What do partial metrics represent?

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Abstract. Partial metrics were introduced in 1992[Ma94] as a metric to allow the distance of a point from itself to be non zero. This notion of self distance, designed to extend metrical concepts to Scott topologies as used in computing, has little intuition for the mainstream Hausdorff topologist. The talk will show that a partial metric over a set X can be represented by a metric over $X \cup \{\phi\}$, for a so-called *base point* ϕ. Thus we establish that a partial metric is essentially a structure combining both a metric space and a skewed view of that space from the base point. From this we can deduce what it is that partial metrics are really all about.

A **partial metric** is a function $p : X \times X \to [0, \infty)$ such that,
(1) $p(x, x) \leq p(x, y)$
(2) $p(x, y) = p(y, x)$
(3) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$
(4) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$

Partial metrics were introduced as a generalisation of the notion of *metric* to allow non zero self distance for the purpose of modelling *partial objects* in reasoning about data flow networks. The self distance $p(x, x)$ is to be understood as a quantification of the extent to which x is unknown.

From a partial metric $p : X \times X \to [0, \infty)$ we have each of the following constructions. A partial ordering $\sqsubseteq_p \; \subseteq \; X \times X$ given by, $x \sqsubseteq_p y \Leftrightarrow p(x, x) = p(x, y)$. A quasi-metric $q : X \times X \to [0, \infty)$ given by, $q(x, y) = p(x, y) - p(x, x)$. q yields the usual quasi-metric topology $\tau_q \subseteq 2^X$. A metric $d : X \times X \to [0, \infty)$ given by, $d(x, y) = 2 \times p(x, y) - p(x, x) - p(y, y)$. d yields the usual metric topology $\tau_d \subseteq 2^X$, being the join of τ_q and $\tau_{q^{-1}}$. Earlier work has established a contraction mapping theorem for partial metrics[Ma95] Partial metrizability into arbitrary value quantales has been defined[Kop04].

The starting point for our representation theorem is the observation that if $p : X \times X \to [0, \infty)$ has *bottom* $\perp \in X$ then $x \sqsubseteq_p y$ if and only if $d(\perp, y) = d(\perp, x) + d(x, y)$. Similarly, if p has *top* $\top \in X$ then $x \sqsubseteq_p y$ if and only if $d(x, \top) = d(x, y) + d(y, \top)$. Thus the order induced by a *bottomed* or *topped*

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partial metric is determined by the associated metric and a *base point*. This raises the question, is each partial metric determined by a metric and a base point? The answer is yes!

A **based metric** (or, **metric with base point**) is defined to be a pair $(d : X \times X \rightarrow [0, \infty), \phi \in X)$ such that d is a metric. The base point is introduced to facilitate asymmetric interpretation of metric spaces. For each based metric $M = (d, \phi \in X)$, $\sqsubseteq_M \subseteq X \times X$ is defined to be the relation such that, $x \sqsubseteq_M y$ iff $d(\phi, x) + d(x, y) = d(\phi, y)$. Then we can show that for each based metric $M = (d, \phi)$, $\sqsubseteq_M$ is a partial ordering with *bottom* ϕ. We can show that for each based metric $M = (d, \phi \in X)$, $q_M : X \times X \rightarrow [0, \infty)$ given by $q_M(x, y) = \frac{d(\phi,x)+d(x,y)-d(\phi,y)}{2}$ is a quasi-metric.

Now for three examples of based metrics. Let $M = (d : [0, \infty)^2 \rightarrow [0, \infty), 0)$ be the non negative real numbers under the usual metric with base point 0, then $\sqsubseteq_M$ is the usual $\leq$. For a second example, consider the usual metric over all the reals with base point 0, then the ordering is, $x \sqsubseteq_M y$ iff $y \leq x \leq 0$ or $0 \leq x \leq y$ where $\leq$ is the usual ordering on the reals. A third example, this time motivated by computer science, is a *flat* domain. For any set X and $\phi \in X$, let $d : X \times X \rightarrow \{0, 1, 2\}$ be the unique metric such that, $d(x, y) = 1$ if $x = \phi$ or $y = \phi$, 2 otherwise. Let $M = (d, \phi)$, then $x \sqsubseteq_M y$ iff $x = \phi$ or $x = y$.

A subset $A \subseteq X$ of a poset $(X, \sqsubseteq)$ is defined to be **directed** if it is non empty and each pair of elements of A has an upper bound in A[AJ94]. A **directed metric** is defined to be a based metric $(d, \phi \in X)$ such that d is complete, and for each directed set $A \subseteq X$ there exists $c_A < \infty$ such that for each $a \in A$, $d(\phi, a) < c_A$. For each directed metric $M = (d, \phi \in X)$ we can then prove each of the following. (1) $(X, \sqsubseteq_M)$ is a dcpo, (2) For each directed set A, $d(\phi, \bigsqcup^{\uparrow} A) = sup\{\ d(\phi, a) \mid a \in A\ \}$, (3) Each member of τ_{q_M} is Scott-open.

We now take the usual definitions for continuous domains. For each dcpo $(X, \sqsubseteq)$, $\ll \subseteq X \times X$ is the **way below** relation such that, $x \ll y$ iff for each directed set A, if $y \sqsubseteq \bigsqcup^{\uparrow} A$ then there exists $a \in A$ such that $x \sqsubseteq a$. A dcpo $(X, \sqsubseteq)$ is defined to be **continuous** if, for each $x \in X$ there exists directed set $A \subseteq \,\downarrow\!\!\!\downarrow x$ such that $\bigsqcup^{\uparrow} A = x$. Then we can prove that for each directed metric $M = (d, \phi \in X)$, (1) $x \ll y$ iff $y \in Int_{\tau_{q_M}}(\uparrow x)$, (2) $x \ll x$ iff there exists $\epsilon > 0$ such that $B^{\epsilon}_{q_M}(x) = \uparrow x$, (3) If $(X, \sqsubseteq_M)$ is a continuous dcpo then τ_{q_M} is the Scott topology.

Now we begin our representation of partial metrics. In general a partial metric need have neither a *bottom* nor a *top*, but each can be added. First we consider adding bottom. For each bounded partial metric $p : X \times X \rightarrow [0, \infty)$, and for some $\perp \notin X$, let $X_\perp = X \cup \{\perp\}$. Let, $\hat{p} = sup\{p(x, y) | x, y \in X\} + 1$. Let $p_\perp : X_\perp \times X_\perp \rightarrow [0, \infty)$ be such that,

$$p_\perp(x, y) \;=\; \begin{cases} p(x, y) & if\ x \in X\ and\ y \in X \\ \hat{p} & otherwise \end{cases}$$

Then $p_\perp$ is a partial metric, being p *lifted* to add $\perp$ as *bottom*.

Now we construct an order preserving representation for bounded partial metrics. Let $d : X_\perp \times X_\perp \rightarrow [0, \infty)$ be such that $d(x, y) = 2 \times p_\perp(x, y) -$

$p_\bot(x,x) - p_\bot(y,y)$. Then $M = (d, \bot_{p_\bot})$ is a based metric such that $\sqsubseteq_M = \sqsubseteq_{p_\bot}$. Let $S = (M, \hat{p})$. Let $p_S : X_\bot \times X_\bot \to [0,\infty)$ be such that,

$$p_S(x,y) \;=\; \hat{p} + \frac{d(x,y) - d(\bot_{p_\bot}, x) - d(\bot_{p_\bot}, y)}{2}$$

Then p_S is a partial metric such that $\sqsubseteq_{p_S} = \sqsubseteq_M$. Let $p' : (X_\bot - \{\bot_{p_S}\})^2 \to [0,\infty)$ be such that $p'(x,y) = p_S(x,y)$. Then $p' = p$. Note that $p(x,x) = \hat{p} - d(\bot, x)$. For a representation unique up to topological equivalence $\hat{p}$ can be replaced by a universal constant.

Now we try representing a partial metric using *top* as the base point. For each bounded partial metric $p : X \times X \to [0,\infty)$, and for some $\top \notin X$, let $X^\top = X \cup \{\top\}$. Let, $\hat{p} = sup\{p(x,y) | x,y \in X\} + 1$. Let $p^\top : X^\top \times X^\top \to [0,\infty)$ be such that,

$$p^\top(x,y) \;=\; \begin{cases} \hat{p} + p(x,y) & if\ x \in X\ and\ y \in X \\ \hat{p} + p(x,x) & if\ x \in X\ and\ y = \top \\ \hat{p} + p(y,y) & if\ x = \top\ and\ y \in X \\ 0 & if\ x = y = \top \end{cases}$$

Then $p^\top$ is a partial metric with *top* $\top$, and such that $\sqsubseteq_{p^\top}$ restricted to $X \times X$ is $\sqsubseteq_p$.

Now for an order reversing representation for bounded partial metrics Let $d : X^\top \times X^\top \to [0,\infty)$ be such that $d(x,y) = 2 \times p^\top(x,y) - p^\top(x,x) - p^\top(y,y)$. Then $M = (d,\ \top_{p^\top} \in X^\top)$ is a based metric such that $\sqsubseteq_M\; = \;\sqsupseteq_{p^\top}$. Let $p_M : X^\top \times X^\top \to [0,\infty)$ be such that,

$$p_M(x,y) = \frac{d(x,y) + d(\top_{p^\top}, x) + d(\top_{p^\top}, y)}{2}$$

Then p_M is a partial metric such that $\sqsubseteq_{p_M} = \sqsupseteq_M$. Let $c_{p_M} = \frac{sup\{p_M(x,y) | x,y \in X^\top\} + 1}{2}$. Let $p' : (X^\top - \{\top_{p_M}\})^2 \to [0,\infty)$ be such that, $p'(x,y) = p_M(x,y) - c_{p_M}$. Then $p' = p$. Note that $p(x,x) \;=\; d(\top, x) - c_{p_M}$.

Now for an order preserving representation for based metrics Let $M = (d, \phi \in X)$ be a based metric. Let $p_M : X \times X \to (-\infty,\infty)$ be such that $p_M(x,y) = \frac{d(x,y) - d(\phi,x) - d(\phi,y)}{2}$. Then p_M is a partial metric (generalised to negative distances) such that $\sqsubseteq_{p_M} = \sqsubseteq_M$. Let $d' : X \times X \to [0,\infty)$ be such that $d'(x,y) = 2 \times p_M(x,y) - p_M(x,x) - p_M(y,y)$ Then $(d',\ \bot_{p_M}) = M$. Note that $d(\phi, x) \;=\; -\, p_M(x,x)$. We can generalise partial metrics to allow negative distances as, for example, Simon O'Neil in his work on semi valuations.

Now for an order reversing representation for based metrics Let $M = (d, \phi \in X)$ be a based metric. Let $p_M : X \times X \to [0,\infty)$ be such that $p_M(x,y) = \frac{d(x,y) + d(\phi,x) + d(\phi,y)}{2}$ Then p_M is a partial metric such that $\sqsubseteq_{p_M}\; = \;\sqsupseteq_M$. Let $d' : X \times X \to [0,\infty)$ be such that $d'(x,y) = 2 \times p_M(x,y) - p_M(x,x) - p_M(y,y)$ Then $(d',\ \top_{p_M}) = M$. Note that $d(\phi, x) \;=\; p_M(x,x)$. Where our order preserving constructions use a partial metric p the order reversing counterpart uses the dual $p^*(x,y) = p(x,y) - p(x,x) - p(y,y)$. Note that $p^{**} = p$.

Why use based metrics? From a computer science perspective it is natural to ask, what is the partial metric distance between two order preserving functions? Such a partial metric is possible, but is now seen to be unnecessary. We can handle functions as follows using based metrics and a familiar metric construction. For based metrics $(d, \phi \in X)$ & $(d', \phi' \in X')$ such that d' is bounded, for $a_0, a_1, \ldots \in X$, and for a set of functions $X \to X'$ such that $f = g$ iff for all i, $f(a_i) = g(a_i)$ let $(D : (X \to X')^2 \to [0, \infty),\ \Phi \in X \to X')$ be the based metric such that,

$$D(f,g) = \sum_{i \geq 0} 2^{-i} \times d'(f(a_i), g(a_i))$$
$$\Phi = x \mapsto \phi'$$

Then $f \sqsubseteq g$ iff for all x, $f(x) \sqsubseteq g(x)$.

So, what do partial metrics represent? Our representation theorem suggests that the partial metric notion of *self distance* is more *presentational* than *substantial* as, (1) ϕ is neither necessarily $\bot$ nor $\top$, (2) *self distance* is reducible to *distance from base point*, (3) a partial metric over order preserving functions is reducible to a familiar metric construction. Now we can argue that a partial metric represents a combined metric space with an asymmetric 'perspective' determined by an explicit or implicit *base point*.

We conclude that based metrics embody notions of *order* and *topology* in the spirit of domain theory. This raises a number of questions. Which metric spaces become domains by identifying (or adding) a suitable base point? Based metrics can produce asymmetry without being founded upon asymmetric generalisations such as quasi-metrics? Which domains are *base point metrizable*? The proposal of *non zero self distance* introduced by partial metrics in 1992 has now matured into the conjecture that a metric space can be *observed*. What can be learnt from notions of *observation* used in process calculus and space-time physics?

References

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