Forgetting and Update – an exploration[0]

Abhaya C Nayak[1], Yin Chen[2] and Fangzhen Lin[3]

[1] Intelligent Systems Group, Department of Computing
Division of ICS, Macquare University, Sydney, NSW 2109, Australia
abhaya@ics.mq.edu.au
[2] Department of Comp. Sc., South China Normal University
Guangzhou, P.R. China
ychen@cs.ust.hk
[3] Department of Comp. Sc., Hong Kong U of Sc. & Tech.
Clear Water Bay, Hong Kong
flin@cs.ust.hk

Abstract. Knowledge Update (respectively Erasure) and Forgetting are two very different concepts, with very different underlying motivation. Both are tools for knowledge management; however while the former is meant for accommodating new knowledge into a knowledge corpus, the latter is meant for modifying – in fact reducing the expressivity – of the underlying language. In this paper we show that there is an intimate connection between these two concepts: a particular form of knowledge update and literal forgetting are inter-definable. This connection is exploited to enhance both our understanding of update as well as forgetting in this paper.

Keywords. Knowledge Update, Erasure, Forgetting, Dalal Distance, Winslett Distance.

Knowledge management involves removal of dated information as well as incorporation of new information. Relevant literature provides many different approaches to accomplishing this tasks, including belief change (revision and contraction) [2,3], knowledge update (update and erasure) [4], belief merging [5,6] and forgetting (forgetting and remembering) [7,8,9]. While these are related but different concepts, there is a crucial difference between the first three on one hand and forgetting on the other. The former three manipulate knowledge – they deal with the stuff deemed known; the beliefs. Forgetting on the other hand manipulates the language in which knowledge is expressed. Consider erasure and forgetting, for instance. We may dramatize the difference between them as follows: while belief erasure purports to answer the question "What should I believe if I can no longer support the belief that the cook killed Cock Robin?", forgetting purports to answer the question "What should I believe if *Killing* was a concept not afforded in my language?".[4]

On the face of it, it would appear that forgetting on the one hand, and belief change (or update) on the other are completely different concepts, and the relation, if any, between the language manipulating function forgetting and the belief manipulating functions such as belief change and belief update, would be a tenuous one. The difference

[0] This work is an extention of results published in [1]

[4] It's a case of Relational forgetting, of which, literal forgetting that we will largely deal with in this paper is a special case – propositions are 0-ary relations.

becomes very explicit from the way these functions are constructed. In a certain sense, the result of forgetting a primitive relation (including 0-ary relations or atomic sentences) is computed by projecting the original knowledge corpus into a less expressive language, one in which the relation in question has been eliminated. On the other hand, the result of discarding a belief is computed by removing the belief in question as well as a judicious selection of its supporting beliefs – a process for which, an extra-logical choice mechanism (such as a distance function between the possible worlds or entrenchment relation among beliefs) is utilised. Perhaps surprisingly, it turns out that belief update (respectively erasure) and forgetting, in a qualified way, are inter-translatable. This paper focuses on this inter-translatability as its central theme, and explores the ways known results about belief manipulators and language manipulators can compliment each other.

We initially restrict out attention to belief update and belief erasure, based on Hamming Distance (or Dalal measure) [10] between worlds on the one hand, and literal forgetting as applied to a finite representation of the knowledge base on the other [9]. Afterwards, we show that the results hold for more general distance measures including Winslett measure [11]. We believe this connection will have bearing upon many interesting related issues, including:

1. In the forgetting literature, the knowledge corpus is primarily represented as a first order theory. Forgetting in the context of propositional theory is largely of derivative interest. On the other hand, the literature on belief manipulating functions primarily assumes a propositional language. Given the intertranslatibility between forgetting and updating (erasure) at propositional level, can we extrapolate an interesting account of first-order knowledge corpus update?
2. Assume that we have an account of first order knowledge base updating. Now, in the caes of forgetting, we have literal forgetting as well as relational forgetting. One would naturally try to accordingly extend the account of update to relational update. Is it meaningful to talk of relational update (erasure)? If, then what sort of relational update do we get?
3. In case of update, the new information that one wants to accommodate, or the old information that needs to be discarded, can be any sentence. On the other hand, in case of forgetting, the stuff to be forgotten is an atomic sentence or a relation. Can we meaningfully talk of forgetting any arbitrary formula? If so, how should it be done.

In the next section, we provide the background material. In section 2, we establish the promised inter-translatability, modulo Dalal distance. In section 3 we show that such results hold for more general distance measures. Finally we conclude with a short discussion and summary.

1 Background

Belief manipulation such as updating and erasing, as well as language manipulation such as forgetting involve representation of an agent's current stock of beliefs in a specified language as well as a manipulator operator. In this section we detail crucial parts of the notation used, as well as how the notions of update, erasure and forgetting are formally captured. We adopt a few other conventions that are not explicated here due to lack of space, but we hope they will be obvious from the context.

1.1 Notation

We shall consider a framework based on a finitary propositional language $\mathcal{L}$, over a set of atoms, or propositional letters, $\mathcal{A} = \{a, b, c, \dots\}$, and truth-functional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. $\mathcal{L}$ also includes the truth-functional constants $\top$ and $\bot$. To clarify the presentation we shall use the following notational conventions. Upper-case Roman characters ($A,\ B, \dots$) denote sets of sentences in $\mathcal{L}$. Lower-case Roman characters ($a,\ b, \dots$) denote arbitrary sentences of $\mathcal{L}$.

An *interpretation* of $\mathcal{L}$ is a function from $\mathcal{A}$ to $\{T, F\}$; Ω is the set of interpretations of $\mathcal{L}$. A *model* of a sentence x is an interpretation that makes x true, according to the usual definition of truth. A model can be equated with its defining set of literals, or alternatively, with the corresponding **truth-vector**. $[x]_{\mathcal{L}'}$ denotes the set of models of sentence x over any language $\mathcal{L}'$, while by $[x]$ we will denote $[x]_{\mathcal{L}}$. For interpretation ω we write $\omega \models x$ for x is true in ω. For $x \in \mathcal{L}$, we will define $\mathcal{L}(x)$, the sub-language of $\mathcal{L}$ in which x is expressed, as comprising the minimum set of atoms required to express x, as follows, where x^p_q is the result of substituting atom q everywhere for p in x:

$$\mathcal{L}(x) = \{p \in \mathcal{A} \mid x^p_\top \not\equiv x^p_\bot\} \cup \{\top, \bot\}$$

This set of atoms is unique. Thus $\mathcal{L}(p \wedge (q \vee \neg q)) = \mathcal{L}(p) = \{p, \top, \bot\}$. This can be extended to sets of sentences in the obvious way. It follows trivially that if $\models x \leftrightarrow y$ then $\mathcal{L}(x) = \mathcal{L}(y)$. Also note that if $\models x$ then $\mathcal{L}(x) = \{\top, \bot\}$.

By a theory or belief set we will mean a subset of $\mathcal{L}$ that is closed under Cn. We denote the set of all theories by $\mathcal{T}$. Since $\mathcal{L}$ is finitary, any theory T can be represented by a finite set $T' \subseteq \mathcal{L}$ such that $Cn(T') = T$, and consequently by a single sentence $\bigwedge_{x_i \in T'} x_i$. Traditionally, in the case of knowledge update, erasure and forgetting, a belief corpus is represented by a single sentence, where as in accounts of belief change, knowledge corpus is represented as a theory. Both update and erasures are defined as functions $\oplus, \ominus :\ \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ where as forgetting is defined as a function $\odot :\ \mathcal{L} \times \mathcal{A} \rightarrow \mathcal{L}$. Thus, for instance, $k \oplus e$ and $k \ominus e$ are respectively the sentences representing the update and erasure of sentence e from the knowledge base represented by sentence k. On the other hand, $k \odot a$ is the theory that results from forgetting atom a from knowledge k.

1.2 Update and Erasure

Let $\leq_\omega$ for any world ω be a preorder (reflexive and transitive relation) over Ω that is *faithful* with respect to ω, ie., for every $\omega' \in \Omega$, we have both $\omega \leq_\omega \omega'$ and if $\omega' \leq_\omega \omega$ then $\omega = \omega'$. If, in addition, either $\omega' \leq_\omega \omega''$ or $\omega'' \leq_\omega \omega'$, for every pair ω', ω'', then it is a *total* preorder. By $\omega_1 \leq_\omega \omega_2$ we intuitively mean that ω_2 is at least as similar to ω as ω_1 is. As usual, the strict part of $\leq_\omega$ is denoted by $<_\omega$ This relation compares worlds with respect to their similarity or proximity to ω. By $\omega' \sim_\omega \omega''$ we will denote that ω' and ω'' are not comparable under $\leq_\omega$; and by $\omega' \equiv_\omega \omega''$ we will understand that both $\omega' \leq_\omega \omega''$ and $\omega'' \leq_\omega \omega'$. Given a set Δ of worlds, by $min_{\leq\ \omega}(\Delta)$ we denote the set

$\{\omega' \in \Delta \mid \omega'' \not<_\omega \omega'$ for any $\omega'' \in \Delta\}$ of worlds in Δ that are closest or most similar to ω.

Definition 1 (Update). $[k \oplus e] = \bigcup_{\omega \in [k]} min_{\leq \omega}([e])$

Intuitively, when we learn that e has been effected, with respect to each world allowed by our current knowledge, we compute what the scenario would be if e was effected in that world; and collating all those worlds results in the models of our updated knowledge. Obtaining a knowledge base k' from these models is not difficult.

Definition 2 (from Update to Erasure). *The erasure operation is defined by reduction to update using the Harper Identity.* $k \ominus e = k \vee (k \oplus \neg e)$

Alternatively, Erasure can be directly defined, and update can be defined from erasure using the Levi Identity.

Definition 3 (Erasure). *Erasure can be directly defined as:*
$[k \ominus e] = [k] \cup \bigcup_{\omega \in [k]} min_{\leq \omega}([\neg e])$

Definition 4 (from Erasure to Update). $k \oplus e = (k \ominus \neg e) \wedge e$

The definitions we have provided of Update and Erasure are constructive. Alternatively these operations can be defined as those (with appropriate signature) that satisfy the respective rationality postulates of update and erasure. The postulates can be found in [4].

It is easily noticed that $[k \oplus e] \subseteq [e]$, equivalently, $k \oplus e \vdash e$, i.e., a successful update by a sentence e leads to e being believed. Furthermore, since we assume that a world closer it itself than any other world, and the worlds in $[e]$ that are $\leq_\omega$-minimal with respect to *any* world ω in $[k]$ constitute $[k \oplus e]$, it is clear that $[k] \cap [e] \subseteq [k \oplus e]$. Since $[k] = ([k] \cap [e]) \cup ([k] \cap [\neg e])$, we get the following as an immediate consequence:

Observation 1 $[k] \subseteq ([k \oplus e] \cup [k \oplus \neg e])$ *wherefore* $k \vdash (k \oplus e) \vee (k \oplus \neg e)$.

Now, it follows from Definition 2 that $(k \ominus e) \vee (k \ominus \neg e) = k \vee (k \oplus e) \vee (k \oplus \neg e)$. However, as we just noticed, $k \vdash (k \oplus e) \vee (k \oplus \neg e)$. This leads to:

Observation 2 $(k \oplus e) \vee (k \oplus \neg e) \equiv (k \ominus e) \vee (k \ominus \neg e)$.

We will call $(k \oplus e) \vee (k \oplus \neg e)$ the *symmetric update*, and $(k \ominus e) \vee (k \ominus \neg e)$ the *symmetric erasure*, of k by e. [5] Thus Observation 2 shows the equivalence between symmetric update and symmetric erasure.

[5] Our terminology is somewhat at variance with that used by Katsuno and Mendelzon [4] who refer to $(k \oplus e) \vee (k \oplus \neg e)$ as the symmetric erasure of e from k, and mention that Winslett, in an unpublished manuscript, calls it the forgetting of e from k.

1.3 Literal Forgetting

Forgetting, as we mentioned earlier, is more about language manipulation than about belief manipulation. While belief erasure (say by e) involves traversing to a judicious belief state where e is not believed (while allowing the possibility of $\neg e$ still believed), forgetting of e involves moving to a state where no information regarding e is still retained. Forgetting has a clear meaning when e in the above case is an atomic sentence – forgetting of an atom a means moving to a state where neither a nor $\neg a$ is believed. Hence it is often called *literal forgetting*.

Let us look at forgetting from an information theoretic point of view. We are considering the forgetting of atom a from knowledge base k. For an interpretation ω of k, we will call ω' the a-dual of ω if it differs from ω exactly on the truth assignment to a. For instance, if the first bit in $\omega = 1011$ is the assignment to a, then the a-dual of ω is 0011. A belief base k has information regarding atom a just in case $[k]$ contains some interpretation ω but not its a-dual.

Definition 5 (a-dual closure). *Given a set of interpretations Δ and atomic sentence a, Δ is closed under a-dual iff the a-dual of every interpretation $\omega \in \Delta$ is also in Δ.*

Clearly, if Δ is closed under a-dual, then it contains no information regarding atom a. Furthermore, if Δ is closed under a-dual for every atom $a \in \mathcal{A}$, then Δ has no information at all. Since forgetting by a from k is meant to removing all information pertaining to a from k, it suggests the operation of closing the set of interpretations $[k]$ under a-dualship. Accordingly we define the dual closure operation $\uplus : 2^{\Omega} \times \mathcal{A} \rightarrow 2^{\Omega}$ as follows:

Definition 6 (dual-closure operation). *Given a set of interpretations $\Delta \subseteq \Omega$ and atomic sentence $a \in \mathcal{A}$, the dual-closure $\uplus(\Delta,\ a)$ of Δ under a is the smallest subset of Ω that includes Δ and is closed under a-dual. In other words, $\uplus(\Delta,\ a) = \Delta' \subseteq \Omega$ such that (1) $\Delta \subseteq \Delta'$, (2) Δ' is closed under a-dual, and (3) if any Δ'' such that $\Delta \subseteq \Delta'' \subseteq \Delta'$ is closed under a-dual, then $\Delta' = \Delta''$.*

This definition of $\uplus$ is non-constructive. It also presumes that $\uplus(\Delta,\ a)$ is unique. Theorem 1 below guides its construction, and justifies the presumption in question. We need one more definition before producing this result.

Definition 7 (Model Counterparts). *For language $L \subseteq \mathcal{L}$, by $[S]_{\Omega_L}$ we denote the set of "model counterparts" of S in Ω_L. It can be represented as the set of interpretations ω_2 in Ω_L that are sub-model of some interpretation or other, $\omega_1 \in [S]$ as: $\{\omega_2 \in \Omega_L \mid \exists_{\omega_1 \in [S]} \omega_2 \subseteq \omega_1$. Alternatively, it is the set of interpretations in Ω that are super models of any such model ω_2 in Ω_L: $\{\omega \in \Omega \mid \exists_{\omega_1 \in [S],\ \omega_2 \in \Omega_L} \omega_2 \subseteq \omega_1$ and $\omega_1 \subseteq \omega\}$.*

Example 1. Consider for example $\mathcal{A} = \{a, b, c\},\ \mathrm{L} = \mathcal{L}(\{b, c\})$ and $S = \{a \leftrightarrow b\}$. Now, $[S] = \{11\star, 00\star\}$, and its submodels in Ω_L are $\{-\star\star\} = \Omega_L$.[6] Alternatively, the desired set is $\{\star\star\star\} = \Omega$.

[6] From here onwards we use $\star$ to indicate atoms with "wild" truth-values, and $-$ to indicate location of "non-applicable atoms" by a given interpretation.

Theorem 1. [7] *Given a set of interpretations $\Delta \subseteq \Omega$, an atomic sentence $a \in \mathcal{A}$ and a sentence d such that $[d] = \Delta$ (a canonical construction of d in DNF*[8] *can be easily carried out taking the composition of Δ as a guide), $\uplus(\Delta,\ a) \ = \ [d]_{\Omega_{\mathcal{L}(\mathcal{A}\setminus\{a\})}}$.*

We introduced the concept of dual-closure being motivated by an intuitive notion of literal forgetting. Now we are in a position to offer a semantic definition of literal forgetting.

Definition 8 (Semantics of literal forgetting). *The forgetting operation $\odot :\ \mathcal{L} \times \mathcal{A} \rightarrow \mathcal{L}$ should be such that $[\odot(k,\ a)] \ = \ \uplus([k],\ a)$.*

The following example illustrates that this definition captures the semantic intuition behind forgetting. It also substantiates our generic claim in the introductory note that forgetting is more about language manipulation than about belief manipulation.

Example 2. Let $\mathcal{A} = \{a, b, c\}$ and $k = (a \wedge \neg b) \vee c$. Thus $[k] = \{001, 101, 100, 111, 011\}$. We tabulate two distinct but equivalent constructions of $[\odot(k,\ a)]$ in accordance with the two definitions of $[S]_{\Omega_L}$. In some truth vectors '$-$' is used to indicate the absence of relevant atoms.

$[k]$	$[\odot_1(k,\ a)]$	$[\odot_2(k,\ a)]$
001	-01	001
101		101
100	-00	100
		000
111	-11	111
011		011

Thus $\odot_1(k,\ a)$ can be expressed as $\neg(b \wedge \neg c)$ where as $\odot_2(k,\ a)$ can be expressed as $\neg(a \wedge b \wedge \neg c)\ \wedge \neg(\neg a \wedge b \wedge \neg c)$. While the latter is expressed in $\mathcal{L}$, the former is couched in a language devoid of the atom a. Nonetheless, both are equivalent, and are equivalent to $(b \rightarrow c)$ as expected.

Now that the semantics of forgetting is in place, we consider its syntactic characterisation. For this purpose, we borrow the definition of forgetting from [9].

Definition 9 (Syntactic representation of literal forgetting). *For any formula x and atom a, denote by $x|_{1\leftarrow a}$ the result of replacing every occurrence of a in x by $\top$, and by $x|_{0\leftarrow a}$ the result of replacing every occurrence of a in x by $\bot$. Forgetting an atom a from a knowledge base represented as sentence k is defined as: $k \odot a = k|_{1\leftarrow a} \vee k|_{0\leftarrow a}$.*

Continuing with the Example 2, we find that $\odot((a \wedge \neg b) \vee c,\ a) \ \equiv \ ((\top \wedge \neg b) \vee c) \ \vee ((\bot \wedge \neg b) \vee c) \ \equiv \ \neg b \vee c \ \equiv (b \rightarrow c)$ matching the result from semantic analysis. The following result shows that this match is not due to a lucky choice of example.

[7] Proofs of results provided in this paper can be obtained from the first named author upon request.

[8] A formula in **D**isjunctive **N**ormal **F**orm is a disjunction of conjunctive clauses, where a *conjunctive clause* is a conjunction of literals.

Theorem 2. *For any sentence* $k \in \mathcal{L}$*, atom* $a \in \mathcal{A}$ *and world* $\omega \in \Omega$*, it holds that* $\omega \in [\odot(k,\ a)]$ *(as defined in Definition 9) if and only if* $\omega \in \biguplus([k], a)$*.*

The Definition 9 also makes it clear that $\oplus$ is very syntax-sensitive. Consider for instance four knowledge bases that are equivalent when conjunctively interpreted: $B_1 = \{a,\ b\}$, $B_2 = \{a \wedge b\}$, $B_3 = \{a \rightarrow b,\ a \vee b\}$, and $B_4 = \{a,\ \neg a \vee b\}$, and we are to forget a. If we were to understand forgetting from a set as piece-wise forgetting from the member-formulas in the set, then we get as result: $B_1' = \{\top,\ b\}$, $B_2' = \{b\}$, $B_3' = \{\top\}$ and $B_4' = \{\top\}$. This explains why we represent the knowledge base as a single sentence. If, however, piece-meal processing is desirable, it can be done by representing the knowledge base as a set of conjunction of literals, disjunctively interpreted (corresponding to DNF), and process the set-members individually. This would closely correspond the semantic account given in Example 2. In contrast, if a base is represented as a set of clauses, it will give the wrong result, as in the case of B_4 above.

2 Update, Erasure and Forgetting – Dalal connection

We have noticed that update and erasure are belief manipulating operations whereas forgetting is a language manipulating operation. Nonetheless, it is clear that both in case of erasure and forgetting, beliefs are lost. We now explore whether, and if so, how they can be inter-defined.

The first obstacle to this is the fact that where as in case of erasure, a particular relational measure is assumed to exist over Ω, no such measure is assumed in case of forgetting. Inter-translatability between these two concepts therefore will therefore push us to either impose such a external measure on Γ and accordingly generalize the definition of forgetting, or restrict us to measures on Γ that are implicitly given by the knowledge base k itself and look at more restricted forms of erasure. Lang and Marqis [7] choose the former option. In this paper we choose the latter. This brings us to specific approaches to erasure (and updating) – in particular one based on Hamming Distance – between interpretations.

2.1 A Concrete Update Operator via Hamming-Distance

Belief update operators presume a preorder over interpretations, as do belief revision operators. Two such classes preorders have drawn much attention from the researchers in the area, one based on the Hamming Distance between the sets of literals representing two interpretations, and the other based on the symmetric difference between them. The former was introduced for belief update by Forbus [12], and by Dalal [10] for belief revision. On the other hand, Winslett [11] introduced the latter for belief update, and Satoh [13] for belief revision. Here we restrict our attention to updates only, and that too based on Hamming Distance. As popularly called in the area, refer to Hamming distance as "Dalal Distance".

Definition 10 (Dalal Distance). *Given* $\omega,\ \omega' \in \Omega_{\mathbf{L}}$ *for some language* $\mathbf{L}$*, the Dalal Distance* $dist_D(\omega,\ \omega')$ *between* ω *and* ω' *is the number of atoms that they assign different values:* $dist_D(\omega,\ \omega') = |\{a \in Atoms(\mathbf{L}) \mid \omega \models a \text{ iff } \omega' \vdash \neg a\}|$.

Thus, $dist_D(101,\ 001) = 1$, $dist_D(101,\ 011) = 2$, and $dist_D(101,\ 110) = 2$.

Definition 11 (Dalal Preorder $\leq^D$). *Given three interpretations ω, ω' and $\omega'' \in \Omega_{\mathbf{L}}$ for some language $\mathbf{L}$, $\omega' \leq^D_\omega \omega''$ iff $dist_D(\omega,\ \omega') \leq dist_D(\omega,\ \omega'')$.*

Thus, we get $001 <^D_{101} 011$, $001 <^D_{101} 110$, and $011 \equiv^D_{101} 110$. It is easily verified that Given a language $\mathbf{L}$ and a world $\omega \in \Omega_{\mathbf{L}}$, the Dalal Preorder $\leq^D_\omega$ is a faithful, total preorder over $\Omega_{\mathbf{L}}$. The definitions of update and erasure we provided earlier (see Definitions 1 – 4) directly or indirectly employ a preorder. We can thus obtain specific update and erasure operators simply by plugging in Dalal and Winslett preorders.

Definition 12 (Dalal Update and Dalal Erasure). *By $\oplus^D$ and $\ominus^D$ we will denote the Dalal Update and Erase operations obtained by plugging in the Dalal Preorder in the definitions of Update and Erasure provided in Definitions 1 and 2 (or alternatively in Definitions 3 and 4).*

Example 3. Let $\mathcal{A} = \{a, b, c\}$ and $k = (a \wedge \neg b) \vee c$ as in Example 2. Thus $[k] = \{001, 101, 100, 111, 011\}$ and $[a] = \{100, 101, 110, 111\}$. We consider updates and erasures of k by a. We need to compute $\bigcup_{\omega \in [k]} min_{\leq\omega}([a])$ for update and $\bigcup_{\omega \in [k]} min_{\leq\omega}([\neg a])$ for erasure with respect to $\leq^D$. Since $\leq^D$ is faithful, for any sentence e,

$$\bigcup_{\omega \in [k]} min_{\leq\omega}([e]) = ([k] \cap [e]) \cup \bigcup_{\omega \in [k] \setminus [e])} min_{\leq\omega}([e]).$$

In the table below, the first two columns give the $\leq_\omega$-ordering of $[a]$ for $\omega \in ([k] \setminus [a])$, and the last three columns give the $\leq_\omega$-ordering of $[\neg a]$ for $\omega \in ([k] \setminus [\neg a])$. The comma's are to be interpreted as $\equiv_D$. The minimal elements of interest are in bold face.

$[\mathbf{a}]^{\leq^D_{001}}$	$[\mathbf{a}]^{\leq^D_{011}}$	$[\neg\mathbf{a}]^{\leq^D_{100}}$	$[\neg\mathbf{a}]^{\leq^D_{101}}$	$[\neg\mathbf{a}]^{\leq^D_{111}}$
110	100	011	010	000
100, 111	101, 110	001, 010	000, 011	001, 010
101	**111**	**000**	**001**	**011**

Accordingly we get:

$[k \oplus^D a] = \{100, 101, 111\}$ and $[k \ominus^D a] = \{001, 011, 100, 101, 111, 000\}$, whereby, $(k \oplus^D a) \equiv a \wedge (b \rightarrow c)$ and $(k \ominus^D a) \equiv b \rightarrow c$. Furthermore, using Definitions 2 and 4 (or going through semantics again) we can also show that $(k \oplus^D \neg a) \equiv \neg a \wedge (b \rightarrow c)$ and $(k \ominus^D \neg a) \equiv (a \wedge \neg b) \vee c \equiv k$.

2.2 Updating, Erasing and Forgetting – Literally

We discussed updating and erasing with Dalal and Winslett preorders so that these accounts of belief manipulation and the account of language manipulation via forgetting are on the same footing, with the hope that we can easily explore the connection between them. Since the account of forgetting involves forgetting of literals, we will also assume in this section that the update and erase in question are those by literals. Hence it is prudent to start with exploring the connection between them, and later on to convert that connection with respect to updates.

Now, both forgetting and erase are operations that *remove* information. In fact, while erase removes some information involving the sentence a, forgetting removes *all* information regarding the atom a – it semantically removes a itself. Hence it would appear that perhaps forgetting of a would involve erasure of both a and $\neg a$. The following result shows that it indeed is the case – forgetting of atom a indeed results in that part of the knowledge that is left untouched by both erasure of a and erasure of $\neg a$ under Dalal Preorder!

Theorem 3 (Dalal Erasure to Forget). *Given k a knowledge base, $k \odot a \equiv (k \ominus^D a) \vee (k \ominus^D \neg a)$ for every atom a.*

Appealing to Observation 2 and Theorems 3, we right away obtain:

Corollary 1 (Dalal Update to Forget). *$k \odot a \equiv (k \oplus^D a) \vee (k \oplus^D \neg a)$ for any knowledge base k and any atom a.*

Putting the results Theorems 3 and Corollary 1 we obtain:

Theorem 4 (Dalal Update/Erasure to Forget). *For any knowledge base k and any atom a, the formula $k \odot a$ is equivalent to each of the following two formulas:*

1. $(k \oplus^D a) \vee (k \oplus^D \neg a)$
2. $(k \ominus^D a) \vee (k \ominus^D \neg a)$

Theorem 4 shows that forgetting of a knowledge base k by atom a can be identified, individually, with both the symmetric update and the symmetric erasure of k by a where the update and erasure operations in question are based on Dalal preorder. The following example illustrates this result.

Example 4. Let's revisit Example 3. We know that

1. $(k \oplus^D a) \equiv a \wedge (b \rightarrow c)$
2. $(k \ominus^D a) \equiv b \rightarrow c$.
3. $(k \oplus^D \neg a) \equiv \neg a \wedge (b \rightarrow c)$.
4. $(k \ominus^D \neg a) \equiv (a \wedge \neg b) \vee c$.

Plugging in these values, we get

- $(k \oplus^D a) \vee (k \oplus^D \neg a) \equiv$
  $(a \wedge (b \rightarrow c)) \vee (\neg a \wedge (b \rightarrow c)) \equiv b \rightarrow c$.
- $(k \ominus^D a) \vee (k \ominus^D \neg a) \equiv$
  $(b \rightarrow c) \vee ((a \wedge \neg b) \vee c) \equiv b \rightarrow c$.

Thus in either case the value obtained for $k \odot a$, reduced to $\oplus^D$ or $\ominus^D$, is $b \rightarrow c$, matching the value of $k \odot a$ independently obtained in Example 2.

Now that we know how Literal Forgetting can be reduced to update (or erase) of a special nature, we consider the question whether the reduction can be done in the reverse direction. Theorem 5 below shows that Dalal literal updates can be defined via forgetting. The similarity between this result, and the definition of updating via erasure (Definition 4) is rather striking.

Theorem 5 (Forget to Update). *Given any knowledge base k and any atom a, the following equivalences hold:*

1. $(k \oplus^D a) \equiv (k \odot a) \wedge a$
2. $(k \oplus^D \neg a) \neg a) \equiv (k \odot a) \wedge \neg a$

This result shows how the update of a knowledge base by a literal, modulo Dalal preorder, can be computed purely by syntactic manipulation in an efficient manner. Syntactic characterisation of most belief update and belief revision operators have been done by del Val [14,15]. del Val's approach assumes conversion of the relevant portion of the knowledge base k, and the evidence e into DNF. However, as pointed out by Delgrande and Schaub [16], this may require exponential time step and exponential space. Clearly update and erasure operations via forgetting are a lot more frugal.

Applying Definition 2, that $k \ominus e = k \vee (k \oplus \neg e)$, to the definition of updates in Theorem 5, and noting that $[k] \subseteq [k \odot a]$, we reduce Dalal erasure to forgetting as well:

Theorem 6 (Forget to Erase). *Given any knowledge base k and any atom a, the following equivalences hold:*

1. $(k \ominus^D a) \equiv (k \odot a) \wedge (a \rightarrow k)$
2. $(k \ominus^D \neg a) \equiv (k \odot a) \wedge (\neg a \rightarrow k)$

In Example 5 below we complete the circle in the sense that the results of updates and erasures computed in accordance with Theorems 5 and 6 are shown to match to those independently computed in Example 3.

Example 5. We know from Example 2 that $(k \odot a) \equiv (b \rightarrow c)$. Applying Theorems 5 and 6 we get

1. $(k \oplus^D a) \equiv (b \rightarrow c) \wedge a$
2. $(k \oplus^D \neg a) \equiv (b \rightarrow c) \wedge \neg a$
3. $(k \ominus^D a)$
   $\equiv (b \rightarrow c) \wedge a \rightarrow ((a \wedge \neg b) \vee c)$
   $\equiv (b \rightarrow c) \wedge (a \rightarrow (b \rightarrow c)) \equiv (b \rightarrow c)$.
4. $(k \ominus^D \neg a)$
   $\equiv (b \rightarrow c) \wedge \neg a \rightarrow ((a \wedge \neg b) \vee c)$
   $\equiv (b \rightarrow c) \wedge ((a \wedge \neg b) \vee (a \vee c))$
   $\equiv (b \rightarrow c) \wedge (a \vee c) \equiv (a \wedge \neg b) \vee c)$.

matching the values of updates and erasures independently computed in the Example 3.

3 Other Connections

Now that we have seen the connection between Dalal erasure and forgetting, let's see whether there is any connection between forgetting and Winslett erasure [11]. As we will see, the connection is no less interesting.

Definition 13 (Winslett Distance). [9] *Given two interpretations* $\omega,\ \omega' \in \Omega_{\mathbf{L}}$ *for some language* $\mathbf{L}$*, the Winslett Distance* $dist_W(\omega,\ \omega')$ *between* ω *and* ω' *is the set of atoms that they assign different values:* $dist_W(\omega,\ \omega') = \{a \in Atoms(\mathbf{L}) \mid \omega \models a \text{ iff } \omega' \vdash \neg a\}$*. It may be taken to be the symmetric difference between* $Literals(\omega)$ *and* $Literals(\omega')$ *as well.*

Definition 14 (Winslett Preorder). *Given three interpretations* ω*,* ω' *and* $\omega'' \in \Omega_{\mathbf{L}}$ *for some language* $\mathbf{L}$*,* $\omega' \leq^W_\omega \omega''$ *iff* $dist_W(\omega,\ \omega') \subseteq dist_W(\omega,\ \omega'')$*.*

As a prelude, we prove the following lemma concerning Winslett distance which claims that if a world ω has an a-dual in ω' in $[a]$, then ω' must be $\leq^W_\omega$-minimal in $[a]$.

Lemma 1 (a-dual and $\leq^W$-minimality for literals). *Given three interpretations* ω, ω' *and* ω'' *such that* $\omega', \omega'' \in [a]$ *for some atom* a*, if* ω' *is the* a*-dual of* ω*, then* $\omega' \leq^W_\omega \omega''$*.*

Proof Sketch: The proof is guided by the following intuition. Since $\omega' \in [a]$ and and is an a-dual of ω, clearly ω assigns 0 to a, and every world in $[a]$ assigns 1 to a. Thus, $\{a, \neg a\} \subseteq Dist_W(\omega, \omega'')$ for every $\omega'' \in [a]$. On the other hand, since ω' is the a-dual of ω, clearly $Dist_W(\omega, \omega') = \{a, \neg a\}$. Thus, $\omega' \leq^W_\omega \omega''$. ■

This lemma effectively implies that, given a world ω and an atom $[a]$, the set of $\leq_\omega$-minimal worlds in $[a]$ under the preorders $\leq^D$ and $\leq^W$ are same. In other words, as long as the input formula is an atom, update and erasure as insensitive to the difference between Dalal distance and Winslett distance. Hence we get:

Theorem 7 (Winslett Measure and Forget). *The inter-translation between forget and update, and between forget and erasure discussed so far (Theorems 3 – 4) holds if we use Winslett Measure instead of Dalal Measure in order to generate the preorders over the worlds.*

Proof Sketch: We know how update, erasure and literal forgetting are inter-definable given that the underlying measure is the Dalal measure. By Lemma 1, in case of atomic input formula, the difference between dalal measure and Winslett measure vanishes as far as update and erasure behaviour is concerned. Hence, it follows that forget, update and erasure can be analogously interdefined in the context of Winslett measure. ■

Theorem 8 (A more general result). *Literal forgetting operation* $\odot$ *and literal updating operation* $\oplus$ *(resp. erasure mutatis mutandis) are interdefinable as*

1. $(k \odot a) = (k \oplus a) \vee (k \oplus \neg a)$*, and*
2. $(k \oplus a) = (k \odot a) \wedge a$

if and only if the operation $\oplus$ *is generated by preorder family* $\leq$ *that is faithful, and* $\omega' <_\omega \omega''$ *given any atom* a' *and interpretations* $\omega', \omega'' \in [a']$ *such that* ω' *is the* a'*-dual of* ω *but* ω'' *is not.*

[9] Strictly speaking Winslett Distance not a distance, not even a pseudo distance. We use the term “distance” here in a very loose sense.

Proof Sketch:

RIGHT TO LEFT ($\Leftarrow$) Assume that the preorder family $\leq$ has the required property Since the preorders are faithful, clearly if $\omega \in [k] \cap [a]$ then $\omega \in min(\leq_\omega)$. On the other hand, if $\omega \in [k] \setminus [a]$ then $min(\leq_\omega) = \{\omega'\}$ where ω' is the a-dual of ω. Thus $[k \oplus_{\leq} a]$ equates

$$([k] \cap [a]) \cup \bigcup_{\omega \in [k] \cap [\neg a]} \{\omega' \in [a] \mid \omega' \text{ is the } a\text{-dual of } \omega\}.$$

Similarly, $[k \oplus_{\leq} \neg a]$ is

$$([k] \cap [\neg a]) \cup \bigcup_{\omega \in [k] \cap [a]} \{\omega' \in [\neg a] \mid \omega' \text{ is the } a\text{-dual of } \omega\}.$$

Thus together we get, $([k \oplus_{\leq} a] \cup [k \oplus_{\leq} \neg a]) =$

$$([k] \cup \bigcup_{\omega \in [k]} \{\omega' \mid \omega' \text{ is the } a\text{-dual of } \omega\} = \biguplus(k, a)$$

leading to the desired result.

LEFT TO RIGHT ($\Rightarrow$) Assume that the preorder family $\leq$ is either

1. not faithful, i.e. there exist $\omega \neq \omega'$ such that either
   (a) $\omega' \leq_\omega \omega$, or
   (b) $\omega \sim_\omega \omega'$
   or
2. there exist $\omega \in \Omega$, atom a' and $\omega', \omega'' \in [a']$ such that
   (a) ω' is the a'-dual of ω,
   (b) ω'' is not the a'-dual of ω, and
   (c) $\omega' \not<_\omega \omega''$.

Case 1 Arbitrarily pick $\omega \neq \omega'$ such that either $\omega' \leq_\omega \omega$, or $\omega \sim_\omega \omega'$. Let sentence k be the sentential representation of ω. Also, wlg, pick atom a such that $\omega, \omega' \in [a]$. Note that $\biguplus(k, a)$ should consist of two interpretations, namely ω and the a-dual of ω. Clearly, either $\{\omega\} \subset \{\omega'' \in [a] \mid \omega''$ is $\leq_\omega$ -minimal in $[a]\}$ or $\omega \notin \{\omega'' \in [a] \mid \omega''$ is $\leq_\omega$ -minimal in $[a]\}$. In other words, $[k \oplus_{\leq} a] \neq \{\omega\}$. Now consider $[k \oplus_{\leq} \neg a] = \{\omega'' \mid \omega''$ is $\leq_\omega$ -minimal in $[\neg a]\}$. Clearly $\omega \notin [k \oplus_{\leq} \neg a] \neq \emptyset$. Furthermore, $[k \oplus_{\leq} a] \cap [k \oplus_{\leq} \neg a] = \emptyset$.
Hence the set $[k \oplus_{\leq} a] \cup [k \oplus_{\leq} \neg a]$ either does not have ω as a member, or has at least three members. In either case, this set would be different from $\biguplus(k, a)$.

Case 2 Pick $\omega, \omega', \omega''$ and atom a' of the appropriate nature. Set $a = a'$. Clealy $\omega \notin [a]$. It is easily verified that either ω' is not $\leq_\omega$ -minimal in $[a]$ or there are at least two elements that are $\leq_\omega$ -minimal in $[a]$. Now, arguing analogously as in (Case 1) we can show that the set $[k \oplus_{\leq} a] \cup [k \oplus_{\leq} \neg a]$ either does not have ω', the $a$dual of a, as a member, or has at least three members, and hence is not identical with $\biguplus(k, a)$.

This completes the proof. ■

How general is the General Result?

One might wonder whether this result buys us anything at all! A case at hand to consider is the preorder generated by the polular distance measures drastic distance:

$$Dist_{Dr}(\omega, \omega') = \begin{cases} 0 & \text{if } \omega = \omega' \\ 1 & \text{other wise} \end{cases}$$

Though faithful, this measure does not satisfy the other attendant condition. So update based on drastic distance cannot be defined in the manner described. This is demonstrated by the following ecample.

Example 6. Assume, as in Example 2 and 3, that $\mathcal{A} = \{a, b, c\}$ and $k = (a \wedge \neg b) \vee c$. Thus $[k] = \{001, 101, 100, 111, 011\}$ and $[a] = \{100, 101, 110, 111\}$. We consider drastic update and drastic erasure of k by a. We need to compute $\bigcup_{\omega \in [k]} min_{\leq^{Dr}\omega}([a])$ for update and $\bigcup_{\omega \in [k]} min_{\leq^{Dr}\omega}([\neg a])$ for erasure. It is easily verified that for any $\omega = 0 \star \star$, $min_{\leq^{Dr}\omega}([a]) = [a]$ and for any $\omega = 1 \star \star$, $min_{\leq^{Dr}\omega}([\neg a]) = [\neg a]$. Accordingly we get

1. $(k \oplus^{Dr} a) \equiv a$ and
2. $(k \ominus^{Dr} a) \equiv k \vee \neg a \equiv (a \rightarrow c)$, where as
3. $(k \oplus a) \equiv (k \odot a) \wedge a \equiv a \wedge (b \rightarrow c)$ and
4. $(k \ominus a) \equiv (k \odot a) \wedge a \rightarrow k \equiv (b \rightarrow c)$.

4 Conclusion

In this paper, we started with intention of showing that belief-manipulation mechanisms such as updating and language manipulation mechanisms like forgetting are interconnected. In Section 1, after introducing the notation, we provided the background information on both knowledge update and erasure, restricted to Dalal measure. In the next section we showed how update and erasure, given Dalal measure and a literal input on the one hand, and forgetting on the other, are inter-definable.

In section 3 we extended these results to more more general measures such as Winslett measure. we also showed that there are popular measures such as Drastic distance do not yeild to such easy definition of forgetting.

It is debatable if interesting accounts of forgetting, where the input is an arbitrary sentence, can be given based on the interconnections provided here. On the face of it, forgetting of an arbitrary formula does not make sense, since it would correspond to removal of a set of atoms from the language. We however prefer to leave it as an unresolved issue at this point. Furthermore, first order belief update, a topic of much interest, is also left to be addressed in future.

References

1. Nayak, A., Chen, Y., Lin, F.: Forgetting and knowledge update. In: AI06: Procs. of the 19th Australian Joint Conference of Artificial Intelligence, Springer, LNAI 4304 (2006) 131–40
2. Alchourrón, C.E., Gärdenfors, P., Makinson, D.: On the logic of theory change: Partial meet contraction and revision functions. Journal of Symbolic Logic **50** (1985) 510–530
3. Gärdenfors, P.: Knowledge in Flux: Modeling the Dynamics of Epistemic States. Bradford Books, MIT Press, Cambridge Massachusetts (1988)
4. Katsuno, H., Mendelzon, A.O.: On the difference between updating a knowledge base and revising it. In Gärdenfors, P., ed.: Belief Revision. Cambridge University Press (1992) 183–203
5. Everaere, P., Konieczny, S., Marquis, P.: Introduuota and gmin merging operators. In: International Joint Conference on Artificial Intelligence. (2005) 424–429
6. Konieczny, S., Pinoperez, R.: Merging information under constraints: a qualitative framework. Journal of Logic and Computation **12(5)** (2002) 773–808
7. Lang, J., Marquis, P.: Resolving inconsistencies by variable forgetting. In: Proceedings of the 8th KR. (2002) 239–250
8. Lang, J., Liberatore, P., Marquis, P.: Propositional independence: Formula-variable independence anf forgetting. J. of Artificial intelligence Research **18** (2003) 391–443
9. Lin, F., Reiter, R.: Forget it! In: Proceedings of the AAAI Symposium of Relevance. (1994) 154–159
10. Dalal, M.: Investigations into a theory of knowledge base revision: Preliminary report. In: Proceedings of the Seventh National Conference on Artificial Intelligence. (1988)
11. Winslett, M.: Reasoning about action using a possible models approach. In: Proceedings of the 7th National Conference in Artificial Intelligence. (1988) 88–93
12. Forbus, K.: Introducing actions into qualitative simulation. In: International Joint Conference on Artificial Intelligence. (1989) 1273–1278
13. Satoh, K.: Nonmonotonic reasoning by minimal belief revision. In: Proceedings of the International Conference of Fifth Generation Computer Systems. (1988) 455–462
14. del Val, A.: Computing knowledge base updates. In: Third Conference on Proncin Principle of Knowledge (KR-92). (1992) ??–??
15. del Val, A.: Syntactic characterizations of belief change operators. In: IJCAI. (1993) 540–547
16. Delgrande, J., Schaub, T.: A consistency-based approach for belief change. Artificial Intelligence **151(1-2)** (2003) 1–41