Ranking Systems: The PageRank Axioms

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Abstract

This paper initiates research on the foundations of ranking systems, a fundamental ingredient of basic e-commerce and Internet Technologies. In order to understand the essence and the exact rationale of page ranking algorithms we suggest the axiomatic approach taken in the formal theory of social choice. In this paper we deal with PageRank, the most famous page ranking algorithm. We present a set of simple (graph-theoretic, ordinal) axioms that are satisfied by PageRank, and moreover any page ranking algorithm that does satisfy them **must** coincide with PageRank. This is the first representation theorem of that kind, bridging the gap between page ranking algorithms and the mathematical theory of social choice.

1 Introduction

The ranking of agents based on other agents' input is fundamental to e-commerce and multi-agent systems (see e.g. [4, 16]). Moreover, the ranking of agents based on other agents' input have become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google's PageRank algorithm[11] and ebay's reputation system[15]. One important set of such ranking systems are page ranking systems. It is well known that page ranking is fundamental for search technology, as well as for other applications. A major problem therefore is the study of the rationale of using a particular page ranking algorithm. What are the properties of a particular page ranking algorithm that characterize and differentiate it from other page ranking algorithms? In order to address this challenge we introduce and adapt the axiomatic approach, adopted in the mathematical theory of social choice, into the context of page ranking.

If we treat the Internet as a graph, where the nodes/pages are agents, and the links originating from node/page p define the preferences of the corresponding agent (i.e. a page that p links to is preferable to a page that p does not link to) then the page ranking problem becomes the problem of aggregating individual rankings into a global (social) ranking. Hence, the problem of page ranking becomes a (novel) problem of social choice. In the classical theory of social choice, as manifested by Arrow[1], a set of agents/voters is called to rank a set of alternatives. Given the agents' input, i.e. the agents' individual rankings, a social ranking of the alternatives is generated. The theory studies desired properties of the aggregation of agents' rankings into a social ranking. In particular, Arrow's celebrated impossibility theorem[1] shows that there is no aggregation rule that satisfies some minimal requirements, while by relaxing any of these requirements appropriate social aggregation rules can be defined. The novel feature of the page ranking setting is that the set of agents and the set of alternatives **coincide**. Therefore, in such setting one may need to consider the transitive effects of voting. For example, if agent (i.e. page) a reports on the importance of (i.e. links to)

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page b then this may influence the credibility of a report by b on the importance of agent c; these indirect effects should be considered when we wish to aggregate the information provided by the agents into a social ranking.

The theory of social choice is an axiomatic theory, and consists of two complementary perspectives:

- The normative perspective: devise a set of requirements that a social aggregation rule should satisfy, and try to find whether there is a social aggregation rule that satisfies these requirements.
- The descriptive perspective: given a particular algorithm r for the aggregation of individual rankings into a social ranking, then r satisfies many properties; the objective is to find a small set of simple properties(aka axioms) that are satisfied by r and has the additional feature that every algorithm that satisfies these properties **must** coincide with r. A result showing such a set of properties is termed a *representation theorem* and captures the exact essence of(and assumptions behind) the use of the particular algorithm.

An excellent example for the normative perspective is Arrow's impossibility theorem mentioned above. In [19] we presented such an approach for ranking systems. Many efforts have been invested in the descriptive approach in the framework of the classical theory of social choice. In that setting, representation theorems have been presented for classical voting rules such as the majority rule over two alternatives[8] (see [9] for an overview). Tackling the descriptive approach in the new Internet context, where the set of voters and the set of alternatives coincide (i.e. the page ranking context) remained an open major challenge.

In our work we address the above challenge by introducing a representation theorem for PageRank. Needless to say that PageRank[11] is the most famous page ranking procedure. In particular, PageRank is the basis for Google's search technology[1][2].If we treat the Internet as a strongly connected graph, where the nodes are the pages and the edges are links between pages, then PageRank can be defined as the limit probability distribution reached in a random walk on that graph. Roughly speaking, page p_1 will be ranked higher than page p_2 if the probability of reaching p_1 is greater than the probability of reaching p_2. We will show several simple properties (called axioms) one may require a page ranking algorithm to satisfy and prove that the PageRank algorithm does satisfy these axioms. Then, we prove our main result: any page ranking algorithm that does satisfy these axioms **must** coincide with PageRank!

The only work that we are familiar with which deals with a related axiomatization is the recent work on the axiomatization of citation indexes [12]. This work deals however with the case of numeric inputs (e.g. the inputs are not only graphs, as in page ranking, but include also numeric measures for the number of citations by each node, and by each node for each other node), and (most importantly) the axioms considered are numeric as well (e.g. when defining the axioms we are allowed for computations such as division or matrix multiplication). Our aim is quite different: we are after ordinal, graph-theoretic requirements that will provide sound and complete axiomatization for PageRank. This creates a most significant challenge: while the PageRank algorithm is numeric and is based on the computation of eigenvectors, we are after simple graph-theoretic properties that will fully characterize the related ranking procedure.

The classical theory of social choice lay the foundations to large part of the rigorous work on the design and analysis of social interactions. Indeed, the most classical results in the theory of mechanism design (e.g. the Gibbard-Satterthwaite [5, 17] theorems) are applications of the theory of social choice. While economic mechanism design had become an extensive line of study in computer science (see e.g. [10]) and electronic commerce (see e.g. [7, 13, 3]), our work introduces another connection between algorithms and Internet technologies to the mathematical theory of social choice.

In the next section we define our setting and some preliminaries, including the PageRank ranking system. In Section 3 we introduce five axioms one may require to hold for any page ranking procedure, and claim that

[1]In fact, ranking based on similar ideas can be found in other contexts as well. See [14] for the use of PageRank-like procedure in the comparison of journals' impact.

PageRank does satisfy these axioms. In Section 4 we show some useful properties implied by the axioms. In Section 5 we use these properties for proving that any page ranking procedure that does satisfy the axioms should coincide with PageRank. Further discussion of the approach taken in this paper is presented in Section 6. This paper is supplemented by an appendix which includes proofs of the theorems.

2 Page Ranking

The current practice of the ranking of Internet pages is based on the idea of computing the limit stationary probability distribution of a random walk on the Internet graph, where the nodes are pages, and the edges are links among the pages. In order for the result of that process will be well defined, we restrict our attention to strongly connected graphs:

Definition 2.1. A directed graph $G = (V, E)$ is called *strongly connected* if for all vertices $v_1, v_2 \in V$ there exists a path from v_1 to v_2 in E.

The output of a page ranking procedure can be viewed as a linear ordering of a set of alternatives:

Definition 2.2. Let A be some set. A relation $R \subseteq A \times A$ is called an *ordering on* A if it is reflexive, transitive, complete and anti-symmetric. Let $L(A)$ denote the set of orderings on A.

Notation: Let $\preceq$ be an ordering, then $\simeq$ is the equality predicate of $\preceq$. Formally, $a \simeq b$ if and only if $a \preceq b$ and $b \preceq a$.

Given the above we can define what a ranking system is:

Definition 2.3. Let $\mathbb{G}_V$ be the set of all strongly connected graphs with vertex set V. A *ranking system* F is a functional that for every finite vertex set V maps every strongly connected graph $G \in \mathbb{G}_V$ to an ordering $\preceq^F_G \in L(V)$.

In order to define the PageRank ranking system, we first recall the following standard definitions:

Definition 2.4. Let $G = (V, E)$ be a directed graph, and let $v \in V$ be a vertex in G. Then: The *successor set* of v is $S_G(v) = \{u|(v, u) \in E\}$, and the *predecessor set* of v is $P_G(v) = \{u|(u, v) \in E\}$.

We now define the PageRank matrix which is the matrix which captures the random walk created by the PageRank procedure. Namely, in this process we start in a random page, and iteratively move to one of the pages that are linked to by the current page, assigning equal probabilities to each such page.

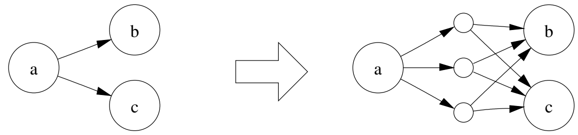
Definition 2.5. Let $G = (V, E)$ be a directed graph, and assume $V = \{v_1, v_2, \ldots, v_n\}$. the *PageRank Matrix* A_G (of dimension $n \times n$) is defined as:

$$[A_G]_{i,j} = \begin{cases} 1/|S_G(v_j)| & (v_j, v_i) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

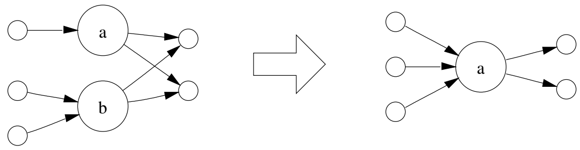
The PageRank procedure will rank pages according to the stationary probability distribution obtained in the limit of the above random walk; this is formally defined as follows:

Definition 2.6. Let $G = (V, E)$ be some strongly connected graph, and assume $V = \{v_1, v_2, \ldots, v_n\}$. Let $\mathbf{r}$ be the unique solution of the system $A_G \cdot \mathbf{r} = \mathbf{r}$ where $r_1 = 1$. The *PageRank* $PR_G(v_i)$ of a vertex $v_i \in V$ is defined as $PR_G(v_i) = r_i$. The *PageRank ranking system* is a ranking system that for the vertex set V maps G to $\preceq^{PR}_G$, where $\preceq^{PR}_G$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq^{PR}_G v_j$ if and only if $PR_G(v_i) \leq PR_G(v_j)$.

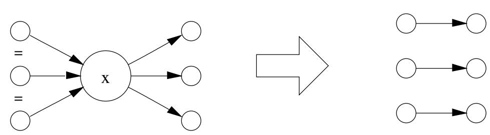
The above defines a powerful heuristic for the ranking of Internet pages, as adopted by search engines[11]. This is however a particular numeric procedure, and our aim is to treat it from an axiomatic social choice perspective, providing graph-theoretic, ordinal representation theorem for PageRank.



(a) Vote by Committee



(b) Collapsing



(c) Proxy

Figure 1: Sketch of several axioms

3 The Axioms

From the perspective of the theory of social choice, each page in the Internet graph is viewed as an agent, where this agent prefers the pages (i.e. agents) it links to upon pages it does not link to. The problem of finding a social aggregation rule will become therefore the problem of page ranking. The idea is to search for simple axioms, i.e. requirements we wish the page ranking system to satisfy. Most of these requirements will have the following structure: page a is preferable to page b when the graph is G if and only if a is preferable to b when the graph is G'. Our aim is to search for a small set of axioms that can be shown to be satisfied by PageRank. The axioms need to be simple graph-theoretic, ordinal properties, which do not refer to numeric computations.

In explaining some of the axioms we will refer to Figure 1. For simplicity, while the axioms are stated as "if and only if" statements, we will sometime emphasize in the intuitive explanation of an axiom only one of the directions (in all cases similar intuitions hold for the other direction).

The first axiom is straightforward:

Axiom 3.1. *(Isomorphism) A ranking system F satisfies* isomorphism *if for every isomorphism function $\varphi : V_1 \mapsto V_2$, and two isomorphic graphs $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq^F_{\varphi(G)} = \varphi(\preceq^F_G)$.*

The isomorphism axiom tells us that the ranking procedure should be independent of the names we choose for the vertices.

The second axiom is also quite intuitive. It tells us that if a is ranked at least as high as b if the graph is G, where in G a does not link to itself, then a should be ranked higher than b if all that we add to G is a link from a to itself. Moreover, the relative ranking of other vertices in the new graph should remain as before. Formally, we have the following notation and axiom:[2]

[2]One may claim that this axiom makes no sense if we do not allow self loops. This is however only a simple technical issue.

Notation: Let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $(v, v) \notin E$. Let $G' = (V, E \cup \{(v, v)\})$. Let us denote $\mathbf{SelfEdge}(G, v) = G'$ and $\mathbf{SelfEdge}^{-1}(G', v) = G$. Note that $\mathbf{SelfEdge}^{-1}(G', v)$ is well defined.

Axiom 3.2. *(Self edge) Let F be a ranking system. F satisfies the* self edge *axiom if for every vertex set V and for every vertex $v \in V$ and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $(v, v) \notin E$, and for every $v_1, v_2 \in V \setminus \{v\}$: Let $G' = \mathbf{SelfEdge}(G, v)$. If $v_1 \preceq^F_G v$ then $v \not\preceq^F_{G'} v_1$; and $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.*

The following, third axiom (titled *Vote by committee*) captures the following idea, which is illustrated in Figure 1(a). If page a links to pages b and c, then the relative ranking of all pages should be the same as in the case where the direct links from a to b and c are replaced by links from a to a new set of pages, which link (only) to b and c. The idea here is that the amount of importance a provides to b and c by linking to them, should not change due to the fact that a assigns its power through a committee of (new) representatives, all of which behave as a. More generally, and more formally, we have the following:

Axiom 3.3. *(Vote by committee) Let F be a ranking system. F satisfies* vote by committee *if for every vertex set V, for every vertex $v \in V$, for every graph $G = (V, E) \in \mathbb{G}_V$, for every $v_1, v_2 \in V$, and for every $m \in \mathbb{N}$: Let $G' = (V \cup \{u_1, u_2, \ldots, u_m\}, E \setminus \{(v, x) | x \in S_G(v)\} \cup \{(v, u_i) | i = 1, \ldots, m\} \cup \{(u_i, x) | x \in S_G(v), i = 1, \ldots, m\})$, where $\{u_1, u_2, \ldots, u_m\} \cap V = \emptyset$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.*

The 4th axiom, termed *collapsing* is illustrated in Figure 1(b). The idea of this axiom is that if there is a pair of pages, say a and b, where both a and b link to the same set of pages, but the sets of pages that link to a and b are disjoint, then if we collapse a and b into a singleton, say a, where all links to b become now links to a, then the relative ranking of all pages, excluding a and b of course, should remain as before. The intuition here is that if there are two voters (i.e. pages), a and b, who vote similarly (i.e. have the same outgoing links), and the power of each one of them stems from the fact a set of other voters have voted for him, where the sets of voters for a and for b are disjoint, then if all voters for a and b would vote only for a (dropping b) then a should provide the same importance to other agents as a and b did together. This of course relies on having a and b voting for the same individuals. As a result, the following axiom is quite intuitive:

Axiom 3.4. *(collapsing) Let F be a ranking system. F satisfies* collapsing *if for every vertex set V, for every $v, v' \in V$, for every $v_1, v_2 \in V \setminus \{v, v'\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $S_G(v) = S_G(v')$, $P_G(v) \cap P_G(v') = \emptyset$, and $[P_G(v) \cup P_G(v')] \cap \{v, v'\} = \emptyset$: Let $G' = (V \setminus \{v'\}, E \setminus \{(v', x) | x \in S_G(v')\} \setminus \{(x, v') | x \in P_G(v')\} \cup \{(x, v) | x \in P_G(v')\})$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.*

The last axiom we introduce, termed the *proxy* axiom, is illustrated in Figure 1(c). Roughly speaking, this axiom tells us that if there is a set of k pages, all having the same importance, which link to a, where a itself links to k pages, then if we drop a and connect directly, and in a 1-1 fashion, the pages which linked to a to the pages that a linked to, then the relative ranking of all pages (excluding a) should remain the same. This axiom captures equal distribution of importance. The importance of a is received from k pages, all with the same power, and is split among k pages; alternatively, the pages that link to a could pass directly the importance to pages that a link to, without using a as a proxy for distribution. More formally, and more generally, we have the following:

Axiom 3.5. *(proxy) Let F be a ranking system. F satisfies* proxy *if for every vertex set V, for every vertex $v \in V$, for every $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $|P_G(v)| = |S_G(v)|$, for all $p \in P_G(v)$: $S_G(p) = \{v\}$, and for all $p, p' \in P_G(v)$: $p \simeq^F_G p'$: Assume $P_G(v) = \{p_1, p_2, \ldots, p_m\}$ and $S_G(v) = \{s_1, s_2, \ldots, s_m\}$. Let $G' = (V \setminus \{v\}, E \setminus \{(x, v), (v, x) | x \in V\} \cup \{(p_i, s_i) | i \in \{1, \ldots, m\}\})$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.*

If we do not allow self loops then the axiom should be replaced by a new one, where the addition of self-loop to a is replaced by the addition of a new page, a', where a links to a' and where a' links only to a. Our results will remain similar.

3.1 Soundness

Although we have provided some intuitive explanation for the axioms, one may argue that particular axiom(s) are not that reasonable. As it turns out however, all the above axioms are satisfied by the PageRank procedure. The proof of the following basic (soundness) proposition appears in the appendix. In Section 5 we show that the above axioms are not only satisfied by PageRank, but also completely and uniquely characterize the PageRank procedure.

Proposition 3.6. *The PageRank ranking system PR satisfies isomorphism, self edge, vote by committee, collapsing, and proxy.*

4 Several Useful Properties

In this section we prove three technical properties which are implied by our axioms. As a result, these three properties are satisfied by the PageRank ranking system. The purpose of presenting them is rather technical: they will be used in the next section, when we show that the PageRank ranking system is the only one that satisfies our axioms.

Notation: Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$. We will use $\mathbf{Del}(G, v)$ to denote the graph $G' = (V', E')$ defined by:

$$
\begin{aligned}
V' &= V \setminus \{v\} \\
E' &= E \setminus \{(p, v), (v, s)\} \cup \{(p, s)\}.
\end{aligned}
$$

The $\mathbf{Del}(\cdot, \cdot)$ operator simply removes a vertex from the graph that has an in-degree and out-degree of 1, replacing it by an edge from its predecessor to its successor. The following lemma says that when our axioms are satisfied then this operator does not change the relative ranking of all (remaining) pages. The proof of this lemma appears in the appendix.

Definition 4.1. Let F be a ranking system. F has the *weak deletion* property if for every vertex set V, for every vertex $v \in V$ and for all vertices $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$: Let $G' = \mathbf{Del}(G, v)$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Lemma 4.2. *Let F be a ranking system that satisfies isomorphism, vote by committee and proxy. Then, F has the weak deletion property.*

We now move to a second deletion property satisfied by the axioms.

Notation: Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s_1, s_2, \ldots, s_t\}$ and $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, and $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$. We will use $\mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots, (s_t, \{p^i_t | i = 0, \ldots m\})\})$ to denote the graph $G' = (V', E')$ defined by:

$$
\begin{aligned}
V' &= V \setminus \{v\} \\
E' &= E \setminus \{(p^i_j, v), (v, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\
&\quad \cup \{(p^i_j, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\}.
\end{aligned}
$$

When the grouping of the predecessors is trivial or understood from context, we will sloppily use $\mathbf{Delete}(G, v)$.

A sketch of the Delete operator can be found in Figure 2. In this figure we see that node x which links to three other nodes, and has two sets of three predecessors, where the nodes in each such set are of the same

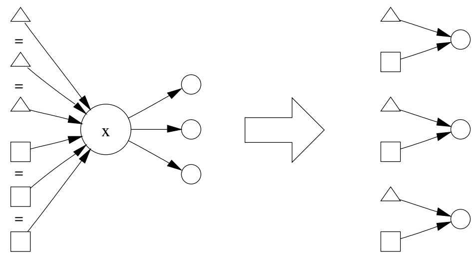


Figure 2: Sketch of **Delete**(G, x).

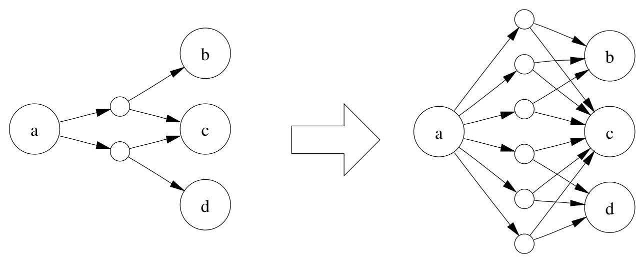


Figure 3: Sketch of **Duplicate**$(G, a, 3)$.

importance. The Delete operator will drop x and connect exactly one element from each of the predecessor sets to exactly one node in the successor set. The following lemma says that when our axioms are satisfied then this operator does not change the relative ranking of all (remaining) pages. The proof of this lemma appears in the appendix.

Definition 4.3. Let F be a ranking system. F has the *strong deletion* property if for every vertex set V, for every vertex $v \in V$, for all $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s_1, s_2, \ldots, s_t\}$, $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$, and $p^i_j \simeq^F_G p^i_k$ for all $i \in \{0, \ldots, m\}$ and $j, k \in \{1, \ldots t\}$: Let $G' = \mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots (s_t, \{p^i_t | i = 0, \ldots m\})\})$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Lemma 4.4. *Let F be a ranking system that satisfies collapsing and proxy. Then, F has the strong deletion property.*

We conclude with a third property which is also satisfied by the axioms.

Notation: Let V be a vertex set and let $G = (V, E) \in \mathbb{G}_V$ be a graph. Let $S(v) = \{s^0_1, s^0_2, \ldots, s^0_t\}$. We will use **Duplicate**(G, v, m) to denote the graph $G' = (V', E')$ defined by:

$$
\begin{aligned}
V' &= V \cup \{s^i_j | i = 1, \ldots, m-1; j = 1, \ldots t\} \\
E' &= E \cup \{(v, s^i_j) | i = 1, \ldots, m-1; j = 1, \ldots t\} \cup \\
&\quad \cup \{(s^i_j, u) | i = 1, \ldots, m-1; j = 1, \ldots t; u \in S_G(s^0_j)\}.
\end{aligned}
$$

A sketch of the Duplicate operator can be found in Figure 3. In this figure we see that a links to two nodes, each of which has its own successor set. Then, each node in the successor set of a is duplicated by a factor of three, i.e. for each node a' in the successor set of a we add two new nodes to the successor set of a, each of which with the same successor set as a'. The following lemma says that when our axioms are satisfied then this operator does not change the relative ranking of the pages, excluding the ones which have been duplicated. The proof appears in the Appendix.

Definition 4.5. Let F be a ranking system. F has the *edge duplication* property if for every vertex set V, for all vertices $v, v_1, v_2 \in V$, for every $m \in \mathbb{N}$, and for every graph $G = (V, E) \in \mathbb{G}_V$: Let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$, and let $G' = \mathbf{Duplicate}(G, v, m)$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Lemma 4.6. *Let F be a ranking system that satisfies isomorphism, vote by committee, collapsing, and proxy. Then, F has the edge duplication property.*

5 Completeness

We are now ready to show that that our axioms fully characterize the PageRank ranking system. We can prove:

Theorem 5.1. *A ranking system F satisfies isomorphism, self edge, vote by committee, collapsing, and proxy if and only if F is the PageRank ranking system.*

Given Proposition 3.6, it is enough to prove the following:

Proposition 5.2. *Let F_1 and F_2 be a ranking systems that have the weak deletion, strong deletion, and edge duplication properties, and satisfy the self edge and isomorphism axioms. Then, F_1 and F_2 are the same ranking system (notation: $F_1 \equiv F_2$).*

The proof of Proposition 5.2 is in the appendix. We shall now describe a sketch of the proof. The basic idea of the proof is to begin with a graph $G = (V, E)$ and two arbitrary vertices a and b in V, and manipulate G by applying $\mathbf{Del}(\cdot, \cdot)$, $\mathbf{Delete}(\cdot, \cdot, \cdot)$, $\mathbf{Duplicate}(\cdot, \cdot, \cdot)$, and $\mathbf{SelfEdge}(\cdot, \cdot)$ to achieve a new graph G_n for which F_1 and F_2 rank a and b the same as in G (Formally $a \preceq_{G_n}^F b \Leftrightarrow a \preceq_G^F b$ for $F \in \{F_1, F_2\}$). Afterwards, G_n is further manipulated to generate $G_{n+\delta}$ for which $a \simeq_{G_{n+\delta}}^F b$, but $a \preceq_{G_n}^F b \Rightarrow b \not\preceq_{G_{n+\delta}}^F a$ for $F \in \{F_1, F_2\}$ or vice versa (with a and b replaced). So, we conclude that $a \preceq_{G_n}^{F_1} b \Leftrightarrow a \preceq_{G_n}^{F_2} b$, and thus $a \preceq_G^{F_1} b \Leftrightarrow a \preceq_G^{F_2} b$.

The steps required to generate G_n from G, and then $G_{n+\delta}$ from G_n may be described algorithmically. These steps are illustrated in Figure 4:

1. Add a new vertex on every edge on the initial graph (Figure 4b), thus splitting each original edge into two new edges. These vertices do not change the relative ranking of a and b due to the weak deletion property.

2. If no original vertices exist in the graph except a and b, go to step 8. Otherwise, select an original vertex $x \notin \{a, b\}$ (in Figure 4 we start by selecting c).

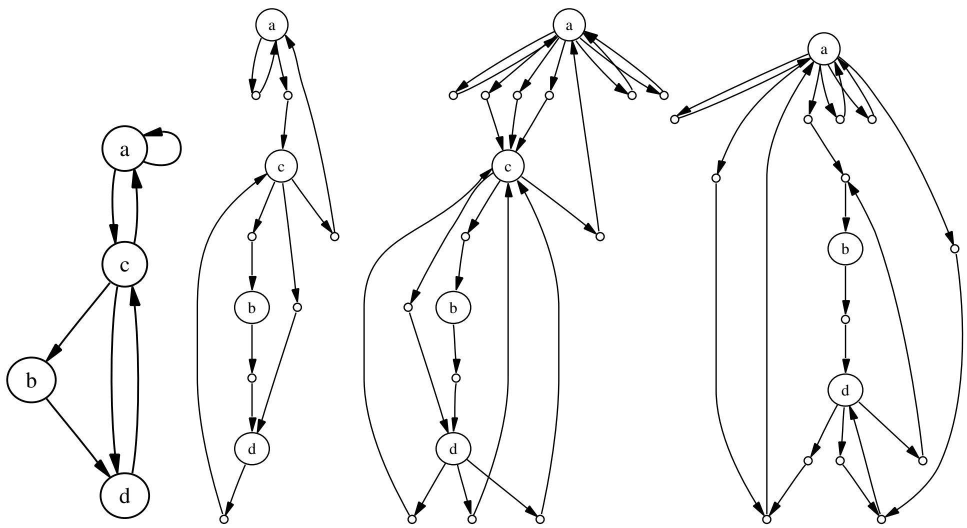
3. Remove all vertices that are both predecessors and successors of x and all edges connected to these vertices. All of these are new vertices, which have an in-degree and out-degree of 1.

   Basically, this step removes all self-edges of x (with an added vertex on them). These deletions do not change the relative ranking of a and b due to the weak deletion property and the self edge axiom.

4. Duplicate all predecessors of predecessors of x by x's out-degree. This does not change the relative ranking of a and b due to the duplication property (Figure 4c).

   Note that all the vertices we duplicate are original ones (possibly a or b, but not x), so to add additional in-between vertices before x, making the in-degree of x a multiple of its out degree, split into groups of isomorphic, and thus equally ranked, vertices.

5. Delete x using $\mathbf{Delete}(G, x)$ (Figure 4d).

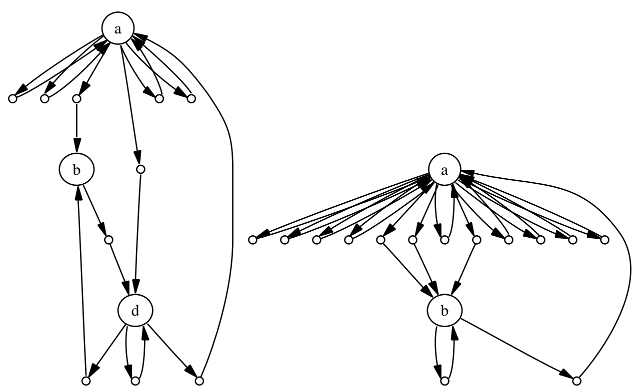


(a) Initial graph

(b) After adding vertices

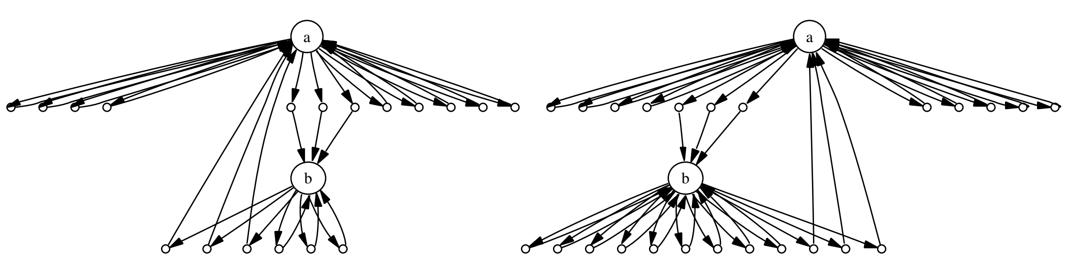
(c) After duplication of c's predecessors

(d) After deletion of c



(e) After deletion of c's intermediate successors

(f) After deletion of d



(g) After duplication of b

(h) Final isomorphic graph

Figure 4: Example run of the completeness algorithm. Here $a \not\preceq b$.

6. Delete the successors of x (new vertices) to retain the state of one new vertex between each pair of original vertices (Figure 4e). These deletions do not change the relative ranking of a and b due to the strong deletion property.

7. Go to step 2 (Figure 4f illustrates the second iteration, where d is selected).

8. Now, a and b are the only original vertices remaining in the graph, and the graph could be defined by the number of vertices (with edges) between a and b, between b and a, between a and a, and between b and b.

9. Duplicate a by the number of edges with vertices from b to a and vice versa, thus equalizing the number of edges with vertices from a to b the number from b to a (Figure 4g). This relative ranking between a and b is retained due to the duplication property.

10. Now, add self edges (with vertices) to the vertex $v \in \{a, b\}$ with fewer self-edges (with vertices), until the number of self edges is equal between a and b (Figure 4h). Let $v' = \{a, b\} \setminus \{v\}$. By the self edge axiom and the weak deletion property, if $v' \preceq^F v$ before adding the self edges, then now $v \not\preceq^F v'$ for $F \in \{F_1, F_2\}$.

11. By the isomorphism axiom, in this graph, $a \simeq b$, therefore in the graph after step 9, $v' \preceq^F v$ for $F \in \{F_1, F_2\}$. But as the relative ranking of a and b did not change until step 10, $v' \preceq^F_G v$ for $F \in \{F_1, F_2\}$, and thus $a \preceq^{F_1}_G b \Leftrightarrow a \preceq^{F_2}_G b$.

□

6 Discussion

Representation theorems are the formal mathematical tool for the justification of decision and choice rules. We have already mentioned the formal theory of social choice, but representation theorems also lay mathematical foundations for other branches of decision and choice theory. For example, the crowning achievement of the theory of (single-agent) choice is Savage's representation theorem [18], which provides sound and complete axiomatization for the expected utility maximization decision criterion. Here also one looks for ordinal requirements, which do not refer to numeric computations, under which an agent can be viewed as an expected utility maximizer. This is similar to our work, where we considered only graph-theoretic ordinal axioms to justify the numeric computations done by PageRank.

Although PageRank is probably the most popular page ranking procedure, it may be interesting to attempt and provide axiomatization for other page ranking procedures, such as Hubs and Authorities [6]. Once such axiomatization is found the different axiomatic systems can be compared as a basis for rigorous evaluation.

We believe that the problem of ranking of Internet pages is indeed a fundamental problem. We see the fact that this central problem is a new type of social choice problem as especially intriguing. In order to provide mathematical foundations to page ranking systems we therefore need to search for basic representation theorems that will provide ordinal, graph theoretic axiomatizations for basic heuristics and approaches for page ranking. Representation theorems isolate the "essence" of particular ranking systems, and provide means for the evaluation (and potentially comparison) of such systems. In this paper we initiated work on this topic by introducing such representation theorem for PageRank. We hope that others will join us in exploring the connections between page ranking algorithms and the mathematical theory of social choice.

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A Proofs

This section includes our proofs. These proofs are not part of the extended abstract, but may be used by the interested reviewer.

A.1 Proof of Proposition 3.6

Proof. The isomorphism axiom is satisfied directly from the definition by the assumption that $V = \{v_1, v_2, \ldots, v_n\}$.

For the vote by committee axiom, let $V = \{v_1, v_2, \ldots, v_n\}$ be a vertex set, let $G = (V, E) \in \mathbb{G}_V$ be a graph, and let $v_s, v_t \in V$ be vertices and let $m \in \mathbb{N}$ be a natural number. Assume $v_s \preceq_G^{PR} v_t$.

Let $G' = (V \cup \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\}, E \setminus \{(v_1, x) | x \in S_G(v_1)\} \cup \{(v_1, v_{n+j}) | j = 1, \ldots, m\} \cup \{(v_{n+j}, x) | x \in S_G(v_1), j = 1, \ldots, m\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $\mathbf{r}'$ be the following vector:

$$\mathbf{r}' = \begin{pmatrix} r_1 \\ \vdots \\ r_n \\ r_1/m \\ \vdots \\ r_1/m \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r}' = \mathbf{r}'$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,n} & a_{1,1} & \cdots & a_{1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,2} & \cdots & a_{n,n} & a_{n,1} & \cdots & a_{n,1} \\ 1/m & & & & & & \\ \vdots & & & 0 & & & \\ 1/m & & & & & & \end{pmatrix}$$

If we multiply, we get: for $i \in \{1, \ldots n\}$:

$$[A_{G'} r']_i \quad = \quad \sum_{j=2}^{n} a_{i,j} r_j + m a_{i,1} \cdot r_1/m = \sum_{j=1}^{n} a_{i,j} r_j = r_i,$$

and for $i \in \{n+1, \ldots n+m\}$, $[A_{G'} r']_i = 1/m \cdot r_1$, as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n+m\}$. Now, $PR_{G'}(v_s) = r'_s = r_s = PR_G(v_s) \leq PR_G(v_t) = r_t = r'_t = PR_{G'}(v_t)$, as required.

For the collapsing axiom, let $V = \{v_1, v_2, \ldots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $S(v_n) = S(v_{n-1})$ and $P(v_n) \cap P(v_{n-1}) = \emptyset$. Let $v_k, v_l \in V$ be vertices ($k, l < n-1$). Assume $v_k \preceq_G^{PR} v_l$.

Let $G' = (V \setminus \{v_n\}, E \setminus \{(v_n, x) | x \in S_G(v_n)\} \setminus \{(x, v_n) | x \in P_G(v_n)\} \cup \{(x, v_{n-1}) | x \in P_G(v_n)\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $\mathbf{r}'$ be the following vector:

$$\mathbf{r}' = \begin{pmatrix} r_1 \\ \vdots \\ r_{n-2} \\ r_{n-1} + r_n \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r'} = \mathbf{r'}$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \\ a_{n-1,1}+a_{n,1} & a_{n-1,2}+a_{n,2} & \cdots & 0 \end{pmatrix}$$

If we multiply, we get for $i \in \{1, \ldots n-2\}$:

$$[A_{G'}r']_i = a_{i,n-1}(r_n + r_{n-1}) + \sum_{j=1}^{n-2} a_{i,j}r_j = a_{i,n-1}r_n + a_{i,n-1}r_{n-1} + \sum_{j=1}^{n-2} a_{i,j}r_j$$

Note that $a_{i,n} = a_{i,n-1} = \frac{1}{|S(v_n)|}$, so

$$\begin{aligned} [A_{G'}r']_i &= \sum_{j=1}^{n-2} a_{i,j}r_j + a_{i,n-1}r_{n-1} + a_{i,n}r_n = \sum_{j=1}^{n} a_{i,j}r_j = r_i. \\ [A_{G'}r']_{n-1} &= \sum_{j=1}^{n-2}(a_{n-1,j} + a_{n,j})r_j = \sum_{j=1}^{n-2} a_{n-1,j}r_j + \sum_{j=1}^{n-2} a_{n,j}r_j \end{aligned}$$

Note that $a_{n-1,n-1} = a_{n-1,n} = a_{n,n-1} = a_{n,n} = 0$, so

$$[A_{G'}r']_{n-1} = \sum_{j=1}^{n} a_{n-1,j}r_j + \sum_{j=1}^{n} a_{n,j}r_j = r_{n-1} + r_n$$

So, we get $A_{G'}\mathbf{r'} = \mathbf{r'}$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n-1\}$. Now, $PR_{G'}(v_k) = r'_k = r_k = PR_G(v_k) \le PR_G(v_l) = r_l = r'_l = PR_{G'}(v_l)$, as required.

For the proxy axiom, let $V = \{v_1, v_2, \ldots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $P(v_n) = \{v_1, v_2, \ldots, v_m\}$, $v_1 \simeq v_2 \simeq \cdots \simeq v_m$, and $S(v_n) = \{v_{t+1}, v_{t+2}, \ldots, v_{t+m}\}$, where $t \in \{0, \ldots, m\}$. Let $v_k, v_l \in V$ be vertices $(k, l < n)$. Assume $v_k \preceq_G^{PR} v_l$.

Let $G' = (V \setminus \{v_n\}, E \setminus \{(x, v_n), (v_n, x) | x \in V\} \cup \{(v_i, v_{t+i}) | i \in \{1, \ldots, m\}\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Since $v_1 \simeq v_2 \simeq \cdots \simeq v_m$, we have $r_1 = r_2 = \cdots = r_m$, and note that because $P_G(v_n) = \{v_1, v_2, \ldots, v_m\}$ and $S(v_i) = \{v_n\}$ for all $i \in \{1, \ldots, m\}$:

$$r_n = \sum_{i=1}^{n} a_{n,i}r_i = r_1 + r_2 + \cdots + r_m = mr_1 = m.$$

Let $\mathbf{r'} = \mathbf{r_{-n}}$. By definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{t,m+1} & a_{t,m+2} & \cdots & a_{t,n-1} \\ 1 & 0 & \cdots & 0 & a_{t+1,m+1} & a_{t+1,m+2} & \cdots & a_{t+1,n-1} \\ 0 & 1 & \cdots & 0 & a_{t+2,m+1} & a_{t+2,m+2} & \cdots & a_{t+2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{t+m,m+1} & a_{t+m,m+2} & \cdots & a_{t+m,n-1} \\ 0 & 0 & \cdots & 0 & a_{t+m+1,m+1} & a_{t+m+1,m+2} & \cdots & a_{t+m+1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1,m+1} & a_{n-1,m+2} & \cdots & a_{n-1,n-1} \end{pmatrix}$$

We multiply can now multiply, and since $a_{i,n} = 0$ for all $i \in \{1, \dots t, t+m+1, \dots, n-1\}$ (because $S(v_n) = \{t+1, \dots, t+m\}$) and $a_{i,j} = 0$ for all $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, m\}$ (because $S(v_j) = \{v_n\}$), we get for $i \in \{1, \dots t, t+m+1, \dots, n-1\}$:

$$[A_{G'}r']_i = \sum_{j=m+1}^{n-1} a_{i,j} r_j = \sum_{j=1}^{n} a_{i,j} r_j = r_i$$

and for $i \in \{t+1, \dots, t+m\}$:

$$\begin{aligned}[A_{G'}r']_i &= \sum_{j=m+1}^{n-1} a_{i,j} r_j + r_{i-t} = \sum_{j=1}^{n-1} a_{i,j} r_j + 1 = \sum_{j=1}^{n-1} a_{i,j} r_j + \frac{1}{m} r_n = \\ &= \sum_{j=1}^{n-1} a_{i,j} r_j + a_{i,n} r_n = \sum_{j=1}^{n} a_{i,j} r_j = r_i\end{aligned}$$

So, we get $A_{G'}\mathbf{r'} = \mathbf{r'}$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \dots, n-1\}$. Now, $PR_{G'}(v_k) = r'_k = r_k = PR_G(v_k) \le PR_G(v_l) = r_l = r'_l = PR_{G'}(v_l)$, as required.

For the self edge axiom, let $V = \{v_1, v_2, \dots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $(v_1, v_1) \notin E$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $G' = (V, E \cup \{(v_1, v_1)\})$ and let $m = |S_G(v_1)|$. Let $\mathbf{r'}$ be the following vector:

$$\mathbf{r'} = \begin{pmatrix} r_1 \\ \frac{m}{m+1} r_2 \\ \vdots \\ \frac{m}{m+1} r_n \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r'} = \mathbf{r'}$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} \frac{1}{m+1} & a_{1,2} & \cdots & a_{1,n} \\ \frac{m}{m+1} a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m}{m+1} a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

If we multiply, we get: for $i \in \{2, \dots n\}$:

$$\begin{aligned}[A_{G'}r']_1 &= \frac{1}{m+1} r_1 + \sum_{j=2}^{n} a_{1,j} \frac{m}{m+1} r_j = \frac{1}{m+1} r_1 + \frac{m}{m+1} \sum_{j=2}^{n} a_{1,j} r_j = \\ &= \frac{1}{m+1} r_1 + \frac{m}{m+1} \sum_{j=1}^{n} a_{1,j} r_j = \frac{1}{m+1} r_1 + \frac{m}{m+1} r_1 = r_1 \\ [A_{G'}r']_i &= \frac{m}{m+1} a_{i,1} r_1 + \sum_{j=2}^{n} a_{i,j} \frac{m}{m+1} r_j = \frac{m}{m+1} \sum_{j=1}^{n} a_{i,j} r_j = \frac{m}{m+1} r_i\end{aligned}$$

So, we get $A_{G'}\mathbf{r'} = \mathbf{r'}$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \dots, n-1\}$.

Assume $v_2 \preceq_G^{PR} v_1$. Then, $PR_{G'}(v_2) = r'_2 < r_2 = PR_G(v_2) \le PR_G(v_1) = r_1 = r'_1 = PR_{G'}(v_1)$, as required.

Now assume $v_2 \preceq_G^{PR} v_3$. Then, $PR_{G'}(v_2) = r'_2 = r_2 = PR_G(v_2) \le PR_G(v_3) = r_3 = r'_3 = PR_{G'}(v_3)$, as required. □

A.2 Proof of Lemma 4.2

Proof. Let V be a vertex set, let $v \in V; v_1, v_2 \in V \setminus \{v\}$ be vertices and let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$. Assume $v_1 \preceq^F_G v_2$. Let $s_0 = v$ and $S(p) = \{s_0, s_1, s_2, \ldots, s_m\}$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{p'\} \\ E_1 &= E \setminus \{(p, s_i) | i = 0, \ldots, m\} \cup \{p, p'\} \cup \\ &\quad \cup \{(p', s_i) | i = 0, \ldots, m\}. \end{aligned}$$

  By the vote by committee axiom with parameter 1, $v_1 \preceq^F_{G_1} v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \cup \{u_i | i = 0, \ldots, m\} \\ E_2 &= E_1 \setminus \{(p, p')\} \cup \\ &\quad \cup \{(p, u_i), (u_i, p') | i = 0, \ldots, m\}. \end{aligned}$$

  By the vote by committee axiom with parameter $m + 1$, $v_1 \preceq^F_{G_2} v_2$.

- Let $G_3 = (V_3, E_3)$, where

$$\begin{aligned} V_3 &= V_2 \setminus \{p'\} \\ E_3 &= E_2 \setminus \{(u_i, p'), (p', s_i) | i = 0, \ldots, m\}. \\ &\quad \cup \{(u_i, s_i) | i = 0, \ldots, m\}. \end{aligned}$$

  By the isomorphism axiom, $u_i \simeq_{G_2} u_j$ for all $i, j \in \{0, \ldots, m\}$. By the proxy axiom, $v_1 \preceq^F_{G_3} v_2$.

- Let $G_4 = (V_4, E_4)$, where

$$\begin{aligned} V_4 &= V_3 \setminus \{v\} \\ E_4 &= E_3 \setminus \{(u_0, v), (v, s)\} \cup \{(u_0, s)\}. \end{aligned}$$

  By the vote by committee axiom with parameter 1, $v_1 \preceq^F_{G_4} v_2$.

- Let $G' = \mathbf{Del}(G, v)$. By the vote by committee, isomorphism, and proxy axioms, as between G and G_3 above, $v_1 \preceq^F_{G'} v_2 \Leftrightarrow v_1 \preceq^F_{G_4} v_2$. Thus, $v_1 \preceq^F_{G'} v_2$ as required.

□

A.3 Proof of Lemma 4.4

Proof. Let V be a vertex set, let $v \in V; v_1, v_2 \in V \setminus \{v\}$ be vertices and let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $S(v) = \{s_1, s_2, \ldots, s_t\}$, $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$, and $p^i_j = p^k_j$ for all $j \in \{1, \ldots t\}$ and $i, k \in \{0, \ldots, m\}$. Assume $v_1 \preceq^F_G v_2$. Denote $u^0 = v$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{u^i | i = 1, \ldots, m\} \\ E_1 &= E \setminus \{(p^i_j, v) | i = 1, \ldots, m; j = 1, \ldots, t\} \cup \\ &\quad \cup \{(p^i_j, u^i), (u^i, s_j) | i = 1, \ldots, m; j = 1, \ldots, t\} \end{aligned}$$

  By the collapsing axiom applied in the reverse direction a total of m times for $\{(u^{i-1}, u^i) | i = 1, \ldots, m\}$, $v_1 \preceq^F_{G_1} v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \setminus \{u^i | i = 0, \ldots, m\} \\ E_2 &= E_1 \setminus \{(p^i_j, u^i), (u^i, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\ &\cup \{(p^i_j, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\}. \end{aligned}$$

  By the proxy axiom applied a total of $m + 1$ times for $\{u^i | i = 0, \ldots, m\}$, $v_1 \preceq^F_{G_2} v_2$.

Note that G_2 is exactly $G' = \mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots (s_t, \{p^i_t | i = 0, \ldots m\})\})$, so $v_1 \preceq^F_{G'} v_2$ as required.

□

A.4 Proof of Lemma 4.6

Proof. Let V be a vertex set, let $v, v_1, v_2 \in V$ be vertices, and let $m' \in \mathbb{N}$ be a natural number. Assume $m' > 1$ (otherwise $G' = G$), and let $m = m' - 1$. Let $G = (V, E) \in \mathbb{G}_V$ be a graph. Assume $v_1 \preceq^F_G v_2$, and let $S(v) = \{s^0_1, s^0_2, \ldots, s^0_t\}$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{u^i_j | i = 0, \ldots, m; j = 1, \ldots t\} \\ E_1 &= E \setminus \{(v, x) | x \in S_G(v)\} \cup \{(v, u^i_j) | i = 0, \ldots, m; j = 1, \ldots t\} \cup \\ &\cup \{(u^i_j, x) | x \in S_G(v), i = 0, \ldots, m; j = 1, \ldots t\}. \end{aligned}$$

  By the vote by committee axiom with parameter $(m + 1)t$, $v_1 \preceq^F_{G_1} v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \cup \{w^i_j | i = 0, \ldots, m; j = 1, \ldots t\} \\ E_2 &= E_1 \setminus \{(v, u^i_j) | i = 0, \ldots, m; j = 1, \ldots t\} \cup \\ &\cup \{(v, w^i_j), (w^i_j, u^i_j) | i = 0, \ldots, m; j = 1, \ldots t\} \end{aligned}$$

  By the vote by committee axiom (applied $(m - 1)t$ times) with parameter 1, $v_1 \preceq^F_{G_2} v_2$.

- Let $G_3 = (V_3, E_3)$, where

$$\begin{aligned} V_3 &= V_2 \setminus \{u^i_j | i = 0, \ldots, m; j = 2, \ldots, t\} \\ E_3 &= E_2 \setminus \{(u^i_j, x | x \in S_G(v); i = 0, \ldots, m; j = 2, \ldots, t\} \setminus \\ &\setminus \{(w^i_j, u^i_j) | i = 0, \ldots, m; j = 2, \ldots, t\} \cup \\ &\cup \{(w^i_j, u^i_1) | i = 0, \ldots, m; j = 2, \ldots, t\}. \end{aligned}$$

  By the collapsing axiom applied a total of $(m+1)(t-1)$ times for $\{(u^i_{j-1}, u^i_j) | j = 2, \ldots, t; i = 0, \ldots m\}$, $v_1 \preceq^F_{G_3} v_2$.

- Let $G_4 = (V_4, E_4)$, where

$$\begin{aligned} V_4 &= V_3 \setminus \{u^i_1 | i = 0, \ldots, m\} \\ E_4 &= E_3 \setminus \{(u^i_1, x) | i = 0, \ldots, m; x \in S_G(v)\} \setminus \\ &\setminus \{(w^i_j, u^i_1) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\ &\cup \{(w^i_j, s^0_j) | i = 0, \ldots, m; j = 1, \ldots, t\}. \end{aligned}$$

By the isomorphism axiom, $w_j^i \simeq w_k^i$ for all $i \in \{0,\ldots,m\}$ and $j,k \in \{1,\ldots,t\}$. By the proxy axiom (applied a total of $m+1$ times for $\{u_1^i | i = 0,\ldots m\}$), $v_1 \preceq_{G_4}^F v_2$.

- Let $G_5 = (V_5, E_5)$, where

$$\begin{aligned} V_5 &= V_4 \cup \{s_j^i | i = 1,\ldots,m; j = 1,\ldots,t\} \\ E_5 &= E_4 \setminus \{(w_j^i, s_j^0) | i = 1,\ldots,m; j = 1,\ldots,t\} \cup \\ &\cup \{(w_j^i, s_j^i) | i = 1,\ldots,m; j = 1,\ldots,t\} \cup \\ &\cup \{(s_j^i, x) | x \in S(s_j^0); i = 1,\ldots,m\}. \end{aligned}$$

  By the collapsing axiom applied in the reverse direction a total of $m \cdot t$ times for $\{(s_j^{i-1}, s_j^i) | i = 1,\ldots,m; j = 1,\ldots,t\}$, $v_1 \preceq_{G_5}^F v_2$.

- Let $G_6 = (V_6, E_6)$, where

$$\begin{aligned} V_6 &= V_5 \setminus \{w_j^i | i = 0,\ldots,m; j = 1,\ldots t\} \\ E_6 &= E_5 \setminus \{(v, w_j^i), (w_j^i, s_j^i) | i = 0,\ldots,m; j = 1,\ldots t\} \cup \\ &\cup \{(v, s_j^i) | i = 0,\ldots,m; j = 1,\ldots t\}. \end{aligned}$$

  By the vote by committee axiom applied in the reverse direction a total of $(m+1) \cdot t$ times for $\{w_j^i | i = 0,\ldots,m; j = 1,\ldots t\}$, $v_1 \preceq_{G_6}^F v_2$.

Note that G_6 is exactly $\mathbf{Duplicate}(G, v, m+1) = \mathbf{Duplicate}(G, v, m') = G'$, so $v_1 \preceq_{G'}^F v_2$ as required.

□

A.5 Proof of Proposition 5.2

Proof. Let V be a vertex set and let $G = (V, E) \in \mathbb{G}_V$ be some graph. If $|V| = 1$, then there exists only one ordering on V, so trivially $\preceq_G^{F_1} \equiv \preceq_G^{F_2}$. Assume $V = \{v_1, v_2, \ldots, v_n\}$. We will show that $v_1 \preceq_G^{F_1} v_2 \Leftrightarrow v_1 \preceq_G^{F_2} v_2$. Without loss of generality we can show only one direction. Let $F \in \{F_1, F_2\}$.

Let $G_2 = (V_2, E_2)$ be the following graph (G with a vertex added on every edge):

$$\begin{aligned} V_2 &= V \cup \{u_{i,j} | (v_i, v_j) \in E\} \\ E_2 &= \{(v_i, u_{i,j}), (u_{i,j}, v_j) | (v_i, v_j) \in E\}. \end{aligned}$$

Note that

$$G = \mathbf{Del}(\mathbf{Del}(\cdots \mathbf{Del}(G_2, u_1) \cdots, u_{|E|-1}), u_{|E|})$$

where $\{u_1, \ldots, u_{|E|}\} = \{u_{i,j} | (v_i, v_j) \in E\}$ and that G_2 satisfies the conditions of weak deletion property for the vertices $\{u_{i,j} | (v_i, v_j) \in E\}$, thus $v_1 \preceq_G^F v_2 \Leftrightarrow v_1 \preceq_{G_2}^F v_2$.

For all strongly connected directed graphs G' such that for all $v \in V$ and for all $v' \in P_{G'}(v) \cup S_{G'}(v)$ s.t. $|S_{G'}(v')| = |P_{G'}(v')| = 1$, let us denote for all $v \in V$: $S_G^2(v) = \{v' \in V : x \in S_{G'}(v), S_{G'}(x) = \{v'\}\}$ and $P_G^2(v) = \{v' \in V : x \in P_{G'}(v), P_{G'}(x) = \{v'\}\}$.

For, $i = 3, \ldots, n$, we recursively define G_i as follows: Let $\{q_1, q_2, \ldots, q_m\} = S_{G_{i-1}}(v_i) \cap P_{G_{i-1}}(v_i)$. Let G'_{i-1} be the graph

$$G'_{i-1} = \mathbf{SelfEdge}^{-1}(\mathbf{Del}(\cdots \mathbf{SelfEdge}^{-1}(\mathbf{Del}(G_{i-1}, q_1), v_i) \cdots, q_m), v_i).$$

Now, let $P^2_{G'_{i-1}}(v) = \{p_1, \ldots, p_k\}$. and let $S_{G'_{i-1}}(v_i) = \{s_1, s_2, \ldots, s_l\}$. Let G''_{i-1} be defined as:

$$G''_{i-1} = \mathbf{Duplicate}(\cdots \mathbf{Duplicate}(G'_{i-1}, p_1, l) \cdots, p_k, l)$$

Let $\{p^i_j | i = 1, \ldots, l\} = S_{G''_{i-1}}(p_j)$ be the duplicated successors of p_j for $j = 1 \ldots k$. Now let $G_i = (V_i, E_i)$ be defined as:

$$\begin{aligned} G'''_{i-1} &= \mathbf{Delete}(G'_{i-1}, v_i, \{(s_1, \{p^1_j | j = 1, \ldots, k\}), \ldots, (s_l, \{p^l_j | j = 1, \ldots, k\})\}) \\ G_i &= \mathbf{Delete}(\cdots \mathbf{Delete}(\mathbf{Delete}(G'''_{i-1}, s_1), s_2) \cdots, s_l). \end{aligned}$$

By the edge duplication and strong deletion properties and the self edge axiom, $v_1 \preceq^F_{G_i} v_2$ for all $i \in \{2, \ldots, n\}$.

We will now prove that for all $i \in \{2, \ldots, n\}$ and for all $v \in V_i \setminus V$: $|P_{G_i}(v)| = |S_{G_i}(v)| = 1$ and $P_{G_i}(v) \cup S_{G_i}(v) \subseteq V$ and for all $v \in V$: $(P_{G_i}(v) \cup S_{G_i}(v)) \cap V = \emptyset$. Proof by induction: G_2 trivially satisfies both requirements. Now assume that for all $v \in V_i \setminus V$: $|P_{G_i}(v)| = |S_{G_i}(v)| = 1$ and $P_{G_i}(v) \cup S_{G_i}(v) \subseteq V$ and for all $v \in V$: $(P_{G_i}(v) \cup S_{G_i}(v)) \cap V = \emptyset$. Clearly, G'_i satisfies the conditions, because we only removed elements from V_i, and not changed the predecessors or successors of any $v \in V \setminus V_i$. Also, all edges added between vertices in V were removed. The $\mathbf{Duplicate}(\cdot, \cdot, \cdot)$ operation adds vertices with in-degree 1 and out-degree equal to the out degree of the successors of v, which is also 1. So, the new vertices added in G''_i satisfy the conditions. Furthermore, no edges were added between elements of V. Thus, G''_i satisfies the conditions. In G_{i+1}, we removed v and all its successors. The predecessors of v in G''_i keep their out-degree 1, and point to elements of $S^2_{G''_i}(v)$, and thus still meet the requirements. Other elements of $V''_i \setminus V$ have not changed their edges, and thus still meet the requirements. Still, no edges were added between elements of V. Therefore, for all $v \in V_{i+1} \setminus V$: $|P_{G_{i+1}}(v)| = |S_{G_{i+1}}(v)| = 1$ and $P_{G_{i+1}}(v) \cup S_{G_{i+1}}(v) \subseteq V$ and for all $v \in V$: $(P_{G_{i+1}}(v) \cup S_{G_{i+1}}(v)) \cap V = \emptyset$.

Specifically, this is true for $G_n = (V_n, E_n)$. Furthermore, $V_n \cap V = \{v_1, v_2\}$. Thus, G_n could be described as:

$$\begin{aligned} V_n &= \{v_1, v_2\} \cup \{v^i_{jk} | j, k \in \{1, 2\}; i = 1, \ldots, n_{jk}\} \\ E_n &= \{(v_j, v^i_{jk}), (v^i_{jk}, v_k) | j, k \in \{1, 2\}; i = 1, \ldots, n_{jk}\}. \end{aligned}$$

The only parameters which affect the structure of G_n are n_{jk} ($j, k \in \{1, 2\}$), so we can denote $G_n = \mathbf{G}[n_{11}, n_{12}, n_{21}, n_{22}]$. Now, let

$$\begin{aligned} G'_n &= \mathbf{Duplicate}(\mathbf{Duplicate}(G_n, v_1, n_{21}), v_2, n_{12}) \\ &= \mathbf{G}[n_{21}n_{11}, n_{21}n_{12}, n_{12}n_{21}, n_{12}n_{22}]. \end{aligned}$$

By the edge duplication property, $v_1 \preceq^F_G v_2 \Leftrightarrow v_1 \preceq^F_{G'_n} v_2$.

Consider the following 3 cases:

- If $n_{21}n_{11} = n_{12}n_{22}$, then the graph is isomorphic to itself, replacing v_1 with v_2 and v^i_{jk} with v^i_{kj}. In this case, by the isomorphism axiom, $v_1 \simeq^F_{G'_n} v_2$ and thus $v_1 \simeq^F_G v_2$, and therefore $v_1 \preceq^F_G v_2$ for $F \in \{F_1, F_2\}$.

- If $n_{21}n_{11} > n_{12}n_{22}$, let $\delta = n_{21}n_{11} - n_{12}n_{22} > 0$. Now we define for $i = n + 1, \ldots n + \delta$:

$$\begin{aligned} G'_i &= \mathbf{SelfEdge}(G_{i-1}, v_2) \\ G_i &= \mathbf{G}[n_{21}n_{11}, n_{21}n_{12}, n_{12}n_{21}, n_{12}n_{22} + i - n]. \end{aligned}$$

  Note that $G'_i = \mathbf{Del}(G_i, v_{22}^{n_{12}n_{22}+i-n})$. Thus, by the self-edge axiom and the weak deletion property, $v_1 \preceq^F_G v_2 \Rightarrow v_2 \not\preceq^F_{G_{n+\delta}} v_1$. Now, note that $G_{n+\delta} = \mathbf{G}[n_{21}n_{11}, n_{12}n_{21}, n_{12}n_{21}, n_{21}n_{11}]$, thus as before, by isomorphism, $v_1 \simeq^F_{G_{n+\delta}} v_2$. Therefore we conclude that $v_1 \not\preceq^F_G v_2$ for $F \in \{F_1, F_2\}$.

- If $n_{21}n_{11} < n_{12}n_{22}$, we can similarly conclude that $v_2 \not\preceq_G^F v_1$, and therefore $v_1 \preceq_G^F v_2$ for $F \in \{F_1, F_2\}$.

We have shown that for every vertex set V, for all $G = (V, E) \in \mathbb{G}_V$, and for every $v_1, v_2 \in V$: $v_1 \preceq_G^{F_1} v_2 \Leftrightarrow v_1 \preceq_G^{F_2} v_2$. Thus, $F_1 \equiv F_2$, concluding the proof of the proposition. □