On the Logic of Constitutive Rules

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Abstract. The paper proposes a logical systematization of the notion of counts-as which is grounded on a very simple intuition about what counts-as statements actually mean, i.e., forms of classification. Moving from this analytical thesis the paper disentangles three semantically different readings of statements of the type X counts as Y in context C, from the weaker notion of contextual classification to the stronger notion of constitutive rule. These many ways in which counts-as can be said are then formally addressed by making use of modal logic techniques. The resulting framework allows for a formal characterization of all the involved notions and their reciprocal logical relationships.

Keywords. Constitutive rules, counts-as, modal logic.

1 Introduction

The term "counts-as" derives from the paradigmatic formulation that in [1] and [2] is attributed to the non-regulative component of institutions, i.e., constitutive rules:

> [...] "institutions" are systems of constitutive rules. Every institutional fact is underlain by a (system of) rule(s) of the form "X counts as Y in context C" ([1], pp.51-52).

In legal theory the non-regulative component of normative systems has been labeled in ways that emphasize a classificatory, as opposed to a normative or regulative, character: *conceptual rules* ([3]), *qualification norms* ([4]), *definitional norms* ([5]). Constitutive rules are definitional in character:

> The rules for checkmate or touchdown must 'define' *checkmate in chess* or *touchdown in American Football* [...] ([1], p.43).

With respect to this feature, a first reading of counts-as is thus readily available: it is plain that counts-as statements express classifications. For example, they express what *is classified* to be a checkmate in chess, or a touchdown in American Football. However, is this all that is involved in the meaning of counts-as statements?

The interpretation of counts-as in merely classificatory terms does not do justice to the notion which is stressed in the label "constitutive rule", that is, the notion of *constitution*. Aim of the paper is to show that this notion, as it is presented in some work in legal and social theory, is amenable to formal characterization and that the theory we

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developed in [6, 7] provides a ground for its understanding. The paper disentangles and analyzes three precise senses in which it can be said that "X counts as Y in context C". For each of these different senses of counts-as a formal semantics is developed by making use of standard modal logic techniques. From a methodological point of view, we will proceed as recommended here:

> "[. . .] it seems to me obvious that the only rational approach to such problems would be the following: [1] We should reconcile ourselves with the fact that we are confronted, not with one concept, but with several different concepts which are denoted by one word; [2] we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way); [3] to avoid further confusions, we should agree to use different terms for different concepts; and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations" ([8], p. 355).

The structure of the paper reflects its method. Section 2 disentangles three different meanings of counts-as statements and exposes a first informal analysis. In Section 3 a modal logic of contextual classification is introduced and by means of it a formal analysis of the classificatory view of counts-as is provided. The two remaining senses of counts-as are formally analyzed in Sections 5 and 6. Finally, the relationships between the three readings are studied in Section 7. Conclusions follow in Section 8.

2 Counts-as between Classification and Constitution

Consider the following reasoning pattern.

Example 1. It is a rule of normative system Γ that conveyances transporting people or goods count as vehicles; it is always the case that bikes count as conveyances transporting people or goods but not that bikes count as vehicles; therefore, in the context of normative system Γ, bikes count as vehicles.

This is an instance of a typical reasoning pattern involving constitutive rules. The counts-as locution occurs three times. However, the second premise states a generally acknowledged classification ("bikes count as conveyances transporting people or goods"), while the conclusion states classification which is considered to hold only with respect to the normative system at issue ("according to normative system Γ, bikes count as vehicles"). The first premise expresses something yet different, a classification which is brought about —constituted— by the normative system: "conveyances transporting people or goods are classified as vehicles" is one of the rules of Γ.

2.1 The classificatory reading of counts-as

The fact that "bikes count as conveyances transporting people or goods" can be readily analyzed as a form of classification: the concept 'bike' is a subconcept of the concept 'conveyance transporting people or goods'. ([6, 9, 10]).

In Example 1 one of the premises was that bikes do not always count as vehicles. In other words, there are contexts in which 'bike' is not a subconcept of 'vehicle'. This suggests that a notion of context is necessary because classifications holding for a normative system are not of a universal kind, they do not hold in general. The classificatory reading of counts-as statements of the form "X counts-as Y in context c" runs thus as follows: "X is a subconcept of Y in context c". Following much literature on context theory (see for instance [11, 12]) we conceive of a context simply as set of situations (possible worlds). What precisely these situations have to be in order to be considered a context will be clarified soon discussing the notion of constitutive rule (Section 2.3).

Classificatory counts-as will be formally studied in Section 3. A more extensive discussion of the intuitions underpinning the classificatory reading of counts-as statements can be found in [6, 7].

2.2 Counts-as statements as proper classifications

The analytic literature on constitutive norms often comes to emphasize the following characteristic feature: counts-as statements are not just classifications but "new" classifications, that is, classifications which would not hold without the normative system stating them:

> "Where the rule is purely regulative, behaviour which is in accordance with the rule could be given the same description or specification (the same answer to the question "What did he do?") whether or not the rule existed, provided the description or specification makes no explicit reference to the rule. But where the rule (or system of rules) is constitutive, behaviour which is in accordance with the rule can receive specifications or descriptions which it could not receive if the rule did not exist" ([1], p.35).

This was the case for the conclusion of the inference in Example 1: "in the context of normative system Γ, bikes count as vehicles" although this is not generally the case. In this view, counts-as statements do not only state contextual classifications, but they state new classifications which would not otherwise hold.

Observation 1 *Counts-as statements are classifications which hold with respect to a context (set of situations) but which do not hold in general (i.e., with respect to all situations).*

We call counts-as statements intended in the sense of Observation 1 *proper contextual classifications*. In other words, X counts as Y in context c because X is classified as Y in c but also because this does not hold in general, i.e., in the global context. They state that something new is brought about and in this sense the notion of proper contextual classification already captures a precise notion of constitution: the fact that X is classified as Y is constituted by context c in the sense that out of context c it might not hold. Proper contextual classifications will be formally studied in Sections 4.1 5. A more detailed exposition of the intuitions behind the proper classificatory view on counts-as can be found in [7].

2.3 Counts-as statements as constitutive rules

Example 1 sketched an inference grounded on a constitutive rule: "It is a rule of normative system Γ that conveyances transporting people or goods count as vehicles". First of all, this statement expresses a classification which is brought about by the normative system Γ ("conveyances transporting people or goods count as vehicles"), that is, what we called in the previous section a proper contextual classification. There is however something more. It explicitly states that a classification is one of the rules of Γ. This semantic ingredient is not captured by the classificatory and proper classificatory readings sketched in the previous sections and it involves two essential aspects.

The first one is that counts-as statements of the constitutive type are always part of a *set* of similar statements, the system of rules Γ.

> "Rules are constitutive if and only if they are part of a set of rules. Strictly speaking, there is no such thing as *a* rule that is constitutive in isolation" ([13], p.5).

The second aspect concerns the relation between, on the one hand, the notion of a set of rules Γ, i.e., normative system or institution, and on the other hand the notion of set of situations c, or context c. A Γ constitutes a context c by means of its rules. The set of classifications stated as constitutive rules by a normative system (for instance, "conveyances transporting people or goods count as vehicles") can be thought of as the set of situations which make that set of classifications true. Hence, the set of constitutive rules of any normative system can be seen as a set of situations. And a set of situations —we have seen— is what is called a context in much literature on context theory (see for instance [11, 12]). To put it in a nutshell, a context is a set of situations, and if the constitutive rules of a given normative system Γ are satisfied by all and only the situations in a given set, then that set of situations is *the context defined by* Γ. This simple observation allows us to think of contexts as "systems of constitutive rules" ([1], p.51). Notice that this is no exotic thought. In fact, this idea has been neatly advanced —informally— in some literature on the theory of institutions:

> "A set of constitutive rules defines a logical space" ([13], p.6).

A logical space is nothing but a set of states, i.e., a context. Getting back to Example 1, consider the statement concluding the argument: "according to Γ, bikes count as vehicles". In this light such a statement just says that "in the set of situations defined by the rules of system Γ, bike is a subconcept of vehicle".

The discussion above is distilled in the following observation.

Observation 2 *A constitutive counts-as statement is a proper contextual classification such that: (a) it is an element of the set of rules specifying a given normative system Γ; (b) the set of rules of Γ define the context (set of situations) to which the counts-as statement pertains.*

Constitutive counts-as statements will be formally studied in Sections 4.2 and 6.

To recapitulate, we distinguished between *constitutive counts-as statements*, *proper classificatory counts-as statements* and *classificatory counts-as statements*. When statements "X counts as Y in the context c of normative system Γ" are read as constitutive rules, what is meant is that the classification of X under Y is considered to be an explicit promulgation of the normative system Γ defining context c. Instead, when they are read as proper classificatory statements they are meant to denote classifications that are constituted, or brought about, by the context at issue in the sense that they might not hold if another context is considered. Finally, when they are read as mere contextual classification, they are meant to denote classificatory statements that are just the case in the given context .

3 Modal logic of Classificatory Counts-as

This section summarizes the results presented in [6]. We first introduce the languages we are going to work with: propositional n-modal languages $\mathcal{ML}_n$ ([14]). The alphabet of $\mathcal{ML}_n$ contains: a countable set $\mathbb{P}$ of propositional atoms p; the set of boolean connectives $\{\neg, \wedge, \vee, \rightarrow\}$; a finite non-empty set of n (context) indexes C, and the operator $[\,]$. Metavariables $i, j, ...$ are used for denoting elements of C. The set of well formed formulas ϕ of $\mathcal{ML}_n$ is then defined by the following BNF:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid [i]\phi.$$

We will refer to formulae ϕ in which at least one modal operator occurs as modalized formulae. We call instead objective formulae in which no modal operator occur and we denote them using the metavariables $\gamma_1, \gamma_2, \ldots$.

3.1 Semantics

Semantics for these languages is given via structures $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$, where:

- $\mathcal{F}$ is a CXT frame, i.e., a structure $\mathcal{F} = \langle W, \{W_i\}_{i \in C} \rangle$, where W is a finite set of states (possible worlds) and $\{W_i\}_{i \in C}$ is a family of subsets of W.
- $\mathcal{I}$ is an evaluation function $\mathcal{I} : \mathbb{P} \longrightarrow \mathcal{P}(W)$ associating to each atom the set of states which make it true.

Such frames model thus n different contexts i which might be inconsistent, if the corresponding set W_i is empty, or global if W_i coincides with W itself. This implements in a straightforward way the thesis developed in context modeling according to which contexts can be soundly represented as sets of possible worlds ([11]).

The satisfaction relation, then, results in the following.

Definition 1. (Satisfaction based on CXT frames)
Let $\mathcal{M}$ be a model built on a CXT *frame.*

$$\mathcal{M}, w \models [i]\phi \text{ iff } \forall\, w' \in W_i : \mathcal{M}, w' \models \phi$$
$$\mathcal{M}, w \models \langle i \rangle\, \phi \text{ iff } \exists w' \in W_i : \mathcal{M}, w' \models \phi.$$

The obvious boolean clauses are omitted. Validity in a model, in a frame and in a class of frames are defined as usual.

It is instructive to make a remark about the $[i]$-operator clause, which can be seen as the characterizing feature of the modeling of contexts as sets of worlds[1]. It states that the truth of a modalized formula abstracts from the point of evaluation of the formula. In other words, the notion of "truth in a context i" is a *global* notion: $[i]$-formulae are either true in every state in the model or in none. This reflects the idea that what is true or false in a context does not depend on the world of evaluation, and this is what we would intuitively expect especially for contexts interpreted as normative systems: what holds in the context of a given normative system is not determined by the point of evaluation but just by the system in itself, i.e., by its rules: the fact that in Γ bikes count as vehicles depends only on the rules of Γ.

3.2 Axiomatics

The multi-modal logic that corresponds, i.e., that is sound and complete with respect to the class of CXT frames, is a system we call here $\mathbf{K45^{ij}_{n}}$. It consists of a logic weaker than the logic $\mathbf{KD45^{ij}_{n}}$ investigated in [6] in that the semantic constraint has been dropped which required the sets in family $\{W_i\}_{i \in C}$ to be non-empty. As a consequence the D axiom is eliminated. To put it in a nutshell, the system is the very same logic for contextual classification developed in [6] except for the fact the we want to allow here the representation of empty contexts as well. In the knowledge representation setting we are working in, where contexts can be identified with the normative systems defining them, this amounts to accept the possibility of normative systems issuing inconsistent constitutive rules.

Logic $\mathbf{K45^{ij}_{n}}$ is axiomatized via the following axioms and rules schemata:

$$
\begin{array}{rl}
(\mathtt{P}) & \text{all tautologies of propositional calculus} \\
(\mathtt{K}) & [i](\phi_1 \rightarrow \phi_2) \rightarrow ([i]\phi_1 \rightarrow [i]\phi_2) \\
(4^{ij}) & [i]\phi \rightarrow [j][i]\phi \\
(5^{ij}) & \neg[i]\phi \rightarrow [j]\neg[i]\phi \\
(\mathtt{Dual}) & \langle i \rangle \phi \leftrightarrow \neg[i]\neg\phi \\
 & \\
(\mathtt{MP}) & \phi_1,\ \phi_1 \rightarrow \phi_2 \ /\ \phi_2 \\
(\mathtt{N}) & \phi \ /\ [i]\phi
\end{array}
$$

where i, j denote elements of the set of indexes C. The system is a multi-modal homogeneous $\mathbf{K45}$ with the two interaction axioms 4^{ij} and 5^{ij}. Soundness and completeness are proven in Section 9.

A remark is in order especially with respect to axiomata 4^{ij} and 5^{ij}. In fact, what the two schemata do, consists in making the nesting of the operators reducible which, leaving technicalities aside, means that truth and falsehood in contexts ($[i]\phi$ and $\neg[i]\phi$) are somehow absolute because they remain invariant even if evaluated from another context ($[j][i]\phi$ and $[j]\neg[i]\phi$). In other words, they express the fact that whether something holds in a context i is not something that a context j can influence. This is indeed the kind of property to be expected given the semantics presented in the previous section.

[1] Propositional logics of context without this clause are investigated in [15, 16].

3.3 Classificatory Counts-as formalized

Using a multi-modal logic $\mathbf{K45^{ij}_n}$ on a language $\mathcal{ML}_n$, the formal characterization of the classificatory view on counts-as statements runs as follows.

Definition 2. (Classificatory counts-as: $\Rightarrow^{cl}_c$)
"γ_1 counts as γ_2 in context c" is formalized in a multi-modal language $\mathcal{ML}_n$ as the strict implication between two objective sentences γ_1 and γ_2 in logic $\mathbf{K45^{ij}_n}$:

$$\gamma_1 \Rightarrow^{cl}_c \gamma_2 \ := \ [c](\gamma_1 \rightarrow \gamma_2)$$

These properties for $\Rightarrow^{cl}_c$ follow.

Proposition 1. (Properties of $\Rightarrow^{cl}_c$)
In logic $\mathbf{K45^{ij}_n}$, the following formulas and rules are valid:

$$\gamma_2 \leftrightarrow \gamma_3 \,/\, (\gamma_1 \Rightarrow^{cl}_c \gamma_2) \leftrightarrow (\gamma_1 \Rightarrow^{cl}_c \gamma_3) \tag{1}$$
$$\gamma_1 \leftrightarrow \gamma_3 \,/\, (\gamma_1 \Rightarrow^{cl}_c \gamma_2) \leftrightarrow (\gamma_3 \Rightarrow^{cl}_c \gamma_2) \tag{2}$$
$$((\gamma_1 \Rightarrow^{cl}_c \gamma_2) \wedge (\gamma_1 \Rightarrow^{cl}_c \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl}_c (\gamma_2 \wedge \gamma_3)) \tag{3}$$
$$((\gamma_1 \Rightarrow^{cl}_c \gamma_2) \wedge (\gamma_3 \Rightarrow^{cl}_c \gamma_2)) \rightarrow ((\gamma_1 \vee \gamma_3) \Rightarrow^{cl}_c \gamma_2) \tag{4}$$
$$\gamma \Rightarrow^{cl}_c \gamma \tag{5}$$
$$(\gamma_1 \Rightarrow^{cl}_c \gamma_2) \wedge (\gamma_2 \Rightarrow^{cl}_c \gamma_3) \rightarrow (\gamma_1 \Rightarrow^{cl}_c \gamma_3) \tag{6}$$
$$(\gamma_1 \Rightarrow^{cl}_c \gamma_2) \wedge (\gamma_2 \Rightarrow^{cl}_c \gamma_1) \rightarrow [c](\gamma_1 \leftrightarrow \gamma_2) \tag{7}$$
$$(\gamma_1 \Rightarrow^{cl}_c \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \Rightarrow^{cl}_c \gamma_2) \tag{8}$$
$$(\gamma_1 \Rightarrow^{cl}_c \gamma_2) \rightarrow (\gamma_1 \Rightarrow^{cl}_c \gamma_2 \vee \gamma_3) \tag{9}$$

We omit the proofs, which are straightforward via application of Definition 2. This system validates all the intuitive syntactic constraints isolated in [17] (validities 1-4). In addition, this semantic-oriented approach to classificatory counts-as enables the four validities 6-9. Besides, this analysis shows that counts-as conditionals, once they are viewed as conditionals of a classificatory nature, naturally satisfy reflexivity (5), transitivity (6), and a form of "contextualized" antisymmetry (7), strengthening of the antecedent (8) and weakening of the consequent (9).

4 Beyond Classificatory Counts-as

Aim of this section is to provide formal counterparts to Observations 1 and 2 which can work as intermediate step towards the development of suitable modal logics for the analysis of proper classificatory counts-as (Section 5) and constitutive counts-as (Section 6).

4.1 From classification to proper classification

As usual, model-theoretic considerations can give us crucial hints. Let us define the set $\mathbb{T}(X)$ of all formulae which, given a model, are satisfied by all worlds in a set of worlds X:

$$\mathbb{T}(X) = \{\phi \mid \forall w \in X : \mathcal{M}, w \models \phi\}.$$

and let $\mathbb{T}^{\rightarrow}(X)$ be the set of all implications between objective formulae γ_1 and γ_2 which are satisfied by all worlds in a set of worlds X:

$$\mathbb{T}^{\rightarrow}(X) = \{\gamma_1 \rightarrow \gamma_2 \mid \forall w \in X : \mathcal{M}, w \models \gamma_1 \rightarrow \gamma_2\}.$$

Obviously, for every X: $\mathbb{T}^{\rightarrow}(X) \subseteq \mathbb{T}(X)$. In the classificatory reading, given a model $\mathcal{M}$ where the set of worlds $W_c \subseteq W$ models context c, the set of all classificatory counts-as statements holding in c, which we denote as $\mathbb{CL}(W_c)$, can be defined as the set $\mathbb{T}^{\rightarrow}(W_c)$:

$$\mathbb{CL}(W_c) := \mathbb{T}^{\rightarrow}(W_c).$$

Hence, it is easy to see that: $\mathbb{T}^{\rightarrow}(W) \subseteq \mathbb{CL}(W_c) \subseteq \mathbb{T}(W_c)$. In other words, the set of classificatory counts-as statements is:

- A subset of all the truths of W_c;
- A superset of all conditional truths of W, that is, of the "global" or "universal" context of model $\mathcal{M}$.

While the first point represents a quite banal semantic constraint to which any formal characterization of counts-as should adhere, the second one is much more questionable. Indeed, what is true anyway is not characteristic of any context (except of the global one), and it cannot be properly said to represent any new truth. In other words, interpreting counts-as statements as mere classifications, as it has been done in Section 3 make them inherit all trivial classifications which hold globally in the model. This is the reason why classificatory counts-as, as shown in Proposition 1, behaves classically enjoying antecedent strengthening as well as transitivity and reflexivity.

These considerations suggest thus a readily available strategy to specify the set of proper classificatory counts-as holding in a context c on the basis of $\mathbb{T}^{\rightarrow}(W_c)$. The problem boils down to eliminate from the set of classificatory counts-as $\mathbb{CL}$ for a context W_c those classifications which hold globally, that is, which hold with respect to the global context W. We obtain, in this way, the set of *proper classificatory counts-as* statements, or *proper contextual classifications*, holding in context c in a CXT model $\mathcal{M}$.

Definition 3. (Set of proper classificatory counts-as in c)
The set $\mathbb{CL}^{+}(W_c)$ of proper classificatory counts-as statements of a context c in a CXT *model $\mathcal{M}$ is defined as follows:*

$$\mathbb{CL}^{+}(W_c) := \mathbb{T}^{\rightarrow}(W_c) \setminus \mathbb{T}(W). \tag{10}$$

Intuitively, the set of proper classificatory count-as holding in c corresponds to the set of implications between objective formulae which hold in c, minus those implications which hold universally. Or, to put it otherwise, the set of proper classificatory count-as holding in c corresponds to the set of classificatory counts as of c, minus those implications which hold universally: $\mathbb{CL}^{+}(W_c) := \mathbb{CL}(W_c) \setminus \mathbb{T}(W)$. This is the most natural amendment of the classificatory view toward the specification of a stronger notion of contextual classification along the lines of Observation 1.

4.2 From proper classification to constitution

Let us now focus on Observation 2. What comes to play a role is the notion of a *definition* of the context of a counts-as statement. A definition of a context c, in a CXT model $\mathcal{M}$, is a set of objective formulae[2] Γ such that $\forall w \in W$:

$$\mathcal{M}, w \models \Gamma \text{ iff } w \in W_c \tag{11}$$

that is, the set of formulae Γ such that all and only the worlds in W_c satisfy Γ in $\mathcal{M}$.

Observation 2 can now get a formal formulation. Given the set of formulae Γ, we say that any formula $\gamma_1 \rightarrow \gamma_2 \in \Gamma$ is a constitutive counts-as statement w.r.t. context c iff Γ defines context c and $\gamma_1 \rightarrow \gamma_2$ belongs to the set of proper contextual classifications of c.

Definition 4. (Set of constitutive counts-as in c w.r.t. definition Γ)
The set $\mathbb{CO}(\Gamma, W_c)$ of constitutive counts-as statements of a context c defined by Γ in a CXT *model $\mathcal{M}$ is:*

$$\begin{aligned}\mathbb{CO}(\Gamma, W_c) := \{\gamma_1 \rightarrow \gamma_2 \in \Gamma \mid \gamma_1 \rightarrow \gamma_2 \in \mathbb{CL}^+(W_c) \\ \textit{and } \forall w(\mathcal{M}, w \models \Gamma \textit{ iff } w \in W_c)\}\end{aligned} \tag{12}$$

Notice that $\mathbb{CO}(\Gamma, W_c)$ is defined taking as domain the set of implicative statements of Γ. Notice also that as a result of this definition if Γ does not define context W_c then $\mathbb{CO}(\Gamma, W_c) = \emptyset$. In fact, Formula 12 can be restated as follows:

$$\mathbb{CO}(\Gamma, W_c) = \begin{cases} \mathbb{CL}^+(W_c) \cap \Gamma, & \text{if } \Gamma \text{ defines } W_c \\ \emptyset, & \text{otherwise.} \end{cases}$$

Section 6 is devoted to the development of a modal logic based on this definition. The definitions discussed are summarized in the table below.

Cxt Classification	$\mathbb{CL}(W_c) = \mathbb{T}^{\rightarrow}(W_c)$
Proper Cxt Classification	$\mathbb{CL}^+(W_c) = \mathbb{CL}(W_c) \setminus \mathbb{T}(W)$
Constitution	$\mathbb{CO}(\Gamma, W_c) = \begin{cases} \mathbb{CL}^+(W_c) \cap \Gamma, & \text{if } \Gamma \text{ defines } W_c \\ \emptyset, & \text{otherwise.} \end{cases}$

The table pinpoints the dependencies between the formal characterizations of the three different senses of counts-as which has been taken into consideration: the notion of constitution builds on the notion of proper contextual classification which in its turn builds on the notion of contextual classification. The modal logic analysis of contextual classification developed in Section 3 can thus be used as a sound starting point for the modal logic analysis of the two notions introduced in this section.

[2] This is no arbitrary choice since it can be easily seen that contextual formulae, since they denote global properties of the models, are as a matter of fact irrelevant for the definition of sets of worlds W_i such that $\emptyset \subset W_i \subset W$, that is, those sets which denote neither the empty nor the universal contexts. It is therefore natural to restrict definitions to objective formulae.

4.3 A methodological note

Before rendering the insights of Sections 4.1 and 4.2 in modal logic, it is worth making a methodological remark. We are here concerned with a term, "counts-as", which appears to have different meanings. At this point we had two main ways to pursue the formal characterization of counts-as we were aiming at. We could proceed axiomatically by trying to single out intuitive syntactic properties of counts-as statements? Or rather semantically, by trying to enrich the semantic characterization of classificatory counts-as exposed in the previous sections in order to capture further semantic nuances? While formal approaches to counts-as ([17–19]) have been, up to now, characterized by an axiomatic perspective, we have instead chosen for a semantics-driven solution. This choice has been inspired by considering the methodological standpoint of fundamental work in philosophical logic such as [8, 20].

The same issue we are facing here in analyzing counts-as lies also at the ground of the Tarskian characterization of the notion of truth and consists in the polysemy of the to-be-analyzed term. Because of the inherent polysemy of the predicate "to be true", Tarski found it unconvincing to proceed introducing the predicate as a primitive and then axiomatizing it:

> "[. . .] the choice of axioms always has rather accidental character, depending on inessential factors (such as e.g. the actual state of our knowledge). [. . .] a method of constructing a theory does not seem to be very natural [. . .] if in this method the role of primitive concepts —thus of concepts whose meaning should appear evident— is played by concepts which have led to various misunderstanding in the past" ([20], pag. 405-406).

Instead, he preferred to first isolate a precise sense of the predicate, i.e., truth as correspondence to reality, and then to define it in terms of a better understood notion, i.e., the notion of satisfaction of a formula by a model. An axiomatic analysis of counts-as statements runs the danger alluded to in the quote: since it is not clear what counts-as statements actually mean, an axiomatization of them could result in mixing under the the same logical representation different semantic flavors that, from an analytical point of view, should be kept separated. A systematic discussion of this issue, specifically in relation with the proposal advanced in [17], can be found in [7].

The work presented in this paper is the result of the application of this method to the notion of counts-as: in Section 2 we first disentangled different meanings of the term "counts-as" providing a first map of its polysemy; in Section 3 we formally analyzed the first and more basic of these meanings explaining it in terms of a better-understood notion (strict implication within a context); in this section we have pointed at a first semantic characterization of the other two meanings and in the coming next two sections we will explain them by making use of better-understood modal logic notions: the negation of global statements (proper classificatory counts-as) and the definition of a context (constitutive counts-as).

5 Modal Logic of Counts-as as Proper Contextual Classification

In the following section a modal logic is developed which implements the definition stated in Formula 10 above. By doing this we will capture the intuitions discussed in

Section 2 concerning the intuitive reading of counts-as statements in proper classificatory terms. At the same time we will maintain the possible worlds semantics of context exposed in Section 3 and developed in order to account for the purely classificatory view of counts-as.

5.1 Expansion of $\mathcal{L}_n$ and semantics

Language $\mathcal{L}_n$ is expanded as follows. The set of context indexes C is such that it always contains the special context index u denoting the universal (or global) context. We call this language $\mathcal{L}_n^u$.

Languages $\mathcal{L}_n^u$ are given a semantics via a special class of CXT frames, namely the class of CXT frames $\mathcal{F} = \langle W, \{W_i\}_{i \in C} \rangle$ such that $W \in \{W_i\}_{i \in C}$. That is, the frames in this class, which we call CXT$^\top$, always contain the global context among their contexts. The definition of the satisfaction relation for language $\mathcal{L}_n^u$ follows.

Definition 5. (Satisfaction based on CXT$^\top$ frames)
Let $\mathcal{M}$ be a model built on a CXT$^\top$ frame.

$$\mathcal{M}, w \models [u]\phi \;\; \textit{iff} \;\; \forall\, w' \in W : \mathcal{M}, w' \models \phi$$
$$\mathcal{M}, w \models [c]\phi \;\; \textit{iff} \;\; \forall\, w' \in W_c : \mathcal{M}, w' \models \phi$$

where u is the universal context index and c ranges on the context indexes in C. The obvious boolean clauses and the clauses for the dual modal operators are omitted.

The new clause states that the $[u]$ operator is interpreted on the universal 1-frame contained in each CXT$^\top$ frame. It is therefore nothing but a **S5** necessity operator.

5.2 Axiomatics

We call $\mathbf{Cxt^u}$ the logic characterizing the class of CXT$^\top$ frames. Logic $\mathbf{Cxt^u}$ results from the union $\mathbf{K45^{ij}_n} \cup \mathbf{S5_u} \cup \{\subseteq .ui)\}$, that is, from the union of $\mathbf{K45^{ij}_n}$ with the $\mathbf{S5_u}$ logic for the $[u]$ operator together with the interaction axiom $\subseteq .ui$ below. The axiomatics runs thus as follows:

(P) all tautologies of propositional calculus

(K^i) $[i](\phi_1 \rightarrow \phi_2) \rightarrow ([i]\phi_1 \rightarrow [i]\phi_2)$

(4^{ij}) $[i]\phi \rightarrow [j][i]\phi$

(5^{ij}) $\neg[i]\phi \rightarrow [j]\neg[i]\phi$

(T^u) $[u]\phi \rightarrow \phi$

$(\subseteq .ui)$ $[u]\phi \rightarrow [i]\phi$

(Dual) $\langle i \rangle\, \phi \leftrightarrow \neg[i]\neg\phi$

(MP) IF $\vdash \phi_1$ AND $\vdash \phi_1 \rightarrow \phi_2$ THEN $\vdash \phi_2$

(N^i) IF $\vdash \phi$ THEN $\vdash [i]\phi$

where i, j denote elements of the set of indexes C and u denotes the universal context index in C. The interaction axiom $\subseteq .ui$ states something quite intuitive concerning the interaction of the $[u]$ operator with all other context operators: what holds in the global context, holds in every context. Soundness and completeness of this axiomatization w.r.t. $\text{CXT}^{\top}$ frames are proven in Section 9.

5.3 Proper classificatory counts-as formalized

Using a multi-modal logic $\mathbf{Cxt^u}$ on a language $\mathcal{L}_n^u$, the proper classificatory reading of counts-as statements can be formalized as follows.

Definition 6. (Proper classificatory counts-as: $\Rightarrow_c^{cl+}$)
"γ_1 counts as γ_2 in context c", with γ_1 and γ_2 objective formulae, is formalized in the logic $\mathbf{Cxt^u}$ on a multi-modal language $\mathcal{L}_n^u$ as:

$$\gamma_1 \Rightarrow_c^{cl+} \gamma_2 \ := \ [c](\gamma_1 \rightarrow \gamma_2) \wedge \neg[u](\gamma_1 \rightarrow \gamma_2)$$

Notice that this definition is nothing but the translation in the $\mathcal{L}_n^u$ language of Formula 10.

What properties of counts-as are lost interpreting it as proper contextual classification? And what properties are instead still valid? The following two propositions answer these questions.

Proposition 2. (Properties of $\Rightarrow_c^{cl+}$: invalidities)
The $\Rightarrow_c^{cl+}$ versions of reflexivity, strengthening of the antecedent, weakening of the consequent, transitivity and cautious monotonicity are not valid:

$$\gamma \Rightarrow_c^{cl+} \gamma \tag{13}$$
$$(\gamma_1 \Rightarrow_c^{cl+} \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \Rightarrow_c^{cl+} \gamma_2) \tag{14}$$
$$(\gamma_1 \Rightarrow_c^{cl+} \gamma_2) \rightarrow (\gamma_1 \Rightarrow_c^{cl+} \gamma_2 \vee \gamma_3) \tag{15}$$
$$((\gamma_1 \Rightarrow_c^{cl+} \gamma_2) \wedge (\gamma_2 \Rightarrow_c^{cl+} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow_c^{cl+} \gamma_3) \tag{16}$$
$$((\gamma_1 \Rightarrow_c^{cl+} \gamma_2) \wedge (\gamma_1 \Rightarrow_c^{cl+} \gamma_3)) \rightarrow ((\gamma_1 \wedge \gamma_2) \Rightarrow_c^{cl+} \gamma_3) \tag{17}$$

We do not provide all the proofs, which can be obtained by constructing appropriate countermodels. We show a countermodel for Formula 16: $\forall w \in W, \mathcal{M}, w \models \gamma_1 \rightarrow \gamma_3$; $\forall w \in W_c$, $\mathcal{M}, w \models \gamma_1 \rightarrow \gamma_2$ and $\mathcal{M}, w \models \gamma_2 \rightarrow \gamma_3$; and $\exists w', w''$ s.t. $\mathcal{M}, w' \models \gamma_1 \wedge \neg\gamma_2 \wedge \gamma_3$ and $\mathcal{M}, w'' \models \neg\gamma_1 \wedge \gamma_2 \wedge \neg\gamma_3$.

It might be instructive to provide at this point also an intuitive example for the failure of transitivity. Before 9/11/2001, it was the case that many legal systems did not specify a legal notion of terrorism. In the context of the legal systems that did, the following were therefore proper contextual classifications since they were not holding in general: "the use or threat of action designed to influence the government and advance a political cause counts as terrorism" and "terrorism counts as a criminal activity". However, it could not be inferred from them that "the use or threat of action designed to influence the government and advance a political cause counts as a criminal activity" was a proper contextual classification, because what stated was anyway the case also in those legal systems disregarding a notion of terrorism. Intuitively, transitivity fails just because it is possible to constitute local middle terms, e.g., terrorism, for classifications which hold globally in the model.

Proposition 3. (Properties of $\Rightarrow^{cl+}_{c}$: validities)
In logic $\mathbf{Cxt^u}$ *the* $\Rightarrow^{cl+}_{c}$ *variants of Formulae 1-4 of Proposition 1 are valid:*

$$\gamma_2 \leftrightarrow \gamma_3 \;/\; (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \leftrightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_3) \tag{18}$$
$$\gamma_1 \leftrightarrow \gamma_3 \;/\; (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \leftrightarrow (\gamma_3 \Rightarrow^{cl+}_{c} \gamma_2) \tag{19}$$
$$((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} (\gamma_2 \wedge \gamma_3)) \tag{20}$$
$$((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge (\gamma_3 \Rightarrow^{cl+}_{c} \gamma_2)) \rightarrow ((\gamma_1 \vee \gamma_3) \Rightarrow^{cl+}_{c} \gamma_2) \tag{21}$$

Contextualized antisymmetry, i.e., Formula 7 of Proposition 1 holds in the following form:

$$(\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge (\gamma_2 \Rightarrow^{cl+}_{c} \gamma_1) \rightarrow [c](\gamma_1 \leftrightarrow \gamma_2) \wedge \neg[u](\gamma_1 \leftrightarrow \gamma_2) \tag{22}$$

Cumulative transitivity (alias cut) is also valid:

$$((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge ((\gamma_1 \wedge \gamma_2) \Rightarrow^{cl+}_{c} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_3) \tag{23}$$

Conditional versions of antecedent strengthening, consequent weakening and transitivity are valid:

$$\neg[u](\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2) \rightarrow ((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \Rightarrow^{cl+}_{c} \gamma_2)) \tag{24}$$
$$\neg[u](\gamma_1 \rightarrow \gamma_2 \vee \gamma_3) \rightarrow ((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2 \vee \gamma_3)) \tag{25}$$
$$\neg[u](\gamma_1 \rightarrow \gamma_3) \rightarrow ((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge (\gamma_2 \Rightarrow^{cl+}_{c} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_3) \tag{26}$$

We provide the deduction of Formula 24 as an example.

1.	(P)	$(\gamma_1 \rightarrow \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2)$
2.	(N), (K), (MP), 1	$[c](\gamma_1 \rightarrow \gamma_2) \rightarrow [c](\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2)$
3.	(P)	$\neg[u](\gamma_1 \wedge \gamma_2 \rightarrow \gamma_3)$ $\rightarrow (\neg[u](\gamma_1 \rightarrow \gamma_3) \rightarrow \neg[u](\gamma_1 \wedge \gamma_2 \rightarrow \gamma_3))$
4.	(P), (MP), (Def. 6), 2, 3	$\neg[u](\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2)$ $\rightarrow ((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \Rightarrow^{cl+}_{c} \gamma_2))$

Propositions 2 and 3, though very simple, are of key importance for putting our characterization of counts-as as proper contextual classification in perspective with other proposals. Such a comparison is elaborated in detail in [7].

Formulae 24-26 are also of interest since they show that some quite standard properties of contextual classifications are inherited by proper contextual classification in a conditionalized form, the condition being an assertion of invalidity ($\neg[u]$). Proper classificatory counts-as statements are still monotonic, provided that the strengthened version of the antecedent does not universally imply the consequent. Similarly they are still transitive, provided that the implication between γ_1 and γ_3 is not a validity of the model. It is worth emphasizing the importance of these results from the perspective of conceptual analysis and their clarifying power. An alleged intuitive example of transitivity for counts-as statements, in a proper classificatory sense, is such only if the appropriate condition is assumed to hold. Consider again the example about terrorism discussed above. The example could be in fact legitimately be read as an instance of transitivity once it is also accepted that "the use or threat of action designed to influence the government and advance a political cause counts as a criminal activity" is not something which is already globally the case. Similar considerations hold in particular for the conditionalized version of antecedent strengthening. This property will be further discussed in Section 7.1.

6 Modal Logic of Constitutive Counts-as

In this section a modal logic is developed which implements Definition 4. Again, the possible world semantics developed in order to account for the classificatory view of counts-as lies at the ground of the proposed framework.

6.1 Expanding $\mathcal{L}_n^u$

Language $\mathcal{L}_n^u$, which has been used in the previous section to deal with proper contextual classification, needs now further expansion to enable the necessary expressivity. The language is expanded along two lines.

First, the set of context indexes C contains now a set K of m atomic indexes c among which the universal context index u, and the set of the negations $-c$ of the atomic contexts, i.e., of the elements of K: $C = K \cup \{-c \mid c \in K\}$. The cardinality n of C is therefore equal to $2m$.

Second, the language needs also to contain a set $\mathbb{N}$ of nominals s disjoint from the set $\mathbb{P}$ of propositional atoms. Nominals are names for states in the model or, in other words, formulae that can be satisfied by only one state in the model. They can be freely combined with propositions to form well-formed formulae. The BNF is therefore extended as follows:

$$\phi ::= \top \mid p \mid s \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid [i]\phi \mid \langle i \rangle\, \phi.$$

Metavariables for nominals are written as $\nu_1, \nu_2, \ldots$. Modal languages containing nominals have been recently object of thorough study and are known as hybrid languages ([21]). The language obtained is called $\mathcal{L}_n^{u,-}$.

Nominals are needed in order to provide a sound and complete axiomatization of the logic based on the semantics presupposed by Definition 4. To be more precise, they are necessary in order to axiomatize the notion of complement of a context[3]. This will become evident by exposing the axiomatics (Section 6.3) and especially, from a technical point of view, in proving its completeness (Section 9).

[3] For this purpose nominals were first introduced by the so-called "Sofia school" of modal logic ([22,23]) in order to axiomatize the complement and the intersection of accessibility relations, especially in a dynamic logic setting. In fact, the axiomatics we present in Section 6.3 is strictly related with the systems studied in their works.

6.2 Semantics

Languages $\mathcal{L}_n^{u,-}$ are given a semantics via a special class of CXT frames, namely the class of CXT frames $\mathcal{F} = \langle W, \{W_i\}_{i \in C}\rangle$ such that there always exists a $W_u \in \{W_i\}_{i \in C}$ s.t. $W_u = W$; and such that for any atomic index $c \in K$ $W_u \backslash W_c \in \{W_i\}_{i \in C}$. That is, the frames in this class, which we call CXT$^{\top,\backslash}$, always contain the global context among their contexts and the complement of every atomic context.

The semantics for $\mathcal{L}_n^{u,-}$ is thus obtained interpreting the formulae on models built on CXT$^{\top,\backslash}$ frames. However, because of the introduction of nominals, the evaluation function $\mathcal{I}$ should be redefined as a function $\mathcal{I} : \mathbb{P} \cup \mathbb{N} \longrightarrow \mathcal{P}(W)$ satisfying the following constraints:

- For all nominals $s \in \mathbb{N}$, $\mathcal{I}(s)$ is a singleton set, that is, nominals always denote one and only one state in the model.
- For all states $w \in W$, there exists a nominal $s \in \mathbb{N}$ such that $\mathcal{I}(s) = w$, that is, each state has a name. In other words, the restriction of the interpretation function $\mathcal{I}$ on the set of nominals ($\mathbb{N}\rceil\mathcal{I}$) is a surjection on the set of all singletons of W.

The definition of the satisfaction relation for language $\mathcal{L}_n^{u,-}$ runs as follows.

Definition 7. (Satisfaction based on CXT$^{\top,\backslash}$ frames)
Let $\mathcal{M}$ be a model built on a CXT$^{\top,\backslash}$ *frame.*

$$\begin{aligned}
\mathcal{M}, w \models s \;\; &\textit{iff} \;\; \mathcal{I}(s) = \{w\} \\
\mathcal{M}, w \models [u]\phi \;\; &\textit{iff} \;\; \forall\, w' \in W_u : \mathcal{M}, w' \models \phi \\
\mathcal{M}, w \models [c]\phi \;\; &\textit{iff} \;\; \forall\, w' \in W_c : \mathcal{M}, w' \models \phi \\
\mathcal{M}, w \models [-c]\phi \;\; &\textit{iff} \;\; \forall\, w' \in W \backslash W_c : \mathcal{M}, w' \models \phi.
\end{aligned}$$

where u is the universal context index and c ranges on the context indexes in C, and s is a nominal. The obvious boolean clauses and the clauses for the dual modal operators are omitted.

The first clause states the satisfaction relation for nominals: a nominal s is true in a state w in model $\mathcal{M}$ iff the evaluation function associates w to s. Nominals are therefore objective formulae which are true in at most one world. The second clause, which was already introduced in Definition 5, states that the $[u]$ operator is interpreted on the universal frame contained in each CXT$^{\top,\backslash}$ frame. The third one is just the standard clause for contextual truth introduced in Definition 1. Finally, the last and new clause states that the $[-c]$ operators range over the complements of the sets W_c on which $[c]$ operators range instead.

Some observations are in order. First of all, let us comment upon the semantics of the $[-c]$-operators. In fact, the $[c]$ operator specifies a lower bound on what holds in context c ('something more may hold in c'), that is, a formula $[c]\phi$ means that ϕ *at least* holds in context c. The $[-c]$ operator, instead, specifies an upper bound on what holds in c ('nothing more holds in c'), and a $[-c]\neg\phi$ formula means therefore that ϕ *at most* holds in c, i.e., $\neg\phi$ *at least* holds in the complement of c. It becomes thus possible in CXT$^{\top,\backslash}$ frames to express context definitions by means of modal $\mathcal{L}_n^{u,-}$ formulae interpreted on $\textsc{Cxt}^{\top,\setminus}$ models. A set of objective formulae Γ defines context c in a $\textsc{Cxt}^{\top,\setminus}$ model $\mathcal{M}$ iff:

$$\mathcal{M} \models [c]\Gamma \wedge [-c]\neg\Gamma \tag{27}$$

where $\neg\Gamma$ has to be intended in the obvious sense of the disjunction of the negations of all formulae in Γ. Formula 27 is an object language modal translation of the property stated in Formula 11.

Proposition 4. *(Equivalence of Formulae 11 and 27)*
Let $\mathcal{M}$ be a $\textsc{Cxt}$ model and $\mathcal{M}'$ be a $\textsc{Cxt}^{\top,\setminus}$ such that: $\mathcal{M}'$ is based on a frame having the same domain of the frame on which $\mathcal{M}$ is based, and containing all its contexts; $\mathcal{M}'$ has the same evaluation function of $\mathcal{M}$. It is the case that, given a set of objective formulae Γ and a context $W_c \in \{W_i\}_{i \in C}$:

$$\mathcal{M}, w \models \Gamma \text{ iff } w \in W_c \quad \text{is equivalent to} \quad \mathcal{M}' \models [c]\Gamma \wedge [-c]\neg\Gamma.$$

Proof. The proof is based on the semantics provided in Definition 7. By construction of $\mathcal{M}'$, the clause "if $w \in W_c$ then $\mathcal{M}, w \models \Gamma$" is equivalent to "if $w \in W_c$ then $\mathcal{M}', w \models \Gamma$", and therefore equivalent to $\mathcal{M}' \models [c]\Gamma$. Analogously, the clause "if $w \notin W_c$ then $\mathcal{M}, w \not\models \Gamma$" is equivalent to "if $w \in W \setminus W_c$ then $\mathcal{M}', w \models \neg\Gamma$", and therefore equivalent to $\mathcal{M}' \models [-c]\neg\Gamma$.

It might be instructive to notice that in practice we are making use, in a different setting but with exactly analogous purposes, of a well-known technique developed in the modal logic of knowledge, i.e., the interpretation of modal operators on "inaccessible states" typical, for instance, of the "all that I know" epistemic logics ([24]). In our case, the set of inaccessible states is nothing but the complement of a context.

6.3 Axiomatics

To axiomatize the above semantics an extension of logic $\mathbf{K45^{ij}_n}$ is needed which can characterize nominals as names for modal states and, consequently, context complementation. The extension, which we call logic $\mathbf{Cxt^{u,-}}$, results from the union $\mathbf{K45^{ij}_n} \cup \mathbf{S5_u}$, that is, from the union of $\mathbf{K45^{ij}_n}$ with the $\mathbf{S5_u}$ logic for the $[u]$ operator together with a group of two axioms (`Least` and `Most`) and one rule (`Name`) which axiomatize nominals, and a group of two axioms (`Covering` and `Packing`) which axiomatize context complementation. The axiomatics runs as follows:

(P)	all tautologies of propositional calculus
($\mathtt{K}^i$)	$[i](\phi_1 \rightarrow \phi_2) \rightarrow ([i]\phi_1 \rightarrow [i]\phi_2)$
(4^{ij})	$[i]\phi \rightarrow [j][i]\phi$
(5^{ij})	$\neg[i]\phi \rightarrow [j]\neg[i]\phi$
($\mathtt{T}^u$)	$[u]\phi \rightarrow \phi$
($\subseteq .ui$)	$[u]\phi \rightarrow [i]\phi$
(Least)	$\langle u \rangle\, \nu$
(Most)	$\langle u \rangle\, (\nu \wedge \phi) \rightarrow [u](\nu \rightarrow \phi)$

(Covering) $[c]\phi \wedge [-c]\phi \rightarrow [u]\phi$
(Packing) $\langle -c \rangle \, \nu \rightarrow \neg \, \langle c \rangle \, \nu$
(Dual) $\langle i \rangle \, \phi \leftrightarrow \neg[i]\neg\phi$

(Name) IF $\vdash \nu \rightarrow \theta$ THEN $\vdash \theta$
(MP) IF $\vdash \phi_1$ AND $\vdash \phi_1 \rightarrow \phi_2$ THEN $\vdash \phi_2$
($\mathtt{N}^i$) IF $\vdash \phi$ THEN $\vdash [i]\phi$

where i, j are metavariables for the elements of K, c denotes elements of the set of atomic context indexes C, u is the universal context index, ν ranges over nominals, and θ in rule Name denotes a formula in which the nominal denoted by ν does not occur. The proofs of soundness and completeness of the axiomatization w.r.t. $\text{CXT}^{\top,\setminus}$ frames are provided in Section 9.

The new axioms and rules deserve some comments. Let us start with the axiomatization of nominals. Axiom Least states just that every nominal denotes *at least* one state. Vice versa, axiom Most states that nominals denote *at most* one state. Intuitively it says that, if there is a state named ν where ϕ holds, then ϕ holds if ν is the case. Finally, rule Name, which we borrowed from standard hybrid logic ([21]), states that all states are nominated. It does that by saying that if it is provable that a formula θ holds at an arbitrary state ν —the state is arbitrary since the rule requires ν not to occur in θ— then θ itself is provable since there is no world that falsifies it. From a technical point of view, as observed in [23], this rules states a sufficient condition for function $\mathbb{N}\rceil\mathcal{I})$ to be a surjection on the set of all singletons of W[4]. To sum up, axioms Least and Most with rule Name axiomatize the conditions holding on the interpretation function $\mathcal{I}$ as exposed in Section 6.2.

Let us now discuss the axioms that are more central to the modeling aim we are pursuing: axioms Covering and Packing. They characterize context complementation. Axiom Covering states that if some formula holds in both c and $-c$, than it holds globally. In other words, it states that the universal context is *covered* by the contexts denoted by c and, respectively, $-c$. Axiom Packing states instead that the contexts denoted by c and $-c$ are strongly disjoint, in the sense that they do not contain the same states or. They *pack* the universal context in two disjoint subcontexts. They are thus just modal formulations of the two properties characterizing the bipartition of a given set. Notice that nominals are necessary in the formulation of the Packing axiom. It is easy to see that, without the possibility of naming individual states, it would be impossible to axiomatize disjointness.

6.4 A remark: $\mathbf{Cxt}^{\mathbf{u},-}$ as hybrid logic

Before putting the formalism at work it might be instructive to make one last technical remark. In logic $\mathbf{Cxt}^{\mathbf{u},-}$ a family of $@_\nu$ operators is definable, by means of which it is possible to express that a formula ϕ holds in the state named by ν: $@_\nu\phi$. This operator

[4] Rule Name plays a central role in the completeness proof for $\text{CXT}^{\top,\setminus}$ (see the proof of Lemma 9 in Section 9).

is known in hybrid logics ([21]) as the *satisfaction operator*. Its semantics is given in terms of the following satisfaction clause:

$$\mathcal{M}, w \models @_\nu \phi \text{ iff } \mathcal{M}, \mathcal{I}(\nu) \models \phi.$$

The property of "holding in a state" is thus a global property, that is, it is independent of the point of evaluation. The clause states more precisely that, whatever the state of evaluation is, it is the case that if ν holds then ϕ also holds. In fact, the satisfaction operator can be defined in any logic enabling nominals and a universal modality ([25], [26]) as follows:

$$@_\nu \phi := [u](\nu \to \phi) \tag{28}$$

where $@_\nu$ is a nominal and ϕ a formula. Leaving technicalities aside, this means that logic $\mathbf{Cxt}^{\mathbf{u},-}$ has sufficient expressive means to represent statements of the type "in situation (or state) ν state-of-affairs ϕ holds". This expressive capability of logic $\mathbf{Cxt}^{\mathbf{u},-}$ will turn out useful to represent intuitive reasoning patterns involving constitutive counts-as statements (see Proposition 6).

6.5 Constitutive Counts-as formalized

Using a multi-modal logic $\mathbf{Cxt}^{\mathbf{u},-}$ on a language $\mathcal{L}_n^{u,-}$, the constitutive reading of counts-as statements can now be formalized.

Definition 8. (Constitutive counts-as: $\Rightarrow^{co}_{c,\Gamma}$)
Given a set of formulae Γ such that $\gamma_1 \to \gamma_2 \in \Gamma$, the constitutive counts-as statement "γ_1 counts as γ_2 in the context c defined by Γ" is formalized in a multi-modal logic $\mathbf{Cxt}^{\mathbf{u},\setminus}$ on language $\mathcal{L}_n^{u,-}$ as follows:

$$\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2 := [c]\Gamma \wedge [-c]\neg\Gamma \wedge \neg[u](\gamma_1 \to \gamma_2)$$

with γ_1 and γ_2 objective formulae.

The definition implements in modal logic the intuition summarized in Observation 2, and formalized in Definition 4: constitutive counts-as statements correspond to those non trivial classifications which are stated by the definition Γ of the context c. In fact the following can be proven.

Proposition 5. *(Equivalence of Definitions 8 and 4)*
Let $\mathcal{M}$ be a $\text{CXT}^{\top,\setminus}$ frame and Γ a set of objective formulae. It is the case that: $\gamma_1 \to \gamma_2 \in \mathbb{CO}(\Gamma, W_c)$ iff $\gamma_1 \to \gamma_2 \in \{\gamma_1 \to \gamma_2 \in \Gamma \mid \mathcal{M} \models \gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2\}$. To put it otherwise:

$$\mathbb{CO}(\Gamma, W_c) = \{\gamma_1 \to \gamma_2 \in \Gamma \mid \mathcal{M} \models \gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2\}$$

Proof. The proof follows from Proposition 4.

A detailed comment of Definition 8 is in order. The most important consequence of it is that it is possible to talk about constitutive counts-as only once a set Γ is given. As already stressed in Section 4.2, there is no formula that is constitutive in isolation. This logic of constitutive rules takes therefore the warning raised in [27] very seriously: "no logic of norms without attention to a system of which they form part" ([27], pag. 29). As a result, constitutive counts-as statements can also be viewed as forms of speech acts creating a context: given that $\gamma_1 \rightarrow \gamma_2$ is a formula of Γ, $\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2$ could be read as "let it be that $\gamma_1 \rightarrow \gamma_2$ with all the statements of Γ and only of Γ or, using the terminology of [28], "fiat Γ and only Γ". On the other hand, a constitutive counts-as is false if either Γ does not define the context denoted by c, or if it expresses a classification which is valid in the model.

This is precisely the distinctive feature of constitutive counts-as with respect to its two classificatory relatives. While the classificatory versions of counts-as express what at least holds in a context (contextual classification) and, respectively, what at least hold in a context which is not globally true (proper contextual classification), the constitutive version expresses also what at most holds in a context, thereby making explicit what the context actually is in terms of a set of formulae of the language. We can have a constitutive counts-as statement only if it is known what the definition is of the context the statement refers to. In the classificatory versions of counts-as this knowledge is absent since it is only partially known what the context explicitly is. Classificatory and proper classificatory counts-as statements presuppose the existence of a context of which only some information is available. This issue is discussed in more detail in [7] where classificatory and proper classificatory counts-as statements are related with the notion of enthymeme, i.e., of argument with unstated premises.

From a technical point of view, this linguistic dependence amounts to the fact that expressions of the form $\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2$ where $\gamma_1 \rightarrow \gamma_2 \notin \Gamma$ are just undefined. Only the classifications that belong to Γ can be evaluated as constitutive counts-as. In other words $\Rightarrow^{co}_{c,\Gamma}$ conditionals are not "logical" in the sense of yielding a truth value for any pair of formulae (γ_1, γ_2). Because of this there is no logic, in a proper sense, of constitutive statements pertaining to one context description. Given a set of $\Rightarrow^{co}_{c,\Gamma}$ statements, nothing can be inferred about $\Rightarrow^{co}_{c,\Gamma}$ statements which are not already contained in the set Γ. It is therefore not possible to study $\Rightarrow^{co}_{c,\Gamma}$ conditionals from a structural perspective like it has been done for the other forms of counts-as in Propositions 1, 2 and 3.

How awkward this might sound it is perfectly aligned with the intuitions on the notion of constitution which backed Definition 8: constitutive counts-as are those classifications which are explicitly stated in the specification of the normative system. In a sense, constitutive statements are just given, and that is it. This does not mean, however, that constitutive statements cannot be used to perform reasoning. The following example depicts the most typical form of reasoning involving constitutive counts-as statements.

Proposition 6. *($\Rightarrow^{co}_{c,\Gamma}$ and $@_\nu$)*
The following formula is valid in $\mathrm{CXT}^{\top,\setminus}$ *frames for any* Γ *containing* $\gamma_1 \rightarrow \gamma_2$*:*

$$\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2 \rightarrow ((@_\nu \Gamma \wedge @_\nu \gamma_1) \rightarrow @_\nu \gamma_2) \tag{29}$$

Proof. Follows from Definition 4, Formula 28 and propositional logic.

This property shows how constitutive rules work in providing grounds for inferring the occurrence of new states-of-affairs: it is a rule of the normative system of Utrecht

University that if the promotor pronounces the PhD. student to be a doctor then this counts as the PhD. student to be a doctor ($\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2$); the current situation ν falls under the rules of Utrecht University ($@_\nu \Gamma$) and in the current situation the promotor pronounces a PhD. student to be a doctor ($@_\nu \gamma_1$), hence in the current situation the PhD. student is a doctor ($@_\nu \gamma_2$). Formula 29 perfectly captures the logical pattern of "conventional generation" as it is described in [29]:

> "Act-token A of agent G conventionally generates act-token B [. . .] only if the performance of A [. . .], together with a rule R saying that A [. . .] counts as B, guarantees the performance of B" ([29], p. 25).

It is instructive to notice that, besides formula $\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2$, what plays an essential role here is formula $@_\nu \Gamma$ (i.e., $[u](\nu \rightarrow \Gamma)$), which states that situation ν is one of the situations in context c. Without the notion of context definition and the availability of nominals, this could not be expressed.

Complex reasoning patterns involving constitutive counts-as statements arise also in relation with the other two notions of counts-as. The following section investigates the logical relationships between the three different senses of counts-as.

7 Relating the many faces of counts-as

This section is devoted to pursuing the last goal mentioned in the quote from [8] mentioned in Section 1: "and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations."

The logical relations between $\Rightarrow^{co}_{c,\Gamma}$, $\Rightarrow^{cl+}_{c}$ and $\Rightarrow^{cl}_{c}$ can be studied in logic $\mathbf{Cxt}^{\mathbf{u},\backslash}$ which extends both $\mathbf{K45^{ij}_n}$, i.e., the logic in which $\Rightarrow^{cl}_{c}$ has been defined, and $\mathbf{Cxt}^{\mathbf{u}}$, i.e., the logic in which $\Rightarrow^{cl+}_{c}$ has instead been defined.

Proposition 7. ($\Rightarrow^{cl}_{c}$ vs $\Rightarrow^{cl+}_{c}$ vs $\Rightarrow^{co}_{c,\Gamma}$)
In logic $\mathbf{Cxt}^{\mathbf{u},\backslash}$ the following formulae are valid:

$$(\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \rightarrow (\gamma_1 \Rightarrow^{cl}_{c} \gamma_2) \tag{30}$$
$$(\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \rightarrow (\gamma_1 \wedge \gamma_3 \Rightarrow^{cl}_{c} \gamma_2) \tag{31}$$
$$((\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \wedge (\gamma_2 \Rightarrow^{cl+}_{c} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl}_{c} \gamma_3) \tag{32}$$
$$(\gamma_1 \Rightarrow^{co}_{c,\Gamma} \gamma_2) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2) \tag{33}$$

provided that $\gamma_1 \rightarrow \gamma_2 \in \Gamma$.

Proofs are omitted and can be easily obtained by application of Definitions 2 and 6 and Proposition 1.

Let us have a look at the intuitive meaning of the formulae just proven. Formula 30 states something very simple: proper contextual classification implies contextual classification. This corresponds, in the model-theoretic notation used in Section 4, to the following inclusion relation: $\mathbb{CL}^+(W_c) \subseteq \mathbb{CL}(W_c)$.

Formulae 31 and 32 are particularly interesting. If we forget that the two operators $\Rightarrow^{cl+}_{c}$ and $\Rightarrow^{cl}_{c}$ denote two different notions and we read both expressions $\gamma_1 \Rightarrow^{cl+}_{c} \gamma_2$

and $\gamma_1 \Rightarrow^{cl}_{c} \gamma_2$ just as "γ_1 counts as γ_2", these formulae would sound as statements of the property of antecedent strengthening and of the transitivity of "counts-as". However, our formal analysis based on the acknowledgment that counts-as hides different senses has shown that transitivity and antecedent strengthening hold for $\Rightarrow^{cl}_{c}$ but not for $\Rightarrow^{cl+}_{c}$. On the other hand, and this is what Proposition 7 shows, their logical interactions display patterns clearly reminiscent of those properties. In a sense, we showed that questions such as "is transitivity a meaningful property for a characterization of counts-as?" are flawed by the possibility of confusing under the label counts-as different notions which enjoy different logical behaviors. This is a concrete example of the methodological concerns raised in Section 4.3.

More specifically, Formula 31 expresses that given a counts-as statement interpreted as a proper classification, a contextual classification can be inferred having as antecedent a strengthened version of the antecedent of the first statement, and this although proper contextual classification does not enjoy antecedent strengthening. In other words, although $\Rightarrow^{cl+}_{c}$ does not enjoy antecedent strengthening, it is nonetheless grounds for performing monotonic reasoning via $\Rightarrow^{cl}_{c}$. Analogous considerations apply to Formula 32. Proper contextual classification does not enjoy transitivity but reasoning via transitivity remains valid shifting from $\Rightarrow^{cl+}_{c}$ to $\Rightarrow^{cl}_{c}$.

Finally, Formula 33 translates the following intuitive fact: the promulgation of a constitutive rule guarantees the possibility of applying specific classificatory rules. If it is a rule of Γ that self-propelled conveyances count as vehicles (constitutive sense) then self-propelled conveyances count as vehicles in the context c defined by Γ in a proper classificatory sense.

With respect to the relation between constitution and classification, another interesting consequence of Definition 6 is the following one.

Proposition 8. (Impossibility of $\Rightarrow^{cl+}_{u}$ and $\Rightarrow^{co}_{u,\Gamma}$)
Proper classificatory counts-as statements and constitutive counts-as statements are impossible with respect to the universal context u. In fact, the following formulae are valid:

$$(\gamma_1 \Rightarrow^{cl+}_{u} \gamma_2) \rightarrow \bot \tag{34}$$

$$(\gamma_1 \Rightarrow^{co}_{u,\Gamma} \gamma_2) \rightarrow \bot \tag{35}$$

provided that $\gamma_1 \rightarrow \gamma_2 \in \Gamma$.

The proof is easily obtained from Definition 6.

Intuitively, Formula 34 states that what holds in general can not be the product of constitution, it can not be a "new" classification. This is indeed a very intuitive property: the fact that apples are classified as fruit cannot be a proper classification because it is something that always holds. Formula 35 states something slightly different: if something holds globally then it can not be used to constitute a context. Universal truths hold in all contexts, and therefore, can not be specific of any context. To put it otherwise, the statement "apple count as fruits" can not be a constitutive rule. Notice that contextual classificatory statements are instead perfectly sound also with respect to the universal context. Formula $\gamma_1 \Rightarrow^{cl}_{u} \gamma_2$ is a satisfiable formula in logic $\mathbf{Cxt}^{\mathbf{u},\backslash}$.

Let us now take into consideration properties displaying more complex reasoning patterns.

Proposition 9. (From $\Rightarrow^{co}_{c,\Gamma}$ to $\Rightarrow^{cl}_{c}$ and $\Rightarrow^{cl+}_{c}$ via $\Rightarrow^{cl}_{u}$)
The following formulae are valid:

$$(\gamma_2 \Rightarrow^{co}_{c,\Gamma} \gamma_3) \rightarrow ((\gamma_1 \Rightarrow^{cl}_{u} \gamma_2) \rightarrow (\gamma_1 \Rightarrow^{cl}_{c} \gamma_3)) \quad (36)$$

$$(\gamma_2 \Rightarrow^{co}_{c,\Gamma} \gamma_3) \rightarrow (((\gamma_1 \Rightarrow^{cl}_{u} \gamma_2) \wedge \neg[u](\gamma_1 \rightarrow \gamma_3)) \rightarrow (\gamma_1 \Rightarrow^{cl+}_{c} \gamma_3)) \quad (37)$$

provided that $\gamma_1 \rightarrow \gamma_2 \in \Gamma$.

The proof is straightforward by application of Definitions 2 and 8, and Propositions 3 and 1. These properties represent typical forms of reasoning patterns involving constitutive rules.

Formula 36: if it is a rule of Γ that $\gamma_2 \rightarrow \gamma_3$ ("self-propelled conveyances count as vehicles") and it is always the case that $\gamma_1 \rightarrow \gamma_2$ ("cars count as self-propelled conveyances"), then $\gamma_1 \rightarrow \gamma_3$ ("cars count as vehicles") holds in the context c defined by normative system Γ. Formula 37: if it is a rule of Γ that $\gamma_2 \rightarrow \gamma_3$ ("conveyances transporting people or goods count as vehicles") and it is always the case that $\gamma_1 \rightarrow \gamma_2$ ("bikes count as conveyances transporting people or goods") but it is not always the case that $\gamma_1 \rightarrow \gamma_3$ ("bikes count as vehicles"), then $\gamma_1 \rightarrow \gamma_3$ ("bikes count as vehicles") holds as a constituted classification in the context c defined by normative system Γ. Notice that while "cars count as self-propelled conveyances" in Formula 36 is a classificatory counts-as, since it might still be the case that cars are globally classified as vehicles, "bikes count as vehicles" in Formula 37 is instead a proper classificatory counts-as since it is explicitly stated that such classification is not a validity. Formula 37 represents nothing but the form of the reasoning pattern that has been used as starting point of our analysis (Example 1).

The very remarkable aspect about these properties is that they neatly show how the three senses of counts-as all play a role in the kind of reasoning we perform with constitutive rules. In particular, they show that the constitutive sense, though enjoying extremely poor logical properties, grounds in fact all the rich reasoning patterns proper of classificatory reasoning.

7.1 The *transfer problem* in the light of $\Rightarrow^{cl}_{c}$, $\Rightarrow^{cl+}_{c}$ and $\Rightarrow^{co}_{c,\Gamma}$

The 'transfer problem' has been introduced in [17] as a landmark for testing the intuitive adequacy of formalizations of counts-as. It can be exemplified as follows: suppose that somebody brings it about —for instance by coercion— that a priest effectuates a marriage, does this count as the creation of a state of marriage? Does anything implying that a priest effectuates a marriage count as the creation of a state of marriage? In other words, is the possibility to create a marriage transferable to anybody who brings it about that the priest effectuates the ceremony? In our framework, these questions get a triple formulation, one for each of the different senses of counts-as.

The transfer problem and $\Rightarrow^{cl}_{c}$. In [17], the transfer problem has been used as grounds for the rejection of the property of antecedent strengthening for counts-as conditionals. It is beyond doubt that a characterization of counts-as which enjoys the strengthening of the antecedent also exhibits the transfer problem: if that property holds, then the fact that the performance of the ceremony counts as the creation of a state of marriage implies that also a coerced performance does. As already noticed in [6], contextual classification ($\Rightarrow_c^{cl}$), which enjoys the strengthening of the antecedent (Proposition 1), does exhibit the transfer problem: whatever situation in which a priest performs a marriage ceremony is classified as a situation in which a marriage state comes to be. And this is precisely what we intuitively expect given the notion of contextual classification as informally introduced in Section 2. In other words, contextual classification *should* exhibit the transfer problem or, to put it another way, it should display a *transfer property*: the bringing about of a state of marriage should be transferable to any state in which a priest performs the ceremony.

The transfer problem and $\Rightarrow_c^{cl+}$. It has been shown that the characterization of proper contextual classification ($\Rightarrow_c^{cl+}$) does not enjoy the strengthening of the antecedent (Proposition 2). Interestingly enough, it still exhibits the transfer problem, as shown in Proposition 3 where Formula 24 has been proven valid: $\neg[u](\gamma_1 \rightarrow \gamma_3) \rightarrow ((\gamma_1 \Rightarrow_c^{cl+} \gamma_2) \wedge (\gamma_2 \Rightarrow_c^{cl+} \gamma_3)) \rightarrow (\gamma_1 \Rightarrow_c^{cl+} \gamma_3)$.

Intuitively, this formula expresses what follows. If the fact that a priest effectuates a marriage (γ_1) under coercion of a third party (γ_3) is not globally classified as giving rise to a state of marriage (γ_2) —which is the case, given the intuitive reading of the scenario at issue— then it is safe to say that if the priest's performance of the marriage counts as (in a proper classificatory sense) a marriage, then a coerced performance of the marriage counts also as a marriage.

Notice that this is again something perfectly intuitive given the assumptions about proper contextual classification exposed in Section 2: if a context c makes a classification $\gamma_1 \rightarrow \gamma_2$ true, which does not hold in general, then also the strengthened version of it $\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2$ is true in that context. Besides, if the strengthened version is also not true in general, it then follows that $\gamma_1 \wedge \gamma_3 \rightarrow \gamma_2$ is also a novel classification which is brought about by context c. Exhibiting the transfer problem is also for proper contextual classification not problematic.

From a technical point of view, Proposition 3 shows that a characterization of counts-as, which does not enjoy the strengthening of the antecedent, can still exhibit the transfer problem. This is equivalent to say that a notion of counts-as which genuinely rejects the transfer problem should not only reject antecedent strengthening, but some yet weaker property.

The transfer problem and $\Rightarrow_{c,\Gamma}^{co}$. The 'transfer property' does not hold, instead, for the constitutive reading of counts-as statements. In this view, counts-as statements represent the rules specifying a normative system. So, all that it is explicitly stated by the 'institution of marriage' is that if the priest performs the ceremony then the couple is married, while no rule belongs to that normative system which states that the action of a third party bringing it about that the priest performs the ceremony also counts as a marriage. Our formalization fully captures this feature. Let the 'marriage institution' c be sketched by the set of rules $\Gamma = \{p \rightarrow m\}$, i.e., by the rule "if the priest performs the ceremony, then the couple is married". Let then t represent the fact that a third party brings it about that p. For Definition 8 the counts-as $(t \wedge p) \Rightarrow_{c,\Gamma}^{co} m$ is just an undefined expression, because $((t \wedge p) \rightarrow m) \notin \Gamma$, that is, because the 'marriage institution' does not state such a classification. This seems to suggest that the transfer problem, rather than having to do with the structural properties of a logical connective, concerns instead whether a rule is part of the promulgations of a normative system or not, that is to say, whether a counts-as statement is a constitutive rule or not.

8 Conclusions

Moving from hints provided by the literature on legal and social theory concerning constitutive rules, the paper has analyzed counts-as statements as forms of contextual classifications. This analytical option, which we have studied from a formal semantics perspective, has delivered three semantically precise senses (Definitions 2, 6 and 8) in which counts-as statements can be interpreted, which we called *classificatory*, *proper classificatory* and *constitutive* readings. The three readings have then been formally analyzed making use of modal logic.

The classificatory reading resulted in a strong logic of counts-as conditionals enabling many properties which are typical of reasoning with concept subsumptions such as, in particular, reflexivity, strengthening of the antecedent and weakening of the consequent (Proposition 1). In fact, the logic obtained is nothing but a modal logic version of the contextual terminological logic we investigated in [9, 10].

The characterization of proper contextual classification resulted, instead, in a much weaker logic rejecting reflexivity, transitivity and antecedent strengthening (Proposition 2), but retaining cumulative transitivity (Proposition 3). Noticeably, this notion corresponds to the counts-as characterized in [17] once transitivity is substituted with cumulative transitivity. Finally, the notion of proper contextual classification has offered some new insights on the transfer problem (Section 7.1) showing that it cannot be genuinely avoided just by means of rejecting the strengthening of the antecedent in a conditional logic setting. This result motivated the investigation of a yet stronger form of counts-as which we developed in [30], and which stems nevertheless from the same analytical option backing the present work.

The formal analysis of constitutive counts-as (Definition 8) has neatly shown, with formal means, in what sense constitutive rules are never constitutive in isolation, but only as parts of systems of rules, and how constitutive rules work in providing grounds for attributing institutional properties to situations (Proposition 6). Constitutive counts-as has also been shown to imply the two classificatory readings (Proposition 7). Other logical interrelationships between the three notions of counts-as have also been studied (Propositions 8 and 9) showing also that the logical relations between them could actually be grounds for fallacies in the formal characterization of counts-as once the polysemy of the term "counts-as" is overlooked.

9 Appendix: Soundness and Completeness Results

The appendix provides soundness and completeness results for the logics introduced in the paper: $\mathbf{K45^{ij}_{n}}$, $\mathbf{Cxt^{u}}$ and $\mathbf{Cxt^{u,-}}$. Completeness will be proven via the canonical model technique.

9.1 Preliminaries

In all the logics considered the axiomatization of every modality $[i]$ contains all tautologies of propositional calculus, axiom K and is closed under rules MP and N. We will therefore make use of some general results of completeness theory for normal modal logics. We refer the reader to [21] for further details.

Recall first some facts about maximal consistent sets. Let Λ be a multi-modal normal logic. A maximal Λ-consistent set of formulae on a multi-modal language $\mathcal{L}_n$ is a set Φ s.t.: (a) $\bot$ is not derivable in Λ from Φ (i.e., Λ-consistency of Φ); (b) every set properly including Φ is Λ-inconsistent. Every maximal Λ-consistent set Φ is such that: $\Lambda \subseteq \Phi$; Φ is closed under rule MP; for all formulae ϕ either $\phi \in \Phi$ or $\neg\phi \in \Phi$; for all formulae $\phi, \psi : \phi \vee \psi \in \Phi$ iff $\phi \in \Phi$ or $\psi \in \Phi$.

We can now report the notion of canonical model for a logic Λ.

Definition 9. *(Canonical model for logic Λ)*
The canonical model $\mathcal{M}^\Lambda$ for a normal modal logic Λ in the multi-modal language $\mathcal{L}_n$ is the structure $\langle W^\Lambda, \{R_i^\Lambda\}_{1\leq i\leq n}, \mathcal{I}^\Lambda \rangle$ where:

1. *The set W^Λ is the set of all maximal Λ-consistent sets.*
2. *The anonical relations $R_i^\Lambda \in \{R_i^\Lambda\}_{1\leq i\leq n}$ are defined as follows: for all $w, w' \in W^\Lambda$, if for all formulae ϕ, $\phi \in w'$ implies $\langle i \rangle \phi \in w$, then $wR_i^\Lambda w'$.*
3. *The canonical interpretation $\mathcal{I}^\Lambda$ is defined by $\mathcal{I}^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}$.*

We briefly recall three key propositions of (modal) completeness theory. For the proofs we refer the reader to [21].

Lemma 1. *(Strong completeness = satisfiability of all consistent sets)*
A normal modal logic Λ is strongly complete w.r.t. a class of frames $\mathfrak{F}$ iff every Λ-consistent set of formulae is satisfiable on some $\mathcal{F} \in \mathfrak{F}$, i.e., it has a model $\mathcal{M}$ built on a frame $\mathcal{F}$ in class $\mathfrak{F}$.

Lemma 2. *(Existence Lemma)*
For any normal modal logic Λ and any state $w \in W^\Lambda$, it holds that: if $\langle i \rangle \phi \in w$ then there exists a state $w' \in W^\Lambda$ such that $wR_i^\Lambda w'$ and $\phi \in w'$.

Lemma 3. *(Truth Lemma)*
For any normal modal logic Λ and any formula ϕ, it holds that: $\mathcal{M}^\Lambda, w \models \phi$ iff $\phi \in w$.

Lemma 4. *(Canonical Model Theorem)*
Any normal modal logic Λ is strongly complete w.r.t. its canonical model $\mathcal{M}^\Lambda$.

We will also make use of the notion of point-generated subframe. Given a frame $\mathcal{F} = \langle W, \{R_i\}_{1\leq i\leq n}\} \rangle$, a point-generated subframe $\mathcal{F}^w$ of a frame $\mathcal{F}$ is a structure $\langle W^w, \{R_i^w\}_{1\leq i\leq n}\} \rangle$ such that: (a) W^w is the set of states $w' \in W$ such that there exists, for any R_i, a finite R_i-path from w to w'; (b) $R_i^w = R_i \cap (W^w \times W^w)$, i.e., each R_i^w is the restriction of R_i on W^w. The following result is of interest.

Lemma 5. *(Generated subframes preserve validity)*
Let $\mathfrak{F}$ be a class of frames and $g(\mathfrak{F})$ be the class of point-generated subframes of the frames in $\mathfrak{F}$. It holds that, for all formulae ϕ on language $\mathcal{L}_n$: $\mathfrak{F} \models \phi$ iff $g(\mathfrak{F}) \models \phi$.

Finally, we need a way to relate context frames (see Section 3.1), that is, structures of the type $\langle W, \{W_i\}_{i \in C}\rangle$ with relational structures of the type $\langle W, \{R_i\}_{i \in C}\rangle$. The bridge is offered by *locally universal* relations. A relation R_i on a set W is locally universal if:

- For all $R_i \in \{R_i\}_{i \in C}$ and $w \in W$, R_i is universal on $r_i(w)$;
- For all $w, w' \in W$, $r_i(w) = r_i(w')$, where r_i is a function associating to each state w the set of reachable states via relation R_i.

The following representation result holds for this family of relations.

Lemma 6. *(Representation of context frames)*
A relation R_i on W is locally universal iff there exists a set $W_i \subseteq W$ such that for all w, w', wR_iw' iff $w' \in W_i$.

Proof. The right to left direction is straightforward. From left to right: for every $w, w' \in W$ it holds, by the definition of function r that wR_iw' iff $w' \in r_i(w)$. Since R_i is locally universal, it holds that for every $w, w'' \in W$, $r_i(w) = r_i(w'')$. It is now enough to stipulate $W_i = r_i(w'')$ for any w'' to obtain the desired result: there exists a set $W_i \subseteq W$ such that for all w, w', wR_iw' iff $w' \in W_i$.

Leaving technicalities aside, the property of local universality forces relations in $\{R_i\}_{i \in C}$ to cluster the domain of the frame in sets of worlds (contexts), one for each accessibility relation, and then defines these accessibility relations in such a way that the sets of accessible worlds correspond, for each world in W, to the clusters.

9.2 Soundness and completeness of $\mathbf{K45^{ij}_n}$

The proof of soundness is routinary. It is well-known that inference rules `MP` and `N` preserve validity on any class of frames[5]. Providing the soundness of $\mathbf{K45^{ij}_n}$ w.r.t. CXT frames boils than down to checking the validity of axioms 4^{ij} and 5^{ij}.

Theorem 1. *(Soundness of $\mathbf{K45^{ij}_n}$ w.r.t.* CXT *frames)*
Logic $\mathbf{K45^{ij}_n}$ is sound w.r.t. the class of CXT *frames.*

Proof. The validity of 4^{ij} is proven showing that its contrapositive has no countermodel. Such countermodel $\mathcal{M}$ would contain a state w such that for a given formula ϕ, $\mathcal{M}, w \models \langle j \rangle \langle i \rangle \phi$ and $\mathcal{M}, w \models \neg \langle i \rangle \phi$. Hence, by the semantics, $\exists w' \in W_i$ s.t. $\mathcal{M}, w \models \phi$ and $\nexists w' \in W_i$ s.t. $\mathcal{M}, w \models \phi$, which is impossible. The validity of 5^{ij} is proven in the same way. Suppose there is a model $\mathcal{M}$ and a state w such that $\mathcal{M}, w \models \langle i \rangle \phi$ and $\mathcal{M}, w \models \neg [j] \langle i \rangle \phi$. Hence, by the semantics, $\exists w' \in W_i$ s.t. $\mathcal{M}, w \models \phi$ and $\nexists w' \in W_i$ s.t. $\mathcal{M}, w \models \phi$.

The proof of completeness is obtained in two steps.

1. First, via the canonical model, it is proven that logic $\mathbf{K45^{ij}_n}$ is complete with respect to the class of i-j transitive (if wR_iw' and $w'R_jw''$ then wR_jw''), and i-j euclidean (if wR_iw' and wR_jw'' then $w'R_jw''$) frames[6].

[5] See [21].
[6] In [31], frames with this property are called, respectively, hyper-transitive and hyper-euclidean.

2. Second, it is proven that if $\mathfrak{F}$ is the class of of i-j transitive and i-j euclidean frames, for every $\phi \in \mathcal{L}_n$: $\mathfrak{F} \models \phi$ iff CXT $\models \phi$.

Theorem 2. *(Completeness of $\mathbf{K45^{ij}_n}$)*
Logic $\mathbf{K45^{ij}_n}$ is strongly complete w.r.t. the class of i-j transitive and i-j euclidean frames.

Proof. By Lemma 1, given a $\mathbf{K45^{ij}_n}$-consistent set Φ of formulae, it suffices to find a model state pair ($\mathcal{M}$, w) such that: (a) $\mathcal{M}, w \models \Phi$, and (b) the frame $\mathcal{F}$ on which $\mathcal{M}$ is based is i-j transitive and i-j euclidean. Let $\mathcal{M}^{\mathbf{K45^{ij}_n}} = \left\langle W^{\mathbf{K45^{ij}_n}}, \{R_i^{\mathbf{K45^{ij}_n}}\}_{i \in C}, \mathcal{I}^{\mathbf{K45^{ij}_n}} \right\rangle$ be the canonical model of logic $\mathbf{K45^{ij}_n}$, and let Φ^+ be any maximal consistent set in $W^{\mathbf{K45^{ij}_n}}$ extending Φ. By Lemma 3 it follows that $\mathcal{M}^{\mathbf{K45^{ij}_n}}, \Phi^+ \models \Phi$, which proves (a). It remains to be proven that $\left\langle W^{\mathbf{K45^{ij}_n}}, \{R_i^{\mathbf{K45^{ij}_n}}\}_{i \in C} \right\rangle$ enjoys i-j transitivity (b.1) and i-j euclidicity (b.2). To prove (b.1) consider three states $w, w', w'' \in W^{\mathbf{K45^{ij}_n}}$ such that $wR_j^{\mathbf{K45^{ij}_n}}w'$ and $w'R_i^{\mathbf{K45^{ij}_n}}w''$. Suppose then that $\phi \in w''$. As $w'R_i^{\mathbf{K45^{ij}_n}}w''$ and $wR_j^{\mathbf{K45^{ij}_n}}w'$, it follows that $\langle i \rangle \phi \in w'$ and then that $\langle j \rangle \langle i \rangle \phi \in w$. Now, w is a maximal consistent set of logic $\mathbf{K45^{ij}_n}$, it therefore contains formula $\langle j \rangle \langle i \rangle \phi \rightarrow \langle i \rangle \phi$ (i.e., the contrapositive of axiom 4^{ij}), hence $\langle i \rangle \phi \in w$ and thus $wR_i^{\mathbf{K45^{ij}_n}}w''$ which completes the proof of (b.1). Analogously, to prove (b.2) consider three states $w, w', w'' \in W^{\mathbf{K45^{ij}_n}}$ such that $wR_j^{\mathbf{K45^{ij}_n}}w'$ and $wR_i^{\mathbf{K45^{ij}_n}}w''$. Suppose then that $\phi \in w''$. It follows that $\langle i \rangle \phi \in w$ and since w is a maximal consistent set of logic $\mathbf{K45^{ij}_n}$, it therefore contains formula $\langle i \rangle \phi \rightarrow [j] \langle i \rangle \phi$ (i.e., axiom 5^{ij}) and hence $[j] \langle i \rangle \phi \in w$. From this and from $wR_i^{\mathbf{K45^{ij}_n}}w''$ it follows that $\langle i \rangle \phi \in w''$, that is to say, for any formula ϕ it is the case that: if $\phi \in w'$ then $\langle i \rangle \phi \in w''$. Now, by Definition 9, this implies that $w'R_i^{\mathbf{K45^{ij}_n}}w''$ which proves (b.2).

Lemma 7. *(Semantic equivalence for* CXT *frames)*
Consider the class $\mathfrak{F}$ of i-j transitive and i-j euclidean frames. For every $\phi \in \mathcal{L}_n$, $\mathfrak{F} \models \phi$ iff CXT $\models \phi$*. That is,* CXT *frames and $\mathfrak{F}$ frames define the same logic.*

Proof. From right to left: for every ϕ, CXT $\models \phi$ implies $\mathfrak{F} \models \phi$. The proof is obtained showing that if $\mathcal{F}$ is a CXT frame then it is i-j transitive and i-j euclidean. By Lemma 6, for all $w, w' \in W$, $w' \in W_i$ iff wR_iw'. To prove i-j transitivity, suppose that wR_iw' ($w' \in W_i$) and $w'R_jw''$ ($w'' \in W_j$). It follows therefore that wR_jw''. The proof of i-j euclidicity is perfectly analogous. Suppose that wR_iw' ($w' \in W_i$) and wR_jw'' ($w'' \in W_j$), hence $w'R_jw''$. From left to right: for every ϕ, $\mathfrak{F} \models \phi$ implies CXT $\models \phi$. In this case, the proof is obtained by showing that every i-j transitive and i-j euclidean frame, which is also point-generated, is a context frame. By Lemma 5, it holds that for every ϕ, $\mathfrak{F} \models \phi$ iff $g(\mathfrak{F}) \models \phi$. Now, let $\mathcal{F}^w$ be any frame in $g(\mathfrak{F})$ generated by some state w. In order to prove the desired result, it suffices to show that every i-j transitive and i-j euclidean frame $\mathcal{F}^w$ generated by state w is a CXT frame. By Lemma 6, this is proven by showing that for every $R_i^w \in \{R_i^w\}_{i \in C}$, $w'R_i^ww''$ iff $w'' \in r_i^w(w)$. This amounts to prove that for every w', w'' if there exists an R_i-path from w to w' and from w to w'', then $w' R_i w''$ iff $w'' \in r_i(w)$. From left to right, if there exists an R_i-path from w to w' and $w' R_i w''$, then by transitivity (which is a special case of i-j transitivity) $w R_i w''$, that is, $w'' \in r_i(w)$. From right to left, if there exists an R_i-path from w to w' and $w'' \in r_i(w)$, then $w R_i w''$ and hence, by euclidicity, $w' R_i w''$.

Corollary 1. *(Completeness of $\mathbf{K45^{ij}_n}$ w.r.t.* CXT *frames)*
Logic $\mathbf{K45^{ij}_n}$ is strongly complete w.r.t. the class of CXT *frames.*

Proof. Follows directly from Theorem 2 and Lemma 7.

9.3 Soundness and completeness of $\mathbf{Cxt^u}$

On the grounds of the results of the previous section, the proof of soundness and completeness of $\mathbf{Cxt^u}$ w.r.t. $\mathrm{CXT}^\top$ can be easily obtained. Soundness boils down to prove that axioms T^u and $\subseteq .ui$ are valid in $\mathbf{Cxt^u}$ frames.

Theorem 3. *(Soundness of $\mathbf{Cxt^u}$ w.r.t.$\mathrm{CXT}^\top$ frames)*
Logic $\mathbf{Cxt^u}$ is sound w.r.t. the class of $\mathrm{CXT}^\top$ frames.

Proof. Trivial, given the interpretation of the $[u]$-operator as universal quantification on all the states in the domain W of the frame.

Let $\mathfrak{TE}^\sim$ be the class of frames satisfying the following properties: they are i-j transitive, i-j euclidean; they contain an equivalence relation R_u such that for all $i \in C$, $R_i \subseteq R_u$. Again, completeness w.r.t. the relevant class of frames is proven in two steps.

1. Logic $\mathbf{Cxt^u}$ is first proven to be complete w.r.t. the class of $\mathfrak{TE}^\sim$ frames.
2. It is then proven that for any formula ϕ on $\mathcal{L}_n$: $\mathfrak{TE}^\sim \models \phi$ iff $\mathrm{CXT}^\top \models \phi$.

Theorem 4. *(Completeness of $\mathbf{Cxt^u}$)*
Logic $\mathbf{Cxt^u}$ is strongly complete w.r.t. the class $\mathfrak{TE}^\sim$ frames.

Proof. By Lemma 1, given a $\mathbf{Cxt^u}$-consistent set Φ of formulae, it suffices to find a model state pair ($\mathcal{M}$, w) such that: (a) $\mathcal{M}, w \models \Phi$, and (b) the frame $\mathcal{F}$ on which $\mathcal{M}$ is based is i-j transitive and i-j euclidean and contains a universal relation. Claim (a) is proven by making use of Lemma 3. It remains to be proven that the frame $\langle W^{\mathbf{Cxt^u}}, \{R_i^{\mathbf{Cxt^u}}\}_{i \in C} \rangle$ of the canonical model enjoys i-j transitivity and i-j euclidicity (b.1) and that there exists a relation $R_u^{\mathbf{Cxt^u}} \in \{R_i^{\mathbf{Cxt^u}}\}_{i \in C}$ such that $R_u^{\mathbf{Cxt^u}}$ is an equivalence relation (b.2) and for every $i \in C$, $R_i \subseteq R_u$ (b.3). Claim (b.1) follows from Theorem 2 since $\mathbf{Cxt^u}$ extends $\mathbf{K45^{ij}_n}$. As to (b.2), it follows from (b.1) that each $R_i^{\mathbf{Cxt^u}}$ is transitive and euclidean and, therefore, so is $R_u^{\mathbf{Cxt^u}}$. The proof of the reflexivity of $R_i^{\mathbf{Cxt^u}}$ is then routinary. Finally, claim (b.3) needs to be proven. Consider two states $w, w' \in W^{\mathbf{Cxt^u}}$ such that $w R_i^{\mathbf{Cxt^u}} w'$. Suppose then that $\phi \in w'$. It follows that $\langle i \rangle \phi \in w$. Since w is a maximal $\mathbf{Cxt^u}$-consistent set, it contains formula $\langle i \rangle \phi \rightarrow \langle u \rangle \phi$ (i.e., the contrapositive of axiom $\subseteq .ui$) and therefore $\langle u \rangle \phi \in w$. Hence, by Definition 9, $w R_u^{\mathbf{Cxt^u}} w'$.

Lemma 8. *(Semantic equivalence for $\mathrm{CXT}^\top$ frames)*
For any formula ϕ on $\mathcal{L}_n$: $\mathfrak{TE}^\sim \models \phi$ iff $\mathrm{CXT}^\top \models \phi$. That is, $\mathrm{CXT}^\top$ frames and $\mathfrak{TE}^\sim$ frames define the same logic.

Proof. The proof is analogous to the proof of Lemma 7. The direction from right to left (for every ϕ, $\textsc{Cxt}^{\top} \models \phi$ implies $\mathfrak{TE}^{\sim} \models \phi$) is straightforwardly proven by observing that every $\textsc{Cxt}^{\top}$ frame represents a frame containing a universal relation R_u. In fact, a relation R_u is universal iff it holds that: for any $w, w' \in W$, wR_uw' iff $w' \in W$ (notice that this is a special case of Lemma 6). But every universal relation is an equivalence relation, which also includes all R_i's for any $i \in C$. That all $\textsc{Cxt}^{\top}$ frames are i-j transitive and i-j euclidean follows from Lemma 7. This completes the proof of the right-to-left direction. From left to right: for every ϕ, $\mathfrak{TE}^{\sim} \models \phi$ implies $\textsc{Cxt}^{\top} \models \phi$. Lemma 7 has proven that every i-j transitive and i-j euclidean frame generated by state w is a $\textsc{Cxt}$ frame. Consider now the relation R^w_u of the point-generated subframe $\mathcal{F}^w$ of a frame $\mathcal{F} \in \mathfrak{TE}^{\sim}$ containing an equivalence relation R_u such that for all $i \in C$, $R_i \subseteq R_u$. To obtain the desired result —via Lemma 5— it suffices to show that the relation R^w_u is universal on W^w, which is trivial.

Corollary 2. *(Completeness of* $\mathbf{Cxt^u}$ *w.r.t.* $\textsc{Cxt}^{\top}$ *frames)*
Logic $\mathbf{Cxt^u}$ *is strongly complete w.r.t. the class of* $\textsc{Cxt}^{\top}$ *frames.*

Proof. Follows directly from Theorem 4 and Lemma 8.

9.4 Soundness and completeness of $\mathbf{Cxt^{u,-}}$

The proof of soundness is routinary.

Theorem 5. *(Soundness of* $\mathbf{Cxt^{u,-}}$ *w.r.t.* $\textsc{Cxt}^{\top,\setminus}$ *frames)*
Logic $\mathbf{Cxt^u}$ *is sound w.r.t. the class of* $\textsc{Cxt}^{\top,\setminus}$ *frames.*

Proof. It is to show that axioms `Covering` and `Packing` are valid in $\textsc{Cxt}^{\top,\setminus}$ frames by just noticing that in $\textsc{Cxt}^{\top,\setminus}$ frames, for any atomic context index c, family $\{W_c, W_{-c}\}$ is a bipartition of the domain W: $W \subseteq W_c \cup W_{-c}$, i.e., family $\{W_c, W_{-c}\}$ is a covering of W; and $W_c \cap W_{-c} = \emptyset$, i.e., $\{W_c, W_{-c}\}$ is a packing of W.

Let $\mathfrak{TE}^{\sim,\setminus}$ be the class of frames satisfying the following properties: they are i-j transitive, i-j euclidean; they contain an equivalence relation R_u such that for all $i \in C$, $R_i \subseteq R_u$; the set of relations $\{R_i\}_{i \in C}$ is such that, for any atomic context index c and states $w, w' \in W$: wR_uw' implies wR_cw' or $wR_{-c}w'$; and wR_cw' implies not $wR_{-c}w'$. Again, completeness w.r.t. the $\textsc{Cxt}^{\top,\setminus}$ frames is proven in two steps.

1. Logic $\mathbf{Cxt^{u,-}}$ is first proven to be complete w.r.t. the class of $\mathfrak{TE}^{\sim,\setminus}$ frames.
2. It is then proven that for any formula ϕ on $\mathcal{L}_n$: $\mathfrak{TE}^{\sim,\setminus} \models \phi$ iff $\textsc{Cxt}^{\top,\setminus} \models \phi$.

For completeness we need to prove some facts about the canonical model of logic $\mathbf{Cxt^{u,-}}$. Before stating and proving the desired lemma consider first that, since logic $\mathbf{Cxt^{u,-}}$ extends logic $\mathbf{Cxt^u}$, we know by Theorem 4 that the canonical model of logic $\mathbf{Cxt^{u,-}}$ contains an equivalence relation $R_u^{\mathbf{Cxt^{u,-}}}$ such that for every $i \in C$, $R_i^{\mathbf{Cxt^{u,-}}} \subseteq R_u^{\mathbf{Cxt^{u,-}}}$. Recall also that every equivalence relation yields a partition on its domain. The cluster of the partition yielded by $R_u^{\mathbf{Cxt^{u,-}}}$ on $W^{\mathbf{Cxt^{u,-}}}$ containing state w is denoted by $r_u^{\mathbf{Cxt^{u,-}}}(s)$, that is, the set of states reachable by w via $R_u^{\mathbf{Cxt^{u,-}}}$.

Lemma 9. *(Properties of maximal $\mathbf{Cxt^{u,-}}$-consistent sets)*
Let $\mathcal{M}^{\mathbf{Cxt^{u,-}}} = \left\langle W^{\mathbf{Cxt^{u,-}}}, \{R_i^{\mathbf{Cxt^{u,-}}}\}_{i \in C}, \mathcal{I}^{\mathbf{Cxt^{u,-}}} \right\rangle$ be the canonical model of logic $\mathbf{Cxt^{u,-}}$.

1. *All maximal $\mathbf{Cxt^{u,-}}$-consistent sets in $W^{\mathbf{Cxt^{u,-}}}$ contain at least one nominal;*
2. *If a nominal is contained in a maximal $\mathbf{Cxt^{u,-}}$-consistent set $w \in W^{\mathbf{Cxt^{u,-}}}$ then it is not contained in any other maximal $\mathbf{Cxt^{u,-}}$-consistent set $w' \in W^{\mathbf{Cxt^{u,-}}}$ which is accessible from w via $R_u^{\mathbf{Cxt^{u,-}}}$. In other words, if two maximal $\mathbf{Cxt^{u,-}}$-consistent sets contain the same nominal, and belong to the same cluster of the partition of $W^{\mathbf{Cxt^{u,-}}}$ yielded by $R_u^{\mathbf{Cxt^{u,-}}}$, then they are the same set.*
3. *Each nominal in $\mathbb{N}$ is contained in at least one maximal $\mathbf{Cxt^{u,-}}$-consistent set.*

Proof. Clause 1. Let Φ be a maximal $\mathbf{Cxt^{u,-}}$-consistent set of $\mathcal{L}_n^{u,-}$ formulae. To prove the first claim, suppose per absurdum that $\forall \nu \in \mathbb{N}, \neg\nu \in \Phi$. It follows that for every ν there exists a finite conjunction θ of formulae from Φ such that: $\vdash \nu \rightarrow \neg\theta$. Now, either ν occurs in θ and thus $\nu \in \Phi$, or ν does not occur in θ and therefore, by rule `Name`, $\neg\theta \in \Phi$ which is impossible. Clause 2 is proven in two steps. (a) Given a nominal $\nu \in \Phi$, for any maximal $\mathbf{Cxt^{u,-}}$-consistent set Φ it is proven that for all ϕ: $\phi \in \Phi$ iff $[u](\nu \rightarrow \phi) \in \Phi$. (b) Given two maximal $\mathbf{Cxt^{u,-}}$-consistent sets Φ and Φ', if $\nu \in \Phi, \Phi'$ and $\Phi R_u^{\mathbf{Cxt^{u,-}}} \Phi'$ then $\Phi = \Phi'$. Let us prove (a). From left to right. We assumed a nominal $\nu \in \Phi$, hence if $\phi \in \Phi$ then $\nu \wedge \phi \in \Phi$, being Φ a maximal $\mathbf{Cxt^{u,-}}$-consistent set. The set Φ also contains formula $\phi \rightarrow \langle u \rangle\, \phi$ (i.e., the contrapositive of axiom $\mathtt{T}^u$) and $\langle u \rangle\, (\nu \wedge \phi) \rightarrow [u](\nu \rightarrow \phi)$ (i.e., axiom `Most`) from which it follows that $\langle u \rangle\, (\nu \wedge \phi) \in \Phi$ and hence that $[u](\nu \rightarrow \phi) \in \Phi$. From right to left: for any $\phi \in \Phi$, if $[u](\nu \rightarrow \phi) \in \Phi$ then by axiom $\mathtt{T}^u$ we obtain $\nu \rightarrow \phi \in \Phi$ and then by `MP` $\phi \in \Phi$. Let us prove (b) per absurdum. Suppose $\Phi \neq \Phi'$. Then there should exist a formula ϕ such that $\phi \in \Phi$ and $\phi \notin \Phi'$ and hence $\neg\phi \in \Phi'$. From (a) it follows that $[u](\nu \rightarrow \phi) \in \Phi$ and since $\Phi R_u^{\mathbf{Cxt^{u,-}}} \Phi'$ we obtain that $\nu \rightarrow \phi \in \Phi'$ and via `MP` $\phi \in \Phi'$, which is impossible. Clause 3 follows easily from Lemma 2 and the fact that every state $w \in W^{\mathbf{Cxt^{u,-}}}$ contains formula $\langle u \rangle\, \nu$ (axiom `Least`).

The lemma concerns some key properties of the interpretation of nominals. Clause 1 guarantees that in the canonical model every maximal $\mathbf{Cxt^{u,-}}$-consistent set contains a nominal, that is, that $\mathcal{I}^{\mathbf{Cxt^{u,-}}}$ is a surjection on the set of singletons of $W^{\mathbf{Cxt^{u,-}}}$. Clause 2 is particularly interesting. It states that the same nominal can in fact belong to different maximal $\mathbf{Cxt^{u,-}}$-consistent sets if these sets are not related via $R_u^{\mathbf{Cxt^{u,-}}}$. To put it otherwise, nominals behave as real names only if they refer to sets in a same cluster in the partition yielded by $R_u^{\mathbf{Cxt^{u,-}}}$. It follows that interpreting nominals on a generated subframe guarantees them to behave like names, and this is precisely enough for our purposes since generated subframes preserve validity (Lemma 5). Finally, Clause 3 states just that all nominals get a denotation.

Theorem 6. *(Completeness of $\mathbf{Cxt^{u,-}}$)*
Logic $\mathbf{Cxt^{u,-}}$ is strongly complete w.r.t. the class of $\mathfrak{T}\mathfrak{E}^{\sim,\setminus}$ frames, that is, frames satisfying the following clauses:

1. *They are i-j transitive, i-j euclidean.*
2. *They contain an equivalence relation R_u such that for all $i \in C$, $R_i \subseteq R_u$.*
3. *The set of relations $\{R_i\}_{i \in C}$ is such that, for any atomic context index c and states $w, w' \in W$: (3.a) wR_uw' implies wR_cw' or $wR_{-c}w'$; and (3.b) $wR_{-c}w'$ implies not wR_cw'.*

Proof. By Lemma 1, given a $\mathbf{Cxt^{u,-}}$-consistent set Φ of formulae, it suffices to find a model state pair ($\mathcal{M}$, w) such that: (a) $\mathcal{M}, w \models \Phi$, and (b) the frame $\mathcal{F}$ on which $\mathcal{M}$ is based satisfies clauses 1-3. Claim (a) is proven by making use of Lemma 3. It remains to be proven that the frame $\left\langle W^{\mathbf{Cxt^{u,-}}}, \{R_i^{\mathbf{Cxt^{u,-}}}\}_{i \in C} \right\rangle$ of the canonical model satisfies clauses 1-3. Clause 1 and Clause 2 are proven to be satisfied by Theorem 4 since $\mathbf{Cxt^{u,-}}$ extends $\mathbf{K45^{ij}_n}$ and $\mathbf{Cxt^u}$. Claims (3.a) and (3.b) of clause 3 remain to be proven. To prove claim (3.a) it has to be shown that: for any atomic context index c and states $w, w' \in W^{\mathbf{Cxt^{u,-}}}$, $wR_u^{\mathbf{Cxt^{u,-}}}w'$ implies $wR_c^{\mathbf{Cxt^{u,-}}}w'$ or $wR_{-c}^{\mathbf{Cxt^{u,-}}}w'$. Consider two states $w, w' \in W^{\mathbf{Cxt^{u,-}}}$ such that $wR_u^{\mathbf{Cxt^{u,-}}}w'$ and suppose that $\phi \in w'$. Since w is a maximal $\mathbf{Cxt^{u,-}}$-consistent set, it contains formula $\langle u \rangle \phi \rightarrow (\langle c \rangle \phi \vee \langle -c \rangle \phi)$ (i.e., the contrapositive of axiom `Covering`) and therefore $\langle c \rangle \phi \vee \langle -c \rangle \phi \in w$. For the properties of maximal consistent sets it follows that either $\langle c \rangle \phi \in w$ or $\langle -c \rangle \phi \in w$, and hence by Definition 9, either $wR_c^{\mathbf{Cxt^{u,-}}}w'$ or $wR_{-c}^{\mathbf{Cxt^{u,-}}}w'$, which proves (3.a). As to (3.b), it should be proven that for any atomic context index c and states $w, w' \in W^{\mathbf{Cxt^{u,-}}}$, $wR_{-c}^{\mathbf{Cxt^{u,-}}}w'$ implies not $wR_c^{\mathbf{Cxt^{u,-}}}w'$. Suppose that $wR_{-c}^{\mathbf{Cxt^{u,-}}}w'$. By Clause 1 in Lemma 9 we know that w' should contain at least one nominal. Since all nominals denote at least one state (Clause 3 in Lemma 9) we can pick a nominal ν and suppose it to be the nominal contained in w'. By Clause 2 of this theorem, from $wR_{-c}^{\mathbf{Cxt^{u,-}}}w'$ it follows that $wR_u^{\mathbf{Cxt^{u,-}}}w'$ and from this, by Clause 2 in Lemma 9, we know that there is no $w'' \in r_u^{\mathbf{Cxt^{u,-}}}(w)$ such that $\nu \in w''$. By Definition 9 it follows that $\langle -c \rangle \nu \in w$. Now, w is a maximal $\mathbf{Cxt^{u,-}}$-consistent set and it contains thus formula $\langle -c \rangle \nu \rightarrow \neg \langle c \rangle \nu$ (i.e., axiom `Packing`). It follows that $\neg \langle c \rangle \nu \in w$ and it is therefore not the case that $wR_c^{\mathbf{Cxt^{u,-}}}w'$, which proves claim (3.b).

Lemma 10. *(Semantic equivalence for* $\text{CXT}^{\top,\setminus}$ *frames)*
For any formula ϕ on $\mathcal{L}_n$: $\mathfrak{TE}^{\sim,\setminus} \models \phi$ iff $\text{CXT}^{\top,\setminus} \models \phi$. That is, $\text{CXT}^{\top,\setminus}$ frames and $\mathfrak{TE}^{\sim,\setminus}$ frames define the same logic.

Proof. The proof is analogous to the proof of Lemmata 7 and 8. From right to left: for every ϕ, $\text{CXT}^{\top,\setminus} \models \phi$ implies $\mathfrak{TE}^{\sim,\setminus} \models \phi$. The results follow by the application of Proposition 6. From $W = W_c \cup W_{-c}$ for any atomic context identifier c, it follows that for every $w, w' \in W$, wR_uw' implies wR_cw' or $wR_{-c}w'$. And from $W_c \cap W_{-c} = \emptyset$ for any atomic context identifier c, it follows that for every $w, w' \in W$, $wR_{-c}w'$ implies not wR_cw'. From left to right: for every ϕ, $\mathfrak{TE}^{\sim,\setminus} \models \phi$ implies $\text{CXT}^{\top,\setminus} \models \phi$. It suffices to show that every point-generated subframe of any $\mathfrak{TE}^{\sim,\setminus}$ frame is a $\text{CXT}^{\top,\setminus}$ frame. The desired result follows then from Lemma 5. Consider a frame $\mathcal{F}^w \in g(\mathfrak{TE}^{\sim,\setminus})$ generated by state w. We show that $\mathcal{F}^w$ is a $\text{CXT}^{\top,\setminus}$ frame. Building on the proofs of Lemma 7 and on the fact that $\mathfrak{TE}^{\sim,\setminus}$ already contain a universal relation, it just needs to be shown that for any atomic index c: (a) $W^w \subseteq r_c(w) \cup r_{-c}(w)$ and (b) $r_c(w) \cap r_{-c}(w) \subseteq \emptyset$. Both claims are straightforwardly proven by observing that for any atomic context index c and states $w', w'' \in W^w$: $w' R^w_u w''$ (i.e., $w'' \in W^w$) implies $w' R^w_c w''$ (i.e., $w'' \in r_c(w)$) or $w' R^w_{-c} w''$ (i.e., $w'' \in r_{-c}(w)$); and $w' R^w_c w''$ (i.e., $w'' \in r_c(w)$) implies not $w' R^w_{-c} w''$ (i.e., $w'' \notin r_{-c}(w)$).

Corollary 3. *(Completeness of $\mathbf{Cxt}^{\mathbf{u},-}$ w.r.t. $\text{CXT}^{\top,\setminus}$ frames)*
Logic $\mathbf{Cxt}^{\mathbf{u},-}$ is strongly complete w.r.t. the class of $\text{CXT}^{\top,\setminus}$ frames.

Proof. Follows directly from Theorem 6 and Lemma 10.

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