Elementary Differential Equations on Discrete, Continuous, and Hybrid Spaces

(Extended Abstract)

The state evolution of complicated real systems frequently involves interdependence among variables that represent a variety of distinctly different types of components. Often enough, mathematical models of such systems combine variables of both discrete and continuous types that may jointly evolve on multiple time scales that span many orders of magnitude. We shall use the term 'hybrid dynamical system' to refer to a system of this kind. For example, in nature the molecular operations of a living cell can be thought of as a hybrid dynamical system. These molecular operations happen on time scales that range from 10^{-15} to 10^{4} seconds and proceed in ways that are dependent on populations of molecules that range in size from as few as approximately 10^{1} to approximately as many as 10^{20}. Molecular biologists have used hybrid Petri nets to make simplified models of portions of these transcription and gene regulation processes. In engineered systems, a variety of microelectronic systems-on-a-chip (SOC) involve the interdependence of digital and analog signals, both clocked and unclocked. Many SOC designers use the VHDL-AMS language to support the description and simulation of analog, digital, and mixed-signal circuits and systems. Mixed signal SOCs are hybrid dynamical systems. Additional examples of hybrid dynamical systems are common in situations involving the digital control of physical systems.

The mathematical description of hybrid dynamical systems raises the matter of the extent to which we can unify a variety of continuous and discrete types of change of state phenomena. As a starter for investigation, a single scheme can be considered whose instances are differential calculi on structures that embrace both topological spaces and graphs as well as hybrid ramifications of such structures. These calculi include the elementary differential calculus on real and complex vector spaces.

One class of spaces that has been increasingly receiving attention in recent years is the class of convergence spaces [cf. Heckmann, R., TCS v.305, (159–186)(2003)] The class of convergence spaces together with the continuous functions among convergence spaces forms a Cartesian-closed category CONV that contains as full subcategories both the category TOP of topological spaces and an embedding of the category DIGRAPH of reflexive directed graphs. (More can importantly be said about these embeddings.) These properties of CONV serve to assure that we can construct continuous products of continuous functions, and that there is always at least one convergence structure available in function spaces with respect to which the operations of function application and composition are continuous. The containment of TOP and DIGRAPH in CONV allows to combine arbitrary topological spaces with discrete structures (as represented by digraphs) to obtain hybrid structures, which generally are not topological spaces.

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The differential calculus scheme in CONV that we exhibit addresses three issues in particular.

1. For convergence spaces X and Y and function $f : X \longrightarrow Y$, the scheme gives necessary and sufficient conditions for a candidate differential $df : X \longrightarrow Y$ to be a (not necessarily "the", depending on the spaces involved) differential of f at x_0.

2. The chain rule holds and the differential relation between functions distributes over Cartesian products: e.g. if Df, Dg and Dh are, respectively, differentials of f at $(g(x_0), h(x_0))$ and g and h at x_0, then $Df \circ (Dg \times Dh)$ is a differential of $f \circ (g \times h)$ at x_0.

3. When specialized to real and complex vector spaces, the scheme is in agreement with ordinary elementary differential calculus on these spaces.

Moreover, with two additional constraints having to do with self-differentiation of differentials and translation invariance (for example, a linear operator on, say, $\mathbf{C}^2$, is its own differential everywhere) there is a (unique) maximum differential calculus in CONV.

The category of convergence spaces with the properties and subcategories mentioned above provides a convenient context for concisely describing the relationships among convergence spaces and the ways of combining convergence structures, particularly topological spaces and directed graphs. However, the arguments for the accompanying results are largely topological in character, not category-theoretic. Finally, it is important to note that when taking an instance of the differential calculus scheme in function spaces, one is not forced to use the default convergence structure used in establishing the Cartesian closure of CONV. Consequently, when one is working in familiar settings, the familiar topological/metric structure of the function spaces of interest is still available.