexity for Continuous Problems

Organized by:

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Overview

The Dagstuhl-Seminar on Algorithms and Complexity for Continuous Problems was attended by 44 scientists from 12 countries. We express our gratitude to the staff of Schloß Dagstuhl for providing a great atmosphere.
Continuous algorithms and complexity is a very rich and rapidly changing area. It relates to many other areas of computer science and mathematics, of both theoretical and practical orientation. This seminar has emphasized some of the most dynamic topics like:

- Linear multivariate problems
- Operator equations
- Zero finding and computation of fixed points
- Global optimization
- Probabilistic analysis of algorithms
- Monte Carlo and quasi Monte Carlo methods
- Models of computation for continuous problems
- Complexity issues for wavelets and neural networks
- Numerical integration
- Large scientific computations
- Large financial computations.

This Seminar-Report contains the abstracts of 40 lectures in alphabetical order. We also had a plenary session on new research directions and open problems.

Abstracts

A Sharp Upper Bound for the Expected Number of Shadow-Vertices in the Rotation-Symmetry-Model

Karl Heinz Borgwardt

Institut für Mathematik, Universität Augsburg

In this talk we sharpen our polynomiality-proof from 1982 resp. 1987 for $E_{m,n}(S)$, which is the expected number of shadow-vertices in linear programming problems of the type

$$\text{maximize } v^T x \text{ s.t. } a_1^T x \leq 1, \ldots, a_m^T x \leq 1 \quad \text{where} \quad x, v, a_1, \ldots, a_m \in \mathbf{R}^n \text{ and } m \geq n.$$

We assume a distribution of the linear programming problems corresponding to the Rotation-Symmetry-Model:

> The vectors $a_1, \ldots, a_m, v$ and an auxiliary vector u are distributed on $\mathbf{R}^n \backslash \{0\}$ independently, identically and symmetrically under rotations.

In 1982 and 1987 we had found an upper bound of $m^{\frac{1}{n-1}} \cdot n^3 \cdot \mathit{Const.}$, valid for all RSM-distributions and all pairs ($m \geq n$). This bound had recently (1994) been improved to $m^{\frac{1}{n-1}} \cdot n^{\frac{5}{2}} \cdot \mathit{Const.}$ by use of a refined evaluation technique for space-angles. Based on these papers and on a reorganization of the proof, which avoids some crude estimations, we are now able to confirm a bound

$$E_{m,n}(S) \leq m^{\frac{1}{n-1}} \cdot n^2 \cdot \mathit{Const.}$$

This result had been desired strongly, because also in 1982 , a bound $m^{\frac{1}{n-1}} \cdot n^2 \cdot \mathit{Const.}$ had been derived only for the situation, where n is fixed and m tends to infinity. All the time the question had been open, whether the asymptotical behavior would be better than the behavior in moderate dimensions, or whether the moderate bound was crude.
In addition, we know – also from asymptotic analysis of 1987, that for a special RSM-distribution, namely the uniform distribution on the unit sphere, there is a lower bound for $E_{m,n}(S)$ of type $m^{\frac{1}{n-1}} \cdot n^2 \cdot \mathit{Const.}$ The combination of this result and the new upper bound shows that the new bound is sharp.

Strongly Optimal Quadrature Rules

Helmut Braß

Technische Universität Braunschweig

Let V be a linear space, I a linear functional defined on V. Our problem is the numerical computation of $I[f]$, where $f \in V$. Any numerical algorithm starts with

some input, this may be interpreted as a set Info $\subset V$, so that $f \in$ Info is our knowledge of f. We make the fullest use of the given information, if we compute the set I[Info], this is the strongly optimal (s.o.) algorithm.
In typical situations I[Info] is an interval, the midpoint is called the s.o. estimation of $I[f]$. It is folklore, that s.o. estimations have a more complicated structure than the better known optimal estimations. It is the object of my lecture, to make this more precise. Furthermore some new examples of s.o. estimations in quadrature theory are given.

A Parallel Algorithm for Global Optimization

James M. Calvin

New Jersey Institute of Technology

We study the problem of approximating the global minimum, and a location where it is attained, for a continuous function defined on the unit interval. The function is evaluated at sequentially selected knots, and the algorithm makes use of a subset of the observations of size at most M. The algorithm is described in terms of the operation of M processors, each with memory of a single observation, and with unidirectional flow of information between the processors. The limiting normalized error distribution is derived under the assumption of Wiener measure.

Randomized Algorithms for the Algebraic Eigenvalue Problem

Gianna M. Del Corso

Dipartimento di Matematica, Università di Milano

We consider the problem of estimating the largest eigenvalue and a corresponding eigenvector of a symmetric, positive definite matrix. To solve this problem, iterative methods based on Krylov information are generally used. Iterative methods are preferable since we want to estimate only an eigenpair and are therefore cheaper than methods based on the factorization of the matrix. Among the class of methods based on Krylov information, we find the well known Power and Lanczos algorithms. To guarantee the convergence of these methods to the desired quantities, the starting vector is usually chosen randomly.
We study the behavior of these methods when the starting vector is chosen accordingly to the uniform distribution over the unit sphere of $\mathbf{R}^n$. In particular, we define the randomized error in the sense of $\mathcal{L}_p$ for $p \in [1, \infty]$, and we analyze the randomized error for eigenvalue and eigenvector estimate.
It is well known that in the deterministic case, that is when the starting vector is fixed, if the starting vector is not orthogonal to the eigenspace corresponding to λ_1, the speed of convergence depends on the two largest eigenvalues. In particular, the convergence is very slow if the first two eigenvalues are close one to each other.

Surprisingly, when the randomized setting is considered for eigenvalue approximation, it is possible to bound the error with a quantity that does not depend on the eigenvalues of the matrix (Kuczyński and Woźniakowski SIMAX 1992). We show that this is no longer the case when the eigenvector estimate is considered. In particular, we prove that for the power method there are worst case matrices for which the method for eigenvector approximation is not even convergent in the randomized case. This negative result can be generalized to show that for every polynomial method there are matrices for which the method fails in approximating an eigenvector when less than n steps are performed. This fact proves that, in the randomized setting, estimating an eigenvector is harder than estimating an eigenvalue.
For both the Power and Lanczos method we give also other bounds for eigenvalues and eigenvectors estimate. These bounds do depend on how the eigenvalues of the matrix are distributed and are therefore more accurate. From these bounds we see that a very important role is played by the norm parameter p and by the multiplicity of the largest eigenvalue. In fact, we get different speed of convergence depending on the relation between the multiplicity r of λ_1 and the parameter p. The convergence is faster for large r and small p.
Part of these results are a joint work with Giovanni Manzini.

Integration of Lipschitz Functions with Gaussian Weights

Francisco Curbera

IBM Research - Columbia University

We have studied the following integration problem:

$$\mathrm{I}(f) \;=\; \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbf{R}} f(x) e^{-x^2/2\sigma}\, dx, \quad f \in \mathrm{F}_{\mathrm{Lip}}(L).$$

Here $\mathrm{F}_{\mathrm{Lip}}(L)$ is the class of functions satisfying a Lipschitz condition with constant L. We characterize the optimal information by a system of non-linear equations which we solve numerically for values of n up to 10^6. Next we show that, for $\sigma = \frac{1}{2}$, an asymptotic solution $\{x_i\}$ to these equations is provided by the following relations:

$$\int_0^{x_i} \exp(-t^2/2)\, dt \;=\; \sqrt{\frac{\pi}{2}}\, \frac{i}{(n+\frac{1}{2})}\,, \quad -n \le i \le n.$$

For arbitrary σ we scale $x_i^\sigma = \sqrt{2\sigma} x_i$. We provide the optimal algorithm that uses optimal information. We show that for n optimal sample points the worst case error of the optimal algorithm is

$$r_n \;=\; \sqrt{\frac{\pi\sigma}{2}}\, \frac{L}{n} \quad \text{as} \quad n \to \infty.$$

Finally, we consider the information defined by the zeroes of Hermite polynomials. We show that the error of any algorithm using this information is quadratically worse than optimal for the class $\mathrm{F}_{\mathrm{Lip}}(L)$.

Blending Approximation in Harmonic Hilbert Spaces

Franz-Jürgen Delvos

Universität Siegen

We introduce the concept of harmonic Hilbert space in a multivariate setting as an extension of periodic Hilbert space introduced by Babuska and Prager for studying universally optimal approximation of linear functionals. In these spaces we investigate the approximation power of Fourier partial integral operators. To obtain the approximation power of the product Fourier partial integral projector in isotropic Hilbert space also in anisotropic Hilbert spaces we apply the method of blending approximation. Moreover the approximation power of intermediate blending approximation is investigated, which contain blending and product approximation as extreme cases.

Computing the L_2-Discrepancy of Smolyak Quadrature Rules II

Karin Frank

FB Informatik, Universität Kaiserslautern

The contribution continues the talk of Stefan Heinrich about our joint work on L_2-discrepancies of Smolyak quadrature rules. In the recursive algorithm presented by Stefan Heinrich, the main effort is enclosed in the start of the recursion, that is, in the computation of the univariate terms. In both cases of the r-smooth periodic and the r-smooth nonperiodic L_2-discrepancy, fast algorithms are available. For arbitrary sequences of univariate quadratures, a considerable reduction of the cost can be achieved simply by ordering the underlying point sets. For sequences of composite quadrature rules, the cost can be further reduced by using the special structure of the nodes. Thus, the effort decreases from $O(N^2)$ (Warnock's algorithm) to $O(N(\log N)^{2-d}+d(\log N)^4)$ for arbitrary sequences of univariate quadratures, and to $O(d(\log N)^4)$ for sequences of composite univariate quadratures, where N is the number of points in the d-variate Smolyak rule.
This allows us to compare discrepancies of Smolyak quadratures with other multivariate methods. Numerical results were presented, comparing the r-smooth L_2-discrepancy of two Smolyak quadrature rules with that of good lattice points of rank 1 and 2 in the periodic case as well as with that of Hammersley points in the nonperiodic case. Furthermore, the expected L_2-discrepancy of truly random points, which can be computed exactly, was included in the experiments.

Computing Discrepancies of Smolyak Quadratures I

Stefan Heinrich

FB Informatik, Universität Kaiserslautern

Using a general approach to discrepancy via reproducing kernel Hilbert spaces, we

show how to split the discrepancy into parts with good tensor product properties. These properties are then used to develop fast algorithms for computing various discrepancies of Smolyak quadrature formulas.

Computability and Complexity Over Arbitrary Structures

Armin Hemmerling

Universität Greifswald

We sketch an approach to computability and (time) complexity of string functions over single-sorted, total algebraic structures (not necessarily of finite signature). Our model generalizes the Blum-Shub-Smale setting over the real numbers and other rings.

Relationships both to classical computability and to Friedman's concept of eds-computability are established. Two kinds of nondeterminism as well as several variants of recognizability are considered with respect to interdependencies on each other and on properties of the underlying structures. For structures of finite signatures, there are universal programs and m-complete sets with the usual characteristics. In the general case, the existence of m-complete sets turns out to be independent on those properties.

The approach allows a straightforward definition of time complexity, in particular of the classes $P \subseteq N_1P \subseteq N_2P$ of polynomially bounded recognition problems. For finite signatures, the satisfiability problem of boolean quasi-expressions over the structure is N_2P-complete, and the binary satisfiability of algebraic circuits over the structure is N_1P-complete. Results by C. Gaßner show that the linear structure over the reals (which has infinite signature) owns also N_iP-complete problems even if it has no universal function.

Continuous Computation Trees

Peter Hertling

FernUniversität Hagen

In order to see how many tests are needed for solving a problem (*topological complexity* of the problem, Smale (1987), Vassiliev (1989)) we consider *continuous* computation trees. These are computation trees with binary tests which may use arbitrary continuous operations. Following the definition of residues of Hausdorff (1914) we introduce the *level* of discontinuity of a function. It is shown that the minimal size of a continuous computation tree for a function is equal to the level of discontinuity of the function. Furthermore it is proved that for a function with so called *analytic regions of continuity* the dimension of the input space plus one is an upper bound for the level. Finally we consider the problem of determining a zero of a continuous function on the unit interval with a change of sign in the endpoints. It is shown that

the minimal number of tests needed in order to solve the problem is approximately equal to the logarithm of the number of tests needed by the bisection algorithm.

Complexity Results in l_1 and l_2 System Identification

Boleslaw Kacewicz

University of Warsaw

Two applications of complexity approach to system identification problems are presented. The first is the problem of l_1 identification, where the identification error is measured in the l_1 norm. The output measurements of the system are corrupted by an l_∞ bounded noise. We show that to achieve the minimal error in the worst case setting, exponential number of input sequences is necessary, which makes the problem intractable.
The second problem is the l_2 identification. In contrast to a standard approach, where approximations are allowed to be placed in whole space of solutions, we consider "restricted" algorithms which give approximations in a given subspace of low dimension (a model set). This reflects situation that engineers often deal with, when approximation dependent only on few parameters is required for a complex system. We show upper and lower bounds on the minimal (local) identification error, which allows to select the best model among a number of models available.

Randomized Lower Bounds for Linear Arrangements (Knapsack and Integer Programming Revisited)

Marek Karpinski

Universität Bonn

We overview recent new techniques for proving nonlinear randomized lower bounds for the problems of linear arrangements and convex polyhedra, solving the long standing open problems of Integer Programming (super-quadratic randomized bound) and Knapsack (quadratic bound). The underlying computation models are randomized algebraic decision trees, and randomized computation trees. For the case of linear arrangements the method also yields the first elementary lower bound technique for deterministic computation trees without making use of Milnor's bound on Betti number of algebraic varieties.

The Quasi-Random Walk

Alexander Keller

FB Informatik, Universität Kaiserslautern

We present the deterministic method of the quasi-random walk for the solution of second kind Fredholm integral equations. The method efficiently uses low discrepancy sequences to integrate the Neumann series and avoids kernel discretization.

This is illustrated for the radiance equation of computer graphics. In this setting the variation is infinite and so the classical error bounds of quasi-Monte Carlo integration are far too pessimistic. We therefore give numerical evidence, showing that the convergence rate of the algorithm is slightly better than that of a real random walk, but without variance since the quasi-random walk is a deterministic procedure. The advantage is a much smoother convergence to the true solution than obtained by random techniques. Extensions presented are the calculation of a wavelet approximation to the solution of the integral equation using the histogram method and the application to bidirectional path tracing.

Neural Networks, Wavelets, and Complexity

Mark A. Kon

Boston University

We consider feed-forward neural networks on the model of radial basis function networks. We assume that the activation functions of such networks are wavelets, in which case a network consisting of three layers with one output node exactly implements the corresponding wavelet expansion. In that case complexity results, i.e. statements regarding numbers of neurons needed in order to implement a given desired input-output map, reduce to questions regarding convergence rates of wavelet expansions. We formulate such results for wavelet expansions between general Banach spaces of functions, and show that convergence rate questions essentially reduce to questions regarding the so-called homogeneous function spaces associated with the original Banach spaces. Homogeneous Banach function spaces are by definition ones whose norms scale as pure powers of scaling constants inside of functions. It is shown that if the error operator $E = I - P$ (where I is the identity and P is the projection onto the space of wavelets larger than a given scale) is bounded between two Banach spaces F and G, then the convergence rate of the wavelet expansion of a function in F in the norm of G is determined by the scaling parameters of the corresponding homogeneous spaces F_h and G_h. Some other results related to identifying such convergence rates are given; these in turn translate to complexity information regarding the number of neurons required in the corresponding wavelet-based neural network. These provide upper bounds on the complexities of neural networks which implement given input-output mappings, and, we hope, also give lower bounds within a reasonable class of models.

On Complexity of Approximation Problems

Nikolaj P. Korneichuk

Ukrainian Academy of Sciences, Kiev

In 1962 on the International Congress of Mathematics Kolmogorov formulated the problem about ε-definition of function.

We consider the problems about complexity ε-definition and ε-recovery of elements of a metric space (X, ϱ). Our approach on the one hand lies in the channel of ideology of the Traub, Wasilkowski, Woźaniakowski books and on the other hand is connected with the width problem and allows to use results accumulated in Approximation Theory. We assume that an information operator is given which maps X in $\mathbf{R}^N$ continuously and which is defined by the set $M_N = \{\mu_1, \ldots, \mu_N\}$ of continuous functionals μ_k on X. Complexity ε-definition $N_\varepsilon(F, X)$ of element $x \in F \subset X$ is defined as minimal $N \in \mathbf{N}$ such that for some M_N operator $\varphi : M_N(x) \to Y_x \subset X$ exists such that $\varrho(x, Y_x) \leq \varepsilon$ for any $x \in F$.
We consider the concrete function space $C = C_{2\pi}$ and we find the value of $N_\varepsilon(W_\infty^m, C)$ where $W_\infty^m = \{x(t) : x \in C,\ x^{(m-1)}(t)$ loc. abs. cont, $\|x^{(m)}\|_\infty \leq 1\}$ and we specify explicitly some extremal information operators.
Moreover we consider the problem about computer complexity of ε-definition and ε-recovery of elements $x \in F \subset X$. It is established that computer complexity of ε-definition of functions $x(t) \in W_\infty^m$ is realized for some $n = n_\varepsilon$ by the operator $M_{2n}^\tau = \{\mu_1^\tau, \ldots, \mu_{2n}^\tau\}$ where $\mu_k^\tau(x) = x(\tau_k)$, $\tau_k = k\pi/n + \beta$, $\beta \in \mathbf{R}$. Some considerations allow to assume that the computer complexity of ε-recovery of $x(t) \in W_\infty^m$ in C is realized by operator M_{2n}^τ and operator $M_{2n}^\tau(x) \to s_n(x, t)$, where $s_n(x, t)$ is the interpolating spline.

On the Asymptotic Average-Case Complexity of Linear Vector Optimization

Karl-Heinz Küfer

FB Mathematik, Universität Kaiserslautern

We consider linear vector optimization problems of type

$$\max_{x \in X} c_1^T x,\ \max_{x \in X} c_2^T x, \ldots, \max_{x \in X} c_k^T x,$$

where the $c_j \in \mathbf{R}^n$ are linearly independent and

$$X := \{x \in \mathbf{R}^n \,|\, a_i^T x \leq 1,\ i = 1, \ldots, m\}$$

for $a_i \in \mathbf{R}^n$ and $1 \leq k \leq n \leq m$. A feasible solution $x \in X$ is called efficient if x maximizes some convex combination of the given objective functionals. The most interesting benchmark for the computational complexity of algorithms that compute all efficient solutions is the number of efficient vertices V_e of X. For any triple (k, n, m), there are instances where all vertices of X are efficient. Thus, the given problem is intractable in worst-case situations.
We investigate the number of efficient vertices probabilistically and generate random instances of linear vector optimization problems with i.i.d. distributed vectors a_i and

a_j spherically symmetric in $\mathbf{R}^n$. For fixed k and n we obtain the asymptotic upper bound

$$\mathrm{E}(V_e) \le C(k,n)\, m^{(k-1)/(n-1)}\,(1+o(1))$$

for $m \to \infty$ where

$$C(k,n) = \frac{\sqrt{\pi}^{k-1}}{2^{k-1}\, k!\, \Gamma(\frac{k+1}{2})}\, n^{2(k-1)}\,(1+o(1))$$

for fixed k and $n \to \infty$. Moreover, it turns out that the given asymptotic upper bound is an asymptotic equivalent and simultaneously a proper lower bound in case of distributions that are concentrated on a unique sphere. Thus, the given asymptotic estimate is sharp and the problem is intractable in the average-case-setting as well if k goes to infinity when n does. For fixed k, we conjecture that the expected number of efficient vertices is polynomial in n and m with upper bound

$$\mathrm{E}(V_e) \le C(k)\, m^{(k-1)/(n-1)}\, n^{2(k-1)}$$

with an appropriate constant $C(k)$ depending exclusively on k.

Efficient Mixing on Product Spaces

Peter Mathé

WIAS Berlin

We study the mixing behavior of (ergodic) Markov chains on products of finite spaces $X = X_1 \times \ldots \times X_d$, where each component has its own random walk. Product type walks are obtained by defining a visiting scheme $\rho = (\varrho_1, \ldots, \varrho_d)$ and letting $P_\varrho = \sum_{j=1}^d \varrho_j P_j$. With an error criterion (total variation distance) we determine the mixing times (upper bounds) and optimize with respect to ϱ. As an example the behavior on d-dimensional grids $\mathbf{Z}_n^d$ is observed.

On Diagonal Sets in Uncountable Structures

Klaus Meer

Lehrstuhl C für Mathematik, RWTH Aachen

In this talk we deal with complexity issues in uncountable domains like $\mathbf{R}$ and $\mathbf{C}$ as introduced by Blum, Shub, and Smale. We consider the question of whether for complexity classes $\mathcal{C}_1 \subset \mathcal{C}_2$ there exist diagonal problems if $\mathcal{C}_1 \neq \mathcal{C}_2$ is assumed (i.e. problems in $\mathcal{C}_2 \setminus \mathcal{C}_1$ being non-complete for $\mathcal{C}_2$).
Two methods to obtain such results are presented: application of a transfer result recently proved by Blum, Cucker, Shub, and Smale in order to eliminate constants in algorithms as well as quantifier elimination.

The talk is based on joint work with C. Michaux and S. Ben-David on one hand and G. Malajovich on the other.

Approximation of Random Fields

Thomas Müller-Gronbach

Freie Universität Berlin

We consider a random field on the p-dimensional cube with a covariance function of tensor product type. The quality of an approximation which is based on finitely many observations of the field is measured by the integrated mean squared error. We use the optimal affine linear approximation and analyze the asymptotic performance of hyperbolic cross designs which are constructed from regular sequences. Under Sacks/Ylvisaker regularity conditions the corresponding asymptotic constants are explicitly determined. The optimal approximation as well as the optimal hyperbolic cross design depends on the mean function and on the covariance function of the field. We overcome this difficulty by providing an adaptive method which only requires information about the smoothness of the covariance function.

This latter part is based on a joint work with Klaus Ritter from the University of Erlangen-Nürnberg.

New and Better Constructions of Low-Discrepancy Sequences

Harald Niederreiter

Austrian Academy of Sciences, Vienna

We give a survey of recent work of the speaker and C. P. Xing in which multi-dimensional low-discrepancy sequences are constructed by new methods based on algebraic geometry. The most powerful of these methods employ algebraic curves over finite fields with many rational points. These methods yield significant improvements on all earlier constructions. In fact, within the context of the theory of (t, s)-sequences, these constructions yield the best possible order of magnitude of the quality parameter t as a function of the dimension s.

Multiple Integration of Smooth Functions

Erich Novak

Universität Erlangen-Nürnberg

Major topics of the seminar were numerical integration, high dimensional problems, and the curse of dimension. We shortly compare the different methods that were constructed either for rather general nonsmooth functions or for smooth functions, respectively. We then discuss the problem of numerical integration,

$$S_d(f) = \int_{[0,1]^d} f(x)\,dx \approx \sum_{k=1}^{n} a_k f(x_k),$$

for the classical spaces

$$C_d^k = \{f : [0,1]^d \to \mathbf{R} \mid \|f^{(\alpha)}\|_\infty \le 1,\ |\alpha| \le k\}$$

and also for the classes

$$F_d^k = \{f : [0,1]^d \to \mathbf{R} \mid \|f^{(\alpha)}\|_\infty \le 1 \text{ if } \alpha_i \le k\}.$$

The classes F_d^k are important in connection with partially separable functions, i.e., f may depend on many variables but can be written as a sum of functions that only depend on a few variables. Similar spaces were studied by Babenko, Korobov, and Nikolskii. With the help of the Clenshaw-Curtis method (for $d = 1$) and the construction of Smolyak (1963) we obtain a method that is "almost universal" for all classes C_d^k and F_d^k, i.e., we obtain, up to logarithmic factors, the order $n^{-k/d}$ for each C_d^k and n^{-k} for each F_d^k. We also present numerical examples and results connected with polynomial exactness.

This is joint work with Klaus Ritter and, for the numerical part, Richard Schmitt and Achim Steinbauer (all in Erlangen).

New Results in Deterministic Pricing of Financial Derivatives

Anargyros Papageorgiou

Department of Computer Science, Columbia University

We have used low discrepancy sequences to price a number of financial derivatives and we have compared our testing results to Monte Carlo. We have found that the Sobol and the generalized Faure sequences (i) beat Monte Carlo by a wide margin, (ii) achieve a small error with a small number of sample points (e.g. for a 360-dimensional Collateralized Mortgage Obligation 170 generalized Faure points yield error 10^{-2}), (iii) can be as much as 1000 times faster than Monte Carlo when the accuracy demand is high.

Recently, we have applied low discrepancy sequences to integration problems in Physics, where the integrand depends on the norm of the variable. We have obtained results far superior to those obtained by the "recommended" classical quadrature formulas.

Joint work with J.F. Traub.

What is the Minimal Radius of Galerkin Information for Ill-Posed Problems?

Sergei V. Pereverzev

Ukrainian Academy of Sciences, Kiev

We investigate the information complexity of Fredholm integral equations

$$Ax(t) \equiv \int_0^1 a(t,\tau)x(\tau)\,d\tau = f(t) \tag{1}$$

with kernels $a(t,\tau)$ having square-summable partial derivatives $\frac{\partial^{i+j}}{\partial t^i \partial \tau^j}$, $i = 0,\dots,r$, $j = 0,\dots,s$, bounded by some constants in L_2-norm. Let us denote the class of integral operators A with such kernels by $H^{r,s}$. Let A^+f be a unique element that has minimal norm among all minimizers of the residual $\|Ax-f\|$. A widely used sufficient condition for the existence of A^+f is: $f \in AM_\varrho(A) := \{g : g = Au,\ u \in M_\varrho(A)\}$, where $M_\varrho(A) = \{u : u = A^*Av,\ \|v\| \le \varrho\}$. However, as a rule, instead of the free term f we have some approximation f_δ such that $\|f - f_\delta\| \le \delta$ and δ is usually known. Assume that only Galerkin information of the following type is available:

$$(b_i, Ab_i), \quad (i,j) \in \Omega, \qquad (b_k, f_\delta), \quad k \in \omega, \tag{2}$$

where b_μ, $\mu = 1,2,\dots$ are the elements of some orthonormal basis B and Ω, ω are some sets of indices. Within the framework of any algorithm R using Galerkin information (2) we obtain some element $x_R(B,\Omega,\omega,f_\delta)$ as approximate solution of (1). The error of a specific algorithm R on the class $H^{r,s}$ is defined as

$$E_\delta(H^{r,s},B,\Omega,\omega,R) = \sup_{A\in H^{r,2}} \sup_{f\in AM_\varrho(A)} \sup_{\|f-f_\delta\|\le\delta} \|A^+f - x_R(B,\Omega,\omega,f_\delta)\|.$$

The minimal radius of Galerkin information on the class $H^{r,s}$ is determined by the quantity

$$R_{N,\delta}(H^{r,s}) = \inf_B \inf_{\mathrm{card}(\Omega)+\mathrm{card}(\omega)\le N} E_\delta(H^{r,s},B,\Omega,\omega,R).$$

This is the minimal error which can be reached using at most N values of Galerkin functionals (2). Our main result is

Theorem For $r > 2$

$$R_{N,\delta}(H^{r,s}) \asymp \begin{cases} \delta^{2/3}, & N \gg \delta^{-2/(3s)} \\ N^{-s} + \delta^{1/3}, & N \asymp \delta^{-2/(3s)} \\ N^{-s}, & N \ll \delta^{-2/(3s)} \end{cases}$$

For $r = 2$, $N \asymp \delta^{-2/(3s)}$ or $N \ll \delta^{-2/(3s)}$

$$c_1\, N^{-s} \le R_{N,\delta}(H^{r,s}) \le c_2\, N^{-s} \log^s N.$$

The sets of indices Ω, ω as well as the basis B and the algorithm R realizing the optimal order are indicated, too.

On the Complexity of Validated Numerical Integration

Knut Petras

Technische Universität Braunschweig

We consider the complexity of numerical integration of bounded functions in $C^k([a,b] \setminus Z)$, where Z varies over all finite subsets of $[a,b]$. Using only function

values or values of derivatives, we usually can not guarantee that the costs for obtaining an ε-approximation are bounded by $O(\varepsilon^{-1/k})$ and we may obtain much higher costs. The situation changes if we also allow estimates of ranges of functions or derivatives on intervals as observations. In a practical implementation, estimation of ranges may be done efficiently with interval arithmetic and automatic differentiation. The cost for each such evaluation (also of ranges of derivatives) is bounded by a constant times the cost for a function evaluation. The mentioned techniques reduce the class of integrands, but still allow numerical integration of a wide class of functions to which conventional integration algorithms may applied reasonably.

A very simple algorithm now yields an ε-approximation with $O(\varepsilon^{-1/k})$-costs for the remaining class of functions.

Average Case L_∞-Approximation in the Presence of Gaussian Noise

Leszek Plaskota

University of Warsaw

We consider the average case L_∞-approximation in $C^r([0,1])$ with respect to the r-fold Wiener measure. An approximation is based on n function evaluations in the presence of Gaussian noise with variance $\sigma^2 > 0$. We show that the nth minimal average error is asymptotically of order $(\sigma/\sqrt{n})^{(2r+1)/(2r+2)} \ln^{1/2} n$, and it can be attained either by the piecewise polynomial approximation using repetitive observations, or by the smoothing spline approximation using non-repetitive observations. This completes the already known results for L_q-approximation with $q < \infty$ and $\sigma \geq 0$, and for L_∞-approximation with $\sigma = 0$.

Multivariate Random Functions: Cubature and Optimal Recovery

Klaus Ritter

Universität Erlangen-Nürnberg

We study the problems of optimal recovery and cubature of functions $f : [0,1]^d \to \mathbf{R}$ in an average case setting. The functions f are considered as realizations $f = X(\cdot, \omega)$ of a stochastic process X with parameter space $[0,1]^d$. Errors are defined in mean square sense.

We derive optimal error bounds for both problems under several isotropic and anisotropic smoothness conditions for X. Furthermore we point to applications in Geostatistics and simulation of stochastic processes.

Reconstruction of Functions from Scattered Data Using Translates of a Basis Function

Robert Schaback

Universität Göttingen

This talk exhibits the reconstruction of functions from scattered data by translates of positive definite "basis" functions as a direct generalization of the well-known univariate spline theory: it yields linear optimal methods via the theory of reproducing kernel Hilbert spaces. Some recent developments are sketched:

a) the construction of compactly supported positive definite functions that generate Sobolev spaces (Wendland 1996),

b) the "Uncertainty Relation" that implies bad condition for low error problems (R.S. 1996),

c) $O(nd)$ computational complexity techniques involving properly scaled compactly supported functions.

Complexity in Neurofunctional Magnetic Resonance Tomography

Walter Schempp

Lehrstuhl für Mathematik I, Universität Siegen

The notion that biological systems are found at the interface between chaos and order is a recurrent theme in the science of complexity and has validity when applied to neuronal dynamics. A balance between chaos and order is implicit in the brain's tendency to diversity, wherein cortical areas preserve the unique and regionally specific dynamics, and in an opposing tendency to integrate regional dynamics into globally coherent patterns of activity. This dialectic of cerebral organization is supported by neurofunctional magnetic resonance tomography. The applications to non-invasive image-guided interventions are indicated.

Interior Ellipsoid Method for Fixed Points

Krzysztof Sikorski

University of Utah

We review worst case results for the complexity of approximating fixed points of contractive and noncontractive mappings. The interior ellipsoid, exterior ellipsoid, and centroid algorithms are analyzed for the contractive multivariate case. The best upper bounds are given by $O(d^2(\log 1/\varepsilon + \log 1/(1-q) + \log d))$ as the number of function evaluations of computing ε-approximation in the d-dimensional case with the contraction factor $q < 1$.

Multiple Integration is Intractable in Korobov Classes

Ian H. Sloan

University of New South Wales

In this talk, which describes joint work with H. Woźniakowski, I show that in a certain worst-case setting the error in multivariate numerical integration is not better than would be obtained with the trivial quadrature rule (which is the rule that predicts all integrals to be zero!), unless the number of quadrature points is at least 2^d, where d is the dimension. The setting is that of the worst-case error in the Korobov classes of periodic functions on the unit cube. On the other hand, when the number of points reaches 2^d the result changes completely, because now even the humble tensor product of the 2-point rectangle rule can achieve arbitrarily small worst-case error by taking the Korobov smoothness parameter to be large enough. It is speculated that this 2^d-point product-rectangle rule is even optimal for the Korobov class with sufficiently large smoothness parameter.

Information Complexity of Some Classes of Ill-Posed Problems

Sergei G. Solodky

Ukrainian Academy of Sciences, Kiev

We consider the Fredholm integral equations of the first kind with linear operators whose kernels have square-summable partial derivatives, bounded by some constant in L_2-norm, and with the free terms from the range of operator $A(A^*A)^\nu$, $\nu \geq 1/2$. The aim of our investigations consists in obtaining the order-optimal estimations for some quantities characterizing the complexity of such equations in the sense of volume of Galerkin functionals and in the sense of number of arithmetic operations on values of these functionals. We propose a new scheme for discretization using the idea of hyperbolic cross. This approach allows to find the optimal order of information complexity for some classes of ill-posed problems. Note that the parameters defining the hyperbolic cross depend on the smoothness of two kinds. Namely, on the differential smoothness of the kernels and on the "operator smoothness" of solution ν. This situation differs from the case of equations of second kind, where as in the classical approximation theory the dimension of the cross is defined by differential smoothness only.

An Application of Numerical Integration

Ulrike Storck

Universität Karlsruhe (TH)

In gas-kinetics very often the engineers are looking for new substances. Then in many applications it's necessary to know the transport coefficients (self-diffusion, viscosity

and heat conductivity) of the new developed substance. These transport coefficients can be obtained by measurements which are very expensive with respect to material and manpower. So one would like to calculate the transport coefficients by a model. This model contains the so-called collision integrals which are six-dimensional integrals. The collision integral depends on the chosen potential function describing the intermolecular forces. A new method for calculating the collision integrals for the more general O'Connell-Prausnitz potential function is presented. The two innermost integrals of the collision integral, that means the angle of deflection and the reduced cross section are determined by integration algorithms which encounter all kinds of numerical errors, the truncation error as well as all rounding errors. These so-called verified algorithms take the singularity of the integrand of the angle of deflection and the oscillating point of the integrand of the reduced cross section into account. As the computation time for the determination of the reduced cross section varies between several minutes and some days, the determination of the collision integrals is performed by an approximative integration algorithm. Some numerical results for the transport coefficients of the substance R152a calculated for the Kihara and the O'Connell-Prausnitz potential function are compared with experimental results. These comparisons show that the O'Connell-Prausnitz potential leads to better results.

Hyperbolic Wavelet and Trigonometric Approximation

Vladimir Temlyakov

University of South Carolina

We study the multivariate approximation by certain partial sums of wavelet bases formed by tensor products of univariate wavelets. We relate the approximation by such partial sums to approximation by trigonometric polynomials with frequencies from the hyperbolic cross. We prove the Jackson and Bernstein inequalities for hyperbolic wavelet approximation. The technique involved is based on the Littlewood-Paley theory. We obtain one new sufficient condition for a bases to have the Littlewood-Paley property.

Van der Corput Sequence in Fibonacci Number System

Shu Tezuka

IBM Tokyo Research Laboratory

We introduce an analogue of the van der Corput sequence in the Fibonacci number system. It is shown that this new sequence can be viewed as a theoretical bridge between the Weyl sequence and the van der Corput sequence.

Polya's Characterization Theorem for the Complex Case

Nicholas N. Vakhania

Institute of Computational Mathematics, Georgian Academy of Sciences

The following characterization property of real-valued random variables is well known:
Theorem (D. Polya, 1923). Let ξ be a real random variable, n be a natural number, $\xi_1, \xi_2, \ldots, \xi_n$ be n independent copies of ξ, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero real numbers such that $\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2 = 1$. If the linear combination $\alpha_1\xi_1+\alpha_2\xi_2+\ldots\alpha_n\xi_n$ has the same distribution as ξ then ξ is a Gaussian random variable.
We give the formulation and prove the complex version of this result.

Optimization of Discrete-Stochastic Numerical Procedures

Anton V. Voytishek

Computing Center SD RAS, Novosibirsk

Problems of convergence and optimization of the numerical discrete-stochastic procedures for estimating integrals dependent on a parameter and solutions of integral equations of the second kind are considered. The procedures include introduction of a discrete grid, estimation of the solution at grid nodes using the Monte Carlo methods and approximation of the solution using the values obtained at grid nodes. The probabilistic metrics of spaces $\mathbf{L}_2$ (along with convergence in the mean) and $\mathbf{C}$ (along with convergence in probability) are used for estimating the error of the considered numerical methods. The independent and dependent estimators, and also the combined procedures (for the integrals dependent on a parameter), and the conjugate wandering method, the local estimator, the polygon of frequencies method, the vector and local-vector estimators (for the integral equations) can be used at grid nodes. We succeeded in expressing corresponding convergence conditions for these stochastic estimators in terms of the under-integral function (for the integral) and the kernel and the free term (for the integral equation), probabilistic densities used to obtain the above stochastic estimators. Some approaches for choice of optimal discrete (number of grid nodes) and stochastic (number of Monte Carlo realizations) parameters of the discrete-stochastic procedures are elaborated.

Worst Case Complexity of Weighted Approximation: Unbounded Domain $\mathbf{R}^d$

Grzegorz W. Wasilkowski

Department of Computer Science, University of Kentucky

There are many results concerning the complexity of approximation and integration problems for classes of functions with bounded domains. On the other hand, almost

nothing is known for problems defined over classes of functions with unbounded domains. In this talk we will present recent results concerning weighted approximation and integration problems defined over $\mathbf{R}^d$.
More specifically, we assume that the class F consists of functions $f : \mathbf{R}^d \to \mathbf{R}$ whose partial derivatives of order r are bounded with respect to a given weight ψ, i.e., $\sup_{x \in \mathbf{R}^d} \psi(\|x\|) \sum_{|\alpha|=r} |f^{(\alpha)}(x)| \leq 1$. For the weighted approximation problem, the error is measured in a ρ-weighted L_q-norm; $\|f\| = (\int_{\mathbf{R}^d} \psi^q(\|x\|)|f(x)|^q \, dx)^{1/q}$ with $q \in [1, \infty]$.
Of course, for some weight functions ψ and ρ the weighted approximation has infinite complexity. Moreover, for any weights, the complexity of the weighted approximation cannot be smaller than the complexity of the classical approximation over a finite domain. We provide a necessary and sufficient condition on the weights so that the corresponding weighted approximation has finite complexity. We also provide a necessary and sufficient condition on the weights so that the complexity of the weighted approximation equals (modulo a constant) the complexity of the classical approximation.
We also have analogous results for the weighted integration problem since it is equivalent to the weighted approximation problem with $q = 1$.
Joint work with H. Woźniakowski

A Physically Realizable Model of Computation for Continuous Problems

Klaus Weihrauch

Fernuniversität Hagen

Turing machines can be realized correctly by digital computers, they are a realistic model of computation. Since real numbers are infinite objects, but every physical information channel is finite and every physical memory is finite, no device can read, store or write an arbitrary real number. Therefore, IBC-algorithms [Nov 95] cannot be realized by physical devices in general. There are IBC-algorithms which cannot even be approximated reasonably by multiple precision floating-point number computations. Furthermore, the unit-cost assumption for the IBC-model of computation is unrealistic since good approximate machine computations usually require very long floating-point numbers. Therefore, the IBC-model of computation is unrealistic. Although, often properties of IBC-algorithms are realistic, IBC does not distinguish realistic from unrealistic results in general.
On the other hand, Computable Analysis based on A. Grzegorczyk's definition of computable real functions ("Polish Computable Analysis") and its extension "TTE" [Wei 95a, Wei 95b] from real numbers to second countable topological spaces extend the Turing machine model and are completely realistic. In principal, the TTE-model can be obtained from the IBC-model of computation by the additional assumption: "every input or output information of an algorithm is finite". The wide applicability

of TTE and the relation between the two models are demonstrated by examples for computability and complexity of real functions and for integration and zero-finding on C[0,1].

[Nov 95] Erich Novak: The real number model in numerical analysis, Journal of Complexity 11, 57-73 (1995).
[Wei 95a] Klaus Weihrauch: A simple introduction to Computable Analysis, Informatik-Berichte 171, Fernuniversität, Hagen, 1995.
[Wei 95b] Klaus Weihrauch: A foundation of Computable Analysis, in: Ker-I Ko and Klaus Weihrauch (eds.), Computability and complexity in Analysis, Informatik-Berichte 190, Fernuniversität, Hagen, 25-40 (1995).

The Complexity of Definite Elliptic Problems with Noisy Data

Arthur G. Werschulz

Fordham University and Columbia University

We study the complexity of $2m$th order definite elliptic problems $Lu = f$ (with homogeneous Dirichlet boundary conditions) over a d-dimensional domain Ω, error being measured in the $H^m(\Omega)$-norm. The problem elements f belong to the unit ball of $W^{r,p}(\Omega)$, where $p \in [2, \infty]$ and $r > d/p$. Information consists of (possibly-adaptive) noisy evaluations of f or the coefficients of L. The absolute error in each noisy evaluation is at most δ. We find that the nth minimal radius for this problem is proportional to $n^{-r/d} + \delta$, and that a noisy finite element method with quadrature (FEMQ), which uses only function values, and not derivatives, is a minimal error algorithm. This noisy FEMQ can be efficiently implemented using multigrid techniques. Using these results, we find tight bounds on the ε-complexity (minimal cost of calculating an ε-approximation) for this problem, said bounds depending on the cost $c(\delta)$ of calculating a δ-noisy information value. As an example, if the cost of a δ-noisy evaluation is $c(\delta) = \delta^{-s}$ (for $s > 0$), then the complexity is proportional to $(1/\varepsilon)^{d/r+s}$.

Tractability of Multivariate Tensor Product Linear Operators

Henryk Woźniakowski

Department of Computer Science, Columbia University, and Institute of Applied Mathematics, University of Warsaw

We study multivariate tensor product linear operators defined over Hilbert spaces. The d-variate linear operator S_d is obtained by taking d tensor products of the continuous linear operator S_1. We consider the worst case setting in which we approximate S_d over the unit ball with error at most ε. Approximations of S_d are obtained by computing a number of continuous linear functionals from a given class of information.

The problem is said to be *tractable* iff the number of linear functionals needed to approximate S_d with error at most ε is polynomial in d and $1/\varepsilon$, and is said to be *strongly tractable* iff the number of linear functionals does not depend on d and is polynomial in $1/\varepsilon$. We are mainly interested in characterizing which problems are tractable and strong tractable.

We consider the *three* classes of information. The first class is the class of all linear functionals. For this class, it is known that tractability is equivalent to strong tractability, and that strong tractability holds iff S_1 is a linear functional, or $\|S_1\| < 1$ and singular values of S_1 go polynomially to zero. (A sequence λ_n goes polynomially to zero iff there exists a positive k such that $\lambda_n = O(n^{-k})$.)

The second class is the *Fourier* class of information. This class consists of inner products with respect to given orthonormal elements. The analysis of the Fourier class seems to be new. If the domain space of S_1 is not separable then we may be not able to approximate S_d even if S_1 is a linear functional. On the other hand, if the domain space of S_1 is separable and spanned by orthonormal elements of the Fourier class then we can approximate S_d by using a finitely many linear functionals iff S_1 is compact. We provide necessary and sufficient conditions for tractability and strong tractability. As for the first class, tractability and strong tractability are equivalent for the Fourier class. Strong tractability holds iff S_1 can be approximated with an arbitrarily small error by one inner product from the Fourier class, or $\|S_1\| < 1$ and the nth minimal errors of approximating S_1 go polynomially to zero. (By the nth minimal error we mean the minimal error of approximations that use at most n linear functionals from the given class of information.)

The third class is the standard class of information which consists of function values. In this case we assume that the domain space of S_1 is a reproducing kernel Hilbert space of univariate functions. The standard class is probably the most important from a practical point of view. There are many papers analyzing this class. In particular, it is known that if S_1 is at least two dimensional, i.e., $\dim(S_1(F_1)) \geq 2$ where F_1 is the domain space of S_1, then tractability is again equivalent to strong tractability, and strong tractability holds iff $\|S_1\| < 1$ and the nth minimal errors of approximating S_1 go polynomially to zero.

The unresolved case for the class of standard information is when S_1 is a linear functional. We show that the results for this case are very rich in the possibilities. First of all, there exist domain spaces of S_1 (even of infinite dimension) such that all problems S_d are strongly tractable. In fact, it is enough to compute only *one* function value to get an ε-approximation, and this holds for arbitrarily small positive ε. Such spaces can be even subspaces of continuous functions and their construction is related to Peano curves.

Let us now assume that one function value is not enough to get an ε-approximation for small positive ε. We then have two cases. The first one is $\|S_1\| < 1$. Then it is known that tractability and strong tractability are equivalent, and strong tractability

holds iff the nth minimal errors for approximating S_1 go polynomially to zero.
The second case is $\|S_1\| \geq 1$. Then the problem is *not* strongly tractable. In fact, to approximate S_d we have to compute at least d linear functionals. The last bound is sharp since for some domain spaces it is enough to compute $d+1$ linear functionals to solve *all* S_d even exactly, i.e., with $\varepsilon = 0$. In this case, we have tractability. On the other hand, for some other domain spaces, *all* problems S_d are intractable. Hence, tractability of linear functionals with $\|S_1\| \geq 1$ depends on the given space of functions. We provide conditions on tractability and intractability of linear functionals. In "typical" function spaces these conditions are satisfied for some linear functionals. That is, the classes of tractable and intractable linear functionals are both in general nonempty. For a given linear functional, such as integration or weighted integration, it is usually hard to verify to which class it belongs.

Joint work with Erich Novak and Ian H. Sloan