Report on Dagstuhl-Seminar 00391:

Algorithms and Complexity for Continuous Problems

September 25 - 29, 2000

The seminar was attended by 36 scientists from 12 countries. It was devoted to the complexity of a range of important problems and efficient algorithms for solving them. Among those problems were numerical integration and approximation, solving integral and operator equations, solving partial differential and stochastic differential equations, optimization. Since the complexity of problems depends on the dimensionality of the underlying class of functions, over 20 talks (out of the total of 32) discussed multivariate problems and the dependence of the complexity on the number d of variables.

Integration of functions of very many variables is of great practical interest in a wide range of applications including statistical mechanics, chemistry, physics and mathematical finance. Although it has been studied for a long time, the classical numerical results provide either very negative results of *exponential* dependence on d or insufficient information of the dependence on d. Since in the problems mentioned above, the value of d could easily be in thousands, millions, or sometimes equal to infinity, the dependence on the dimension is of great importance. Sometimes, it is even more important than the dependence on the error demand. Not surprisingly, 14 talks presented new results on high-dimensional integration and we will concentrate on some of them here. There was a number of results dealing with construction and/or properties of low discrepancy points as well as of (Quasi-) Monte Carlo methods. The model of quantum computation and its power for numerical integration was presented. Applications to specific finance problems were discussed, as well as complexity and optimal methods for integrating singular functions or functions over unbounded domains. Monte Carlo and/or Quasi-Monte Carlo methods were also discussed in the context of other than integration problems including differential equations and computer vision. In particular, an optimal randomized method was proposed to solve integral equations with singular kernel.

In addition to presentations, one afternoon was devoted to the *Open Problems Session* where a series of new important research problems has been proposed.

We very much profited from the excellent working and living conditions during the seminar and would like to express our thanks to the staff of Schloss Dagstuhl as well as to the office in Saarbrücken for their efficiency in organizing and handling our meeting.

S. Heinrich, S. Pereverzev, J. Traub, and G. Wasilkowski

Abtracts in alphabetical order by speaker's surname:

Abstracts of Talks
(in alphabetical order by speaker's surname)

Probabilistic Analysis of Interior Point Methods for Linear Programming

Karl Heinz Borgwardt
joint work with Petra Huhn

It is our aim to make a fair comparison of the Simplex Method and Interior Point Methods on the basis of their average case behavior in the solution of Linear Programming problems possible. Therefore we use the same stochastic model (the Rotation Symmetry Model) for the average case analysis of both solution methods.

Interior Point Methods run in three Phases. In Phase I it is the aim to get close to the analytic center of the feasible region. In Phase IIa we perform an iteration process which reduces the distance to the optimum at least with linear convergence. And in Phase IIb one starts a search for a closeby vertex from the last iteration point of Phase IIa.

We demonstrate that for worst-case and for average-case behavior the following geometric figures determine the number of iterations:

1. The maximal distance of a vertex to the origin (the maximal vertex norm) is the crucial measure for the effort in Phase I.
2. The difference of the objective values at the best and the second best vertex determines the number of iterations required in Phase IIa.

In worst case polynomiality-proofs these two figures can only be bounded by use of the encoding length L of the problem. And this is the reason why worst case bounds contain this extremely high factor and that polynomiality but not strong polynomiality can be shown.

In the average case analysis we calculate the distribution functions of the two geometric figures mentioned above. And this can be - via the evaluation of integral formulas - used to achieve bounds on the expected number of iterations for all Phases. Our result of that stochastic analysis leads to a proof that the expected number of iterations for the whole method is not only polynomial in the encoding length but also **strongly polynomial** in the dimensions of the problem.

Impossibility of Exponential Convergence Rate for Optimization on the Wiener Space

Jim Calvin

It is possible to approximate the minimum of a unimodal function on an interval with the worst case error converging to zero at an exponential rate (for example, using the Fibonacci search algorithm). Without strong assumptions such as unimodality, no such worst case bound is available.

In a stochastic setting we can obtain probabilistic bounds for large classes of functions. Let P be the Wiener measure, and choose a sequence of numbers $c_n \downarrow 0$ arbitrarily slowly. Then there exists an algorithm such that

$$P(\Delta_n \leq \exp(-c_n n)) \to 1,$$

where Δ_n is the approximation error after n observations. The purpose of this talk is to show that

$$P(\Delta_n \leq \exp(-cn)) \to 0$$

for any algorithm and any $c > 0$.

Spherical Product Algorithms and the Integration of Smooth Functions with One Singular Point

Ronald Cools
joint work with Erich Novak

We consider the problem of numerical integration for multivariate functions with respect to a radial symmetric weight. We prove that suitable spherical product algorithms have the optimal rate of convergence $n^{-k/d}$ for C^k-functions. We also study classes of integrands with a singularity that are C^k outside the origin. Standard algorithms have a high cost for such functions, because they require that the function is smooth everywhere. We construct suitably modified spherical product algorithms with optimal rate of convergence $n^{-k/d}$ also in this case. In the compact case we can use modified spherical product Gauss formulas with a nonalgebraic degree of precision.

It is available as Report TW 308, Dept. Computer Science, K.U.Leuven, July 2000.

URL: http://www.cs.kuleuven.ac.be/publicaties/rapporten/tw/TW308.abs.html

Complexity of Financial Problems

Francisco Curbera

The complexity of quasi-Monte Carlo algorithms and the behavior of these algorithms for problems with very high dimensionality have attracted much attention in recent years. Several results have been obtained which provide necessary and sufficient conditions for the tractability and strong tractability of quasi-Monte Carlo algorithms. Some of the most challenging applications of quasi-Monte Carlo algorithms occur in the field of mathematical finance. In this talk we show that there are fundamental problems that prevent the application of most upper bounds and tractability results to financial problems. We begin showing that the Koksma-Hlawka inequality cannot be applied to most financial problems, and propose a modified inequality that does apply to these problems. Next we review the application of weighted tensor product (WTP) space techniques to financial problems, and identify two major difficulties. First, the lack of smoothness of most integrands typically found in finance puts these integrands outside the scope of WTP space techniques; we show, however, that this lack of smoothness is of a very particular kind, which may lend itself to more targeted approaches. Second, even when smooth integrands are considered, the discretization techniques commonly used by practitioners yield problems whose tractability cannot be proven using WTP techniques. However, we show examples of smooth financial integrands which yield strongly tractable integration problems when the Karhunen-Loéve expansion is used instead of the standard discretization scheme. We conclude by presenting numerical results of the computation of the weighted discrepancy for high dimensions (100) and large sample sizes (100000).

Complexity results for system identification with random noise

Alexander Goldenshluger

We consider estimating impulse response sequence of a stable linear time-invariant system under stochastic assumptions. We adopt a nonparametric minimax approach for measuring estimation accuracy. The quality of an estimator is measured by ots worst case error over a family of impulse responses. The families with polynomially and exponentially decaying impulse responses are considered. We establish nonasymptotic upper bounds on accuracy of the least squares estimator. Lower bounds on estimation accuracy are derived. An adaptive estimator that does not exploit any a priori information about the 'true' system is developed.

Optimality in Simulation for Balance Equations: Discrete vs Continuous Problems

Nina Golyandina

We study the Monte Carlo solution of the balance differential equation $d\mu_t/dt = G(\mu_t)$ in measures on the space of probability measures given on the compact Polish space D. As examples of the balance equation, we consider the nonlinear Boltzmann-like equation and the linear balance equation. We say that the sequence of jump Markov processes $\xi_n(t)$ solves the balance equation if $\mathbf{E}\psi(\xi_n(t)) \to \psi(\mu_t(\mu))$ as $n \to \infty$ for any smooth functional ψ. The theory of semigroups allows to find conditions for the convergence, its rate and the form of approximation errors (Golyandina and Nekrutkin 1999).

The main result on convergence rate is the following: under natural conditions, errors can be represented as $c_1(\psi,t)/n + c_2(\psi,t)/n + o(1/n)$, where the first term is caused by the errors in initial data, the second term is caused by the errors due to process simulation, and coefficients can be written down in a way explicitly. The term $c_1(\psi,t)$ can be reduced to 0 by stratification of the initial distribution (Golyandina and Nekrutkin 2000). In the case of the discrete phase set $D = \{1,\dots,k\}$ the process with minimal $c_2(\psi,t)$ can be constructed. In the continuous case the optimal process formally exists but is not realizable from the simulation viewpoint. That is why we should consider Markov jump processes $\xi_n(t)$ in the form $\xi_n(t) = \sum_{i=1}^{n} \delta_{\zeta_i(t)}/n$, that is, n-particle processes. The simple examples demonstrate an appreciable advantage of the optimal estimator in comparison with the n-particle estimator.

Using the class of n-particle processes (they are realizable though non-optimal) we can construct new algorithms and compare algorithms on their complexity. In particular, we prove that for the Boltzmann-like equations algorithms with binary collisions have advantage. In the same manner, we can obtain advantage in complexity for the linear equations with the help of introducing artificial collisions.

Monte Carlo Approximation of Weakly Singular Operators

Stefan Heinrich

We study the complexity of solving the integral equation

$$u(s) - \int_G K(s,t)u(t)dt = f(s),$$

where $G \subseteq \mathbf{R}^d$ by randomized (Monte Carlo) methods. We are interested in approximations of the solution u on a submanifold $G_1 \subseteq G$ of dimension $d_1 \le d$. The kernel is supposed to be of weakly singular type

$$K(s,t) = \frac{k(s,t)}{|s-t|^\sigma}$$

for some $\sigma, 0 \le \sigma < d$, and $k \in C^{r,0}(G^2)$, $f \in C(G)$. Previous results for the smooth case $\sigma = 0$ are reviewed. Then a new result is stated for the case $\sigma > 0$ matching upper and lower bounds (up to log-factors).

Approximation of stochastic differential equations with linear functionals

Norbert Hofmann

The talk is concerned with pathwise approximation of scalar stochastic differential equations with respect to the global error in the L_2-norm. We study methods that are based on sequentially chosen linear functionals of the driving Brownian motion W. The minimal error in the class of arbitrary methods that are based on n functional

evaluations on the average tends to zero like $c/\pi \cdot n^{-1/2}$, where c is the average of the diffusion coefficient in space and time. For comparison we consider the class of methods that use n discrete observations of W on the average. The minimal error in this class behaves like $c/\sqrt{6} \cdot n^{-1/2}$. Hence methods with linear functionals yield only a small improvement on the level of asymptotic constants.

Application To Higher Dimensional Problems for (t,m,s)-nets and Scrambled (t,m,s)-nets.

Hee Sun Regina Hong
joint work with Peter Mathé

Monte Carlo methods are widely used in multidimensional integration. Quasi-Monte Carlo methods improve the accuracy of Monte Carlo methods by choosing a low-discrepancy set such as a (t,m,s)-net or a (t,s)-sequence. Scrambled nets were proposed by Art Owen as a hybrid of Monte Carlo method and a quasi-Monte Carlo method which achieve the superior accuracy of the quasi-Monte Carlo method while allowing the simple error estimation of Monte Carlo method.

We tested on higher dimensional problem with different covariance structures and different difficulty of problems with (t,m,s)-nets with their scrambled nets as well as a Monte-Carlo method. Here, we implemented scrambled digital net as simple matrices multiplication which is a variation of Art Owen's scrambling but as general as his.

Quasi-Monte Carlo methods perform well for moderately high dimension but their performance getting worse as dimension getting higher, however scrambled nets perform well in overall dimensions.

Strictly Deterministic Sampling in Computer Graphics

Alexander Keller

We introduce two new techniques for parallel image synthesis. Exploiting the structure of $(0,2n,2)$-nets in base 2 and efficient computation schemes for these nets, we introduce the interleaved method of dependent tests and dependent splitting. Both techniques are realized by *deterministic* low discrepancy samples that allow for an efficient parallelization on heterogenous computer architectures providing a superior performance without the standard correlation problems of parallel pseudo-random number generation. The techniques are implemented in an industrial renderer.

A General Approach to Inference and Optimality

Mark A. Kon

Continuous complexity theory is a way of quantifying the amount of work done to solve problems. As a result, different algorithms can be compared and notions of optimal algorithms can be studied and developed. At this point there are a large number of modeling techniques which study the ill-posed problem of extrapolating functions $f(x)$ from partial information $Nf = (L_1 f, ..., L_n f)$, but which do not have any means of normative comparison of the complexities of the algorithms involved. Many of these technologies find their way into the field of data mining, and include statistical learning theory, neural network theory, computational learning theory, regularization theory, regression, maximum entropy, V-C theory, and decision tree theory. A goal of the present work is to identify potential issues and techniques in adapting current complexity theory to the comparative study of complexities of the various algorithms and strategies mentioned here. The final goal is a normative index of function extrapolation

methods, according to their commonly measured optimality properties. An important basis for the comparison and contrast of extrapolatory methodologies involves the teasing out of their *a priori* and *a posteriori* assumptions. Some of these are examined in a few of the above methodologies, and the following theorem is proved, relating the complexity theory of regularization methods and average case complexity under Gaussian measures.

Theorem: The ε-complexity of the regularization approach for any problem with regularization functional $||Af||^2$ (under basic Bayesian assumptions) is equal to the ε-complexity of the average case setting, assuming a Gaussian measure with covariance operator A^{-2}.

This theorem gives some definition to ε-complexity in the regularization approach, and leads to the following

Conclusion: Complexities are independent of the choice of specific a priori assumptions in regularization problems. All that is necessary is the regularization assumption that "$||Af||$ should be small"; complexities consistent with this assumption can be computed from the average case setting in information-based complexity theory.

Numerical Analysis of Runge-Kutta Quasi-Monte Carlo Methods

Christian Lécot

We are interested in the numerical solution of a system of differential equations $y'(t) = f(t, y(t))$ when f is irregular or may vary rapidly in t. We describe a family of numerical schemes which is akin to the Runge-Kutta family. The schemes use quasi-Monte Carlo estimates of integrals. We focus on third order schemes with three stages. They use low-discrepancy point sets in dimension three. An error bound is shown, which involves the power three of the step size as well as the discrepancy of the point set used for the quasi-Monte Carlo approximations. The results of numerical experiments are described. The influence of the variation of f on the convergence of the schemes is studied. It is shown that the performances of the Runge Kutta method degrade when f varies rapidly in t. The performances of the Runge Kutta Monte Carlo or quasi-Monte Carlo methods are not so dependent on the rate of variation. In addition Runge Kutta quasi-Monte Carlo methods outperform Runge Kutta Monte Carlo methods.

Polynomials of bounded tree-width

Klaus Meer
joint work with J.A. Makowsky

We study subclasses of computationally hard problems in the framework of algebraic complexity theory (BSS model of computation). These subclasses are defined by means of a generalization of the tree-width parameter well known in graph theory. We redefine this parameter w.r.t. meta-finite structures (instead of graphs) including algebraic issues and show: Properties expressible in existential MSO logic (a logic to be defined) over meta-finite structures of bounded tree-width can be decided resp. computed in linear time w.r.t. the BSS model of computation.

The results apply to problems like solvability of polynomial equations over finite fields, computation of the permanent of a real square matrix or optimization of an LP problem.

On the representation of band-limited signals using finitely many bits

H. N. Mhaskar

Let K be a compact subset of a metric space X. For an integer $n \geq 1$, an n-bit encoder is a mapping from K into $\{0,1\}^n$, and n-bit decoder is a mapping from $\{0,1\}^n$ into X. For $\varepsilon > 0$, let

$$L_\varepsilon(K,X) := \min\{n \ : \ \text{there exist } n\text{-bit encoder } E \text{ and decoder } D \text{ such that } \sup_{f\in K} d(f, D(E(f))) < \varepsilon\}.$$

Let $\tau, \lambda \in (0,\infty)$, and $B_{\tau,\lambda}$ be the set of all entire functions f such that $|f(z)| \le \exp(-\tau|z|^\lambda)$ for all $z \in \mathbf{C}$. Let $\alpha > \max(\lambda, 1)$, $1 \le p \le \infty$, and X be the space of all functions g such that, with $w(x) := \exp(-|x|^\alpha/2)$, $wg \in L^p(\mathbf{R})$, equipped with the norm $\|g\| := \|wg\|_p$. We show that

$$L_\varepsilon(B_{\tau,\lambda}, X) = \frac{1}{2(1/\lambda - 1/\alpha)} \frac{(\log(1/\varepsilon))^2}{\log\log(1/\varepsilon)} (1 + o(1)).$$

We give explicit constructions for the asymptotically optimal encoders and decoders, based on finitely many samples of the target function. The number of samples is at most a constant multiple of the optimal number theoretically necessary. If the sampling nodes are equidistant with separation $1/\sigma$, the reconstruction error is $O(\sigma^{-c\sigma^\gamma})$ for some $\gamma > 0$. The decoders are polynomials having an asymptotically optimal degree.

Uniform Approximation of SDE's

Thomas Mueller-Gronbach

We analyze numerical methods for the pathwise approximation of a system of stochastic differential equations. As a measure of performance we consider the q-th mean of the maximum distance between the solution and its approximation on the whole unit interval. We introduce an adaptive discretization that takes into account the local smoothness of every trajectory of the solution. The resulting adaptive Euler approximation performs asymptotically optimal in the class of all numerical methods that are based on a finite number of observations of the driving Brownian motion

Quantum Complexity of Integration

Erich Novak

So far it is known that quantum computers yield a speed-up for certain *discrete* problems. We want to know whether quantum computers are useful for *continuous* problems.

We study the computation of the integral of functions from the classical Hölder classes with d variables. The optimal orders for the complexity of deterministic and (general) randomized methods are known. We obtain the respective optimal orders for quantum algorithms and also for restricted Monte Carlo methods (where we allow only coin tossing instead of general random numbers).

For the classes $F_d^{k,\alpha}$ on $[0,1]^d$ we put $\gamma = (k+\alpha)/d$. The known optimal orders for the complexity of deterministic and (general) randomized methods are

$$\mathrm{comp}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/\gamma}$$

and

$$\mathrm{comp}^{\mathrm{random}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-2/(1+2\gamma)}.$$

For a quantum computer we prove

$$\mathrm{comp}^{\mathrm{quant}}_{\mathrm{query}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/(1+\gamma)}$$

and

$$\mathrm{comp}^{\mathrm{quant}}(F_d^{k,\alpha}, \varepsilon) \le C\varepsilon^{-1/(1+\gamma)} (\log \varepsilon^{-1})^{2/(1+\gamma)}.$$

For restricted Monte Carlo we prove

$$\mathrm{comp}^{\mathrm{coin}}(F_d^{k,\alpha}, \varepsilon) \le C\varepsilon^{-2/(1+2\gamma)} (\log \varepsilon^{-1})^{1/(1+2\gamma)}.$$

To summarize the results one can say that

- there is a (roughly) quadratic speed-up of quantum algorithms over randomized classical methods, if γ is small;
- there is an exponential speed-up of quantum algorithms over deterministic (classical) algorithms, if γ is small.

URL of the paper: http://xxx.lanl.gov/abs/quant-ph/0008124.

The Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration

Anargyros Papageorgiou

The Brownian bridge has been suggested as an effective method for reducing the quasi-Monte Carlo error for problems in finance. We give an example of a digital option where the Brownian bridge performs worse than the standard discretization. Hence, the Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration. We consider integrals of functions of d variables with Gaussian weights such as the ones encountered in the valuation of financial derivatives and in risk management. Under weak assumptions on the class of functions, we study quasi-Monte Carlo methods that are based on different covariance matrix decompositions. We show that different covariance matrix decompositions lead to the same worst case quasi-Monte Carlo error and are, therefore, equivalent.

On the Information Complexity of severely ill-posed problems

Sergei Pereverzev and Eberhard Schock

We study the information complexity of so-called severely ill-posed equations with infinitely smoothing operators but with a solution having only a finite smoothness p. We note that for moderately ill-posed problems when the smoothness of the operator is also finite one can not keep the value of the constant α near the best possible order of accuracy for all $p > 0$ without knowledge of the value of p because of the uncertain principle: $\frac{\alpha}{\sqrt{p}} \geq c$, where c does not depend on α and p. For severely ill-posed problems the information about the exact value of p is not so important because the combination of Morozov's discrepancy principle and a finite dimensional version of the ordinary Tikhonov regularization is order-optimal for any $p > 0$ (It is well-known that for moderately ill-posed problems such a combination has the saturation property).

Asymptotically minimal Smolyak integration and computational aspects in Smolyak's method

Knut Petras

Smolyak's algorithm has proved to be successful for many tensor product problems. In the first part of this talk, it is shown how to obtain Smolyak methods with asymptotically minimal number of nodes for a given degree of polynomial exactness.

The second part of the talk is devoted to the calculation of the coefficients of Smolyak cubature rules for increasing dimension d. Taking the hitherto used formula and symmetries, it is demonstrated that almost all computation time is used for coefficient calculation. Using the principle 'divide and conquer' this calculation time

reduces to at most a small constant factor times the time for the whole algorithms. If we use all symmetries, the contribution of coefficient calculation to the computation time for the whole algorithm asymptotically vanishes.

A Note on the Discrepancy of Digital Nets with Fixed Quality Parameter

Fritz Pillichshammer
joint work with Gerhard Larcher

One of the most powerful concept for the construction of low- discrepancy point sets in the s-dimensional unit cube is the concept of digital (t,m,s)-nets in base b. (This is a special subconcept of the concept of (t,m,s)-nets in base b introduced by Niederreiter.) The discrepancy of digital nets can be estimated by the discrepancy bounds given for arbitrary (t,m,s)-nets in base b (see for example Niederreiter). Most of these discrepancy estimates only use the information given by the quality parameter t of the net. The aim of my talk is to point out, that there are, sometimes considerable, differences in the discrepancy of different digital (t,m,s)-nets with fixed quality parameter t. To support this claim two examples in the two-dimensional case are given. The first example deals with the L_2-discrepancy of symmetrized digital $(0,m,2)$-nets in base 2 and the second example is concerned with the $*$-discrepancy of digital $(0,m,2)$-nets in base 2.

Average case complexity of weighted approximation on $[0,+\infty)$

Leszek Plaskota

We study the average case complexity of weighted approximation of Gaussian stochastic processes X defined on the half-line $[0,\infty)$ whose rth derivatives satisfy the Hölder condition with exponent β in the mean square sense, such as the r-fold integrated fractional Brownian motion with parameter β. Any approximation uses only samples of X at finitely many points. The error of an approximation $\mathcal{A}X$ is given as $\sqrt{\mathbb{E}\int_0^\infty (X(t)-\mathcal{A}X(t))^2\rho^2(t)\,dt}$, where ρ is a weight function. We show that if ρ is *monotonically decreasing* and $\|\rho^{1/\gamma}\|_{L_1(0,\infty)}<\infty$ where $\gamma=r+\beta+1/2$, then the complexity is proportional to $(\|\rho^{1/\gamma}\|^\gamma_{L_1(0,\infty)}/\varepsilon)^{1/(r+\beta)}$. We also give complexity formulas in cases where the corresponding integral is infinite. In particular, if $\rho(t)\asymp t^{-p}\ln^{-q}$, $p,q>0$, then the complexity is infinite for $p<\gamma$, proportional to $\varepsilon^{-1/(r+\beta)}$ for $p>\gamma$, and for $p=\gamma$ we have

$$\mathrm{comp}(\varepsilon)\ \asymp\ \begin{cases} +\infty & q\le 1/2 \\ \varepsilon^{-1/(q-1/2)} & 1/2<q<\gamma \\ \varepsilon^{-1/(r+\beta)}\ln^{\gamma/(r+\beta)}(\varepsilon^{-1}) & q=\gamma \\ \varepsilon^{-1/(r+\beta)} & q>\gamma \end{cases}$$

Examples show that the monotonicity assumption is crucial.

Average Case Complexity of Weighted Integration and Approximation over $\mathbf{R}_+$

Klaus Ritter
joint work with Leszek Plaskota and Greg Wasilkowski

We study weighted integration and L_2-approximation for zero mean Gaussian processes $X(t)$, $t\in[0,\infty[$, based on observations of X at suitably chosen knots $t_1,\dots,t_n$. We analyze the ε-complexity, i.e., the minimal number of observations needed to achieve an average error at most ε. Results include necessary and sufficient conditions for

- the complexity to be finite,
- the complexity to be of the same order as in the classical compact case.

Our results are valid on classes of processes that are defined by second order properties. Examples include r-fold integrated fractional Brownian motions, Sacks-Ylvisaker processes, and stationary processes with spectral densities of prescribed decay. For approximation, we provide constructions of order optimal methods. For integration, we often rely on nonconstructive proofs of the upper bounds.

Wavelet Approximation for Integral Equations

Reinhold Schneider

Boundary integral formulations offer an appropriate tool for the numerical solution of certain boundary value problems in engineering. A major drawback of this approach is the fact that the arising system matrices are densely populated, which is limiting the discretization of realistic 3D problems with complex geometry. Like panel-clustering and multi-pole expansion, biorthogonal wavelet bases remedy this situation by approximating the discrete scheme in an efficient way. Multi-scale methods achieve this by approximating the system matrix relative to a biorthogonal wavelet basis by a sparse matrix. We propose a fully discretized Galerkin Wavelet Methods to discretizes boundary integral equations for static or low frequency problems which is based on parametric surface representation. Another important feature is that preconditioning is is rather simple due to the additive Schwarz decomposition of functions with respect to different scales. We compute a sparse approximation of the system matrix directly causing only an error proportional the optimal error bound of the Galerkin discretization. This can be done such that the total number of nonzero matrix coefficients increases only linearly with the total number of unknowns N. In order to compute the nonzero matrix coefficients directly, we apply an adaptive quadrature method, with the desired accuracy requiring totally $O(N)$ floating point operations.

Bisection-Envelope Algorithms for Multivariate Fixed Points

S. Shellman, K. Sikorski

We review recent complexity results for approximating fixed points of contractive, nonexpanding, and expanding functions that satisfy a Lipschitz condition with respect to the second and infinity norms.

New, bisection-envelope algorithms are presented for the case of bivariate functions that are nonexpanding with respect to the infinity norm. The upper bounds on the number of function evaluations to compute ε-residual approximations are $O((log(1/\varepsilon))^2)$. We believe that these bounds can be improved to $O(log(1/\varepsilon))$. In the d-dimensional case we conjecture that the upper bounds are $O((log(1/\varepsilon))^p)$, where $1 \leq p \leq d$.

Complexity and complexities of Monte Carlo algorithms for the Burgers equation

Nikolai Simonov

We consider one-dimensional Burgers equation

$$u_t + uu_x = \nu u_{xx} + f$$

and suppose that the source function f is not equal to zero. To solve the initial problem $u(x,0) = u_0(x)$ we apply the Monte Carlo method based on simulation of dynamics of a cloud of interacting particles. To take into account

the source term, we recalculate the weights of all the particles at every time step. The rate of convergence of such a procedure does not differ from that in the case $f = 0$. Let ε be the desired error threshold. Hence, the number of particles has to be taken proportional to ε^{-2} and K, number of time steps, proportional to ε^{-2} too. So, the computational cost of this Monte Carlo algorithm, which is equal to $O(N\ln NK)$, is proportional to $\varepsilon^{-4}\ln(\varepsilon)^{-1}$.

The situation radically changes when f is considered to be a random field. Implementation of spectral model representation for f into the algorithm leads to the need of $O(N(\ln N + M)K)$ arithmetic operations, M being the number of terms in the representation of f. It means that the computational cost of the Monte Carlo algorithm essentially depends on the spectral properties of the random source function.

On the construction of quasi-Monte Carlo algorithms that achieve strong QMC Tractability

Ian H. Sloan

It is by now well known that for the class of quasi-Monte Carlo (or QMC) algorithms, numerical integration over the d-dimensional unit cube is intractable in the (unweighted) Sobolev space of functions whose mixed first derivatives are square integrable . That is to say, the minimal number n of quadrature points in rules of the form

$$Q_{n,d} f = \frac{1}{n} \sum_{j=1}^{n} f(\mathbf{t}_j), \qquad \mathbf{t}_j \in [0,1]^d,\ j = 1,\ldots,n,$$

that are needed to reduce the initial worst-case error for functions in the unit ball by a factor of $\varepsilon > 0$ is **not** bounded by a polynomial in ε^{-1} and d. Indeed, the minimal number of quadrature points grows exponentially wi th d.

Recently it has been shown by Sloan and Wozniakowski that the story is quite different for 'weighted' Sobolev spaces in which the successive coordinate directions have 'weights' $\gamma_j,\ j = 1,\ldots,d$, with

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d \geq \cdots > 0,$$

provided that

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

If this condition is satisfied then the integration problem becomes 'strongly QMC tractable' (i.e. the minimal value of n is bounded independently of d), and indeed there exist QMC rules for which the minimal number of quadrature points is bounded abo ve by C/ε^2.

The original proof of Sloan and Wozniakowski was completely non-constructive, but a more recent proof shows that this bound (indeed, even the better bound $C/\varepsilon^{1-\delta}$ for arbitrary $\delta > 0$) can be achieved (if n is prime) by sequences o f rules $Q_{n,d}$ from the smaller class of 'shifted lattice rules'. However, a complete search is still infeasible for large d, because the number of lattice rules with fixed n grows exponentially in d.

Current work, jointly with F. Kuo and S. Joe, has made feasible the task of computing shifted lattice rules which achieve strong QMC tractability, by showing that good shifted lattice rules can be constructed by adding one component at a time, while all existing components are held unchanged.

The talk will outline all these ideas, up to the algorithm for computing good shifted lattice rules one component at a time.

Adaptive approach to the discretization of inverse problems

Sergei G. Solodky

We construct effective algorithms for solving inverse problems, that can be written in the form of operator equation of the first kind with a compact linear operator. These algorithms consist in the combination of our new projection

scheme, Morozov discrepancy principle and some regularization methods (namely, iterated Tikhonov and Landweber methods). Our projection scheme uses the idea of hyperbolic cross and adaptive approach to discretization. It is shown the advantage of proposed algorithms in comparison with standard methods.

Quasi-Monte Carlo for Yield Optimization in Circuit Design

Shu Tezuka

In this talk, I will discuss an application of Quasi-Monte Calro methods to yield optimization in digital circuit design. First, I overview Monte Carlo simulation in the design centering method for computing theoretical yields of digital circuits. I show that the yield can be written as multi-dimensional integration with Gaussian weights, where the integrand is an indicator function. Then, I apply generalized Faure sequences to a 17-dimensional integration problem associated with small size of flip-flop superconductive circuits, and show successful numerical results. Finally, some research issues are discussed.

Discrepancy theory and pair correlations

Robert F. Tichy

In the first part of the lecture a survey on recent developments of discrepancy theory is given. R. Baker's improvement of Roth's lower bound is discussed. Furthermore, applications of low discrepancy sequences to numerical integration are mentioned and various methods for the construction of low discrepancy sequences are presented. In particular, an improvement of Atanassov on the discrepancy of Hammersley sequences is mentioned and the construction of digital (t,m,s)-nets is sketched.

In the second part of the talk several metric and probabilistic discrepancy bounds are presented. Starting from classical results some very recent contributions to the discrepancy of pair correlations are discussed (joint work with I. Berkes and W. Philipp). In particular uniform error bounds are proved in the situation when the random variables satisfy an Erdős gap condition. In the case of independent random variables the bounds are sharp.

Oracle inequalities for inverse problems

Alexandre Tsybakov
joint work with L.Cavalier, Y.Golubev, and D.Picard

We consider a sequence space model of statistical linear inverse problems where we need to estimate a function f from indirect noisy

observations. Let a finite set Λ of linear estimators be given. Our aim is to mimic the estimator in Λ that has the smallest risk on the true f. Under general conditions, we show that this can be achieved by simple minimization of unbiased risk estimator, provided the singular values of the operator of the inverse problem decrease as a power law. The main result is a nonasymptotic oracle inequality that is shown to be asymptotically exact. This inequality can be also used to obtain sharp minimax adaptive results. In particular, we apply it to show that minimax adaptation on ellipsoids in multivariate anisotropic case is realized by minimization of unbiased risk estimator without any loss of efficiency with respect to optimal non-adaptive procedures.

On Weighted Approximation over Unbounded Domain

G. W. Wasilkowski

In this talk we will present recent results concerning the complexity of weighted approximation and weighted integration over $\mathbf{R}^d$. We will discuss both the worst case and the average case settings. In particular, we will present sharp complexity bounds and almost optimal algorithms for the approximation problem in the average case setting where the weight function has a tensor rpoduct norm and the probability measure is either isotropic or tensor product (e.g., r-fold Wiener) measure. This talk is a continuation of talks by Leszek Plaskota and Klaus Ritter where average case complexity results were presented for the univariate case.

How does tractability of multivariate integration depend on norm and periodicity?

Henryk Woźniakowski
joint work with Ian H. Sloan

We study strong tractability and tractability of multivariate integration for deterministic algorithms in the worst case setting and for the classical Monte Carlo algorithm in the randomized setting. This problem is considered in weighted tensor product reproducing kernel Hilbert spaces. We analyze three variants of the classical Sobolev space both for the non-periodic and periodic cases. We obtain necessary and sufficient conditions on strong tractability and tractability in terms of the weights of the spaces.

For the three Sobolev spaces, the conditions on the Monte Carlo strong tractability and tractability are more lenient than for deterministic algorithms. For general reproducing kernel Hilbert spaces, the opposite may happen as well.

For the three Sobolev spaces periodicity has no significant effect on strong tractability and tractability of deterministic algorithms, whereas for Monte Carlo it does. For general reproducing kernel Hilbert spaces, anything can happen: we may have strong tractability or tractability for the non-periodic case and intractability for the periodic one, or vice versa.