Characterizing Coxian Distributions of Algebraic Degree q and Triangular Order p (Research Note)

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In this research note we present a procedure to characterize the set of all Coxian distributions of algebraic degree q with real poles that have Coxian representations of order p where $p > q$.

Coxian distributions (Cox [3]) are a particular class of phase-type (PH) distributions (Neuts [8]) whose representations have the form

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_p \end{pmatrix} \tag{1}$$

$$\boldsymbol{S} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_p \end{pmatrix}, \tag{2}$$

where, without loss of generality, $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$.

Consider a general order p PH representation $(\boldsymbol{\alpha}, \boldsymbol{T})$. For $s < -\lambda_1$, where λ_1 is the dominant eigenvalue of $\boldsymbol{T}$, the Laplace-Stieltjes transform of the PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{T})$ is

$$\phi(s) = -\boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{T}\boldsymbol{e} + \alpha_0 \tag{3}$$

$$= \frac{a_q s^{q-1} + \ldots + a_2 s + a_1}{s^q + b_q s^{q-1} + \ldots + b_2 s + b_1} + \alpha_0, \tag{4}$$

where the numerator and denominator are coprime polynomials. Here, the coefficients are real, and the zeros of the denominator polynomial are a subset of the eigenvalues of $\boldsymbol{T}$. The *point mass at zero* is $\alpha_0 = 1 - \boldsymbol{\alpha}\boldsymbol{e}$. We define the *algebraic degree* of the PH distribution to be the degree of the denominator polynomial, q, see O'Cinneide [9].

Typically, representations for PH distributions are not unique, and we say that any representation for a PH distribution of minimal order is a *minimal* representation. The *order* of the PH distribution itself, is defined to be the order of any of its minimal representations. The order of a PH distribution is greater than or equal to its algebraic degree, see O'Cinneide [9].

O'Cinneide [11] showed that any PH distribution whose generator $\boldsymbol{T}$ has only real eigenvalues, has an equivalent Coxian representation of *some* order. This order is called

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the *triangular order* of the distribution. The order of such a *PH* distribution is less than or equal to its triangular order. Indeed, He and Zhang [6] showed that any order 3 *PH* distribution with only real eigenvalues has an equivalent Coxian representation of order no more than 4.

We can represent any *PH* distribution with Laplace-Stieltjes transform (4) with an alternative, so-called, *matrix-exponential* representation:

$$\boldsymbol{\gamma} = \begin{pmatrix} a_1 & a_2 & \cdots & a_q \end{pmatrix} \tag{5}$$

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_1 & -b_2 & -b_3 & \cdots & -b_q \end{pmatrix} \tag{6}$$

$$\boldsymbol{e}_q = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}', \tag{7}$$

see Asmussen and Bladt [1]. This representation is minimal, and its order (that is, q) is equal to the algebraic degree of the *PH* distribution. Clearly, the representation in (5)–(7) is *not* a *PH* representation. We have alternative expressions for the density function and Laplace-Stieltjes transform of the *PH* distribution:

$$f(u) = \boldsymbol{\gamma} \exp(\boldsymbol{R}u)\boldsymbol{e}_q \tag{8}$$

$$\phi(s) = \boldsymbol{\gamma}\,(s\boldsymbol{I} - \boldsymbol{T})^{-1}\,\boldsymbol{e}_q + \alpha_0. \tag{9}$$

From now on we will assume that $\alpha_0 = 0$ since it makes the analysis slightly simpler, but the case when $\alpha_0 > 0$ can be dealt with similarly.

O'Cinneide [9] showed that the rational Laplace-Stieltjes transform (4) (with $\alpha_0 = 0$) corresponds to a *PH* distribution if and only if

1. the pole of $\phi(s)$ of maximal real part is real, negative, and unique,
2. $a_1 = b_1$, and
3. $f(u) = \boldsymbol{\gamma} \exp(\boldsymbol{R}u)\boldsymbol{e}_q > 0, \quad$ for all $u > 0$.

The first two conditions are simple to check. Equivalently, the third condition can be expressed as: $\phi(s)$ corresponds to a *PH* distribution if and only if the point $(a_2, \ldots, a_q)$ is in the set

$$\Omega_q = \bigcap_{u>0} \{\boldsymbol{x} \in \mathbb{R}^{q-1} \mid a_1 \boldsymbol{e}_i \exp(\boldsymbol{T}u)\boldsymbol{e}_q + \sum_{i=1}^{q-1} x_i \boldsymbol{e}_{i+1}(u) \exp(\boldsymbol{T}u)\boldsymbol{e}_q > 0\}. \tag{10}$$

In Bean, Fackrell, and Taylor [2] or Fackrell [5] a complete characterization of the boundary of Ω_3 when $p = 3$ was given. (In fact, because the paper was concerned with matrix-exponential distributions, the inequality in (10) was not strict, and hence there, the boundary was included in the set.) Figure 1 shows Ω_3 when the eigenvalues of $\boldsymbol{T}$ are $-1, -2, -3$. Any point (x_1, x_2) in Ω_3 corresponds to a such a *PH* (or Coxian as the eigenvalues are all

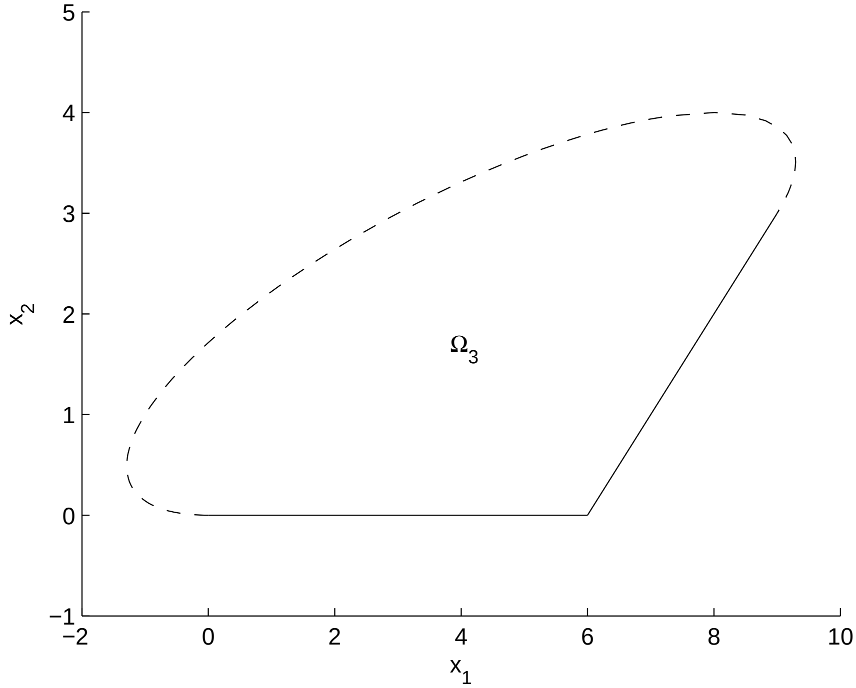

Figure 1: The set Ω_3 contains all points (x_1, x_2) that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$.

real) distribution. Here, because the curved boundary is not in the set it is indicated by a dashed line.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$. For any subset $\{i_1, i_2, \ldots, i_r\}$ of $\{1, 2, \ldots, p\}$, let $F_{i_1,i_2,\ldots,i_r}$ represent the point in $\mathbb{R}^{p-1}$ that corresponds to the Coxian distribution with Laplace-Stieltjes transform

$$\phi_{i_1 i_2 \ldots i_r}(s) \;=\; \frac{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}}{(s+\lambda_{i_r})(s+\lambda_{i_2})\ldots(s+\lambda_{i_r})}. \tag{11}$$

Every Coxian distribution with (triangular) order p (or less) and generator with eigenvalues (diagonal entries) in $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is contained in the $(p-1)$-dimensional hypertetrahedron (a subset of Ω_p) with vertices $F_{12\ldots p}, F_{23\ldots p}, \ldots, F_p$, see Dehon and Latouche [4]. Let T_p denote this hypertetrahedron. Figure 2 shows the 2-dimensional tetrahedron (triangle) with vertices F_{123}, F_{23}, and F_3 - the set T_3 contains all points (x_1, x_2) that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$ and triangular order 3. We note here that some points in T_3 correspond to Coxian distributions of algebraic degree 1 or 2. For example, F_3 and F_{23} have algebraic degrees 1 and 2, respectively. Also, the line $\overline{F_{23}F_3}$ contains points that correspond to distributions of algebraic degree 2, except for F_2 whose algebraic degree is 1.

The question now is: how do we characterize all Coxian distributions of algebraic degree q, with poles $-\lambda_1, -\lambda_2, \ldots, -\lambda_q$, that have triangular order p with Coxian representation whose generator has eigenvalues (diagonal entries) $-\lambda_1, -\lambda_2, \ldots, -\lambda_q, -\lambda_{q-1}, \ldots - \lambda_p$ (ordered from largest to smallest)? The idea is to embed Ω_q in $\mathbb{R}^{p-1}$, and intersect the $(q-1)$-dimensional hyperplane generated by the q vertices of T_q with the hypertetrahedron T_p. The resultant set will contain all points (in $\mathbb{R}^{p-1}$) that correspond to Coxian

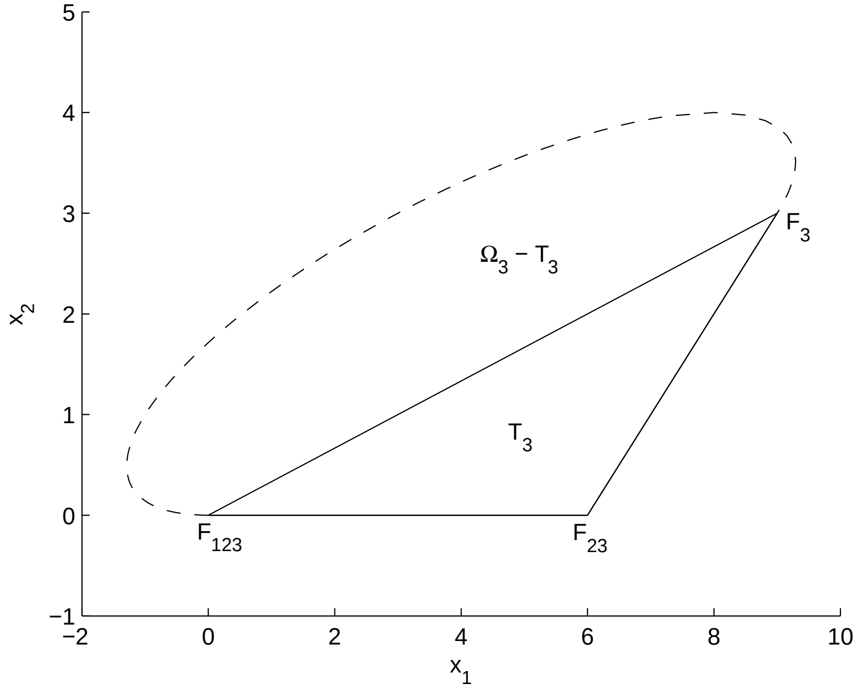

Figure 2: Plot of Ω_3 and T_3. T_3 contains all points that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$ and triangular order 3.

distributions of algebraic degree q (with the designated poles), that have triangular order p (whose Coxian generators have the designated eigenvalues). Projecting this intersection back onto Ω_q gives the required subset of Ω_q, that is, the set of all points in $\mathbb{R}^{q-1}$ that correspond to Coxian distributions of algebraic degree q and triangular order p.

The approach is best illustrated with a simple example. We find all points in Ω_3 that correspond to Coxian distributions with poles -1, -2, -3 that have a Coxian representation with the four eigenvalues -1, -2, -3, and $-\lambda \leq -3$. We start by considering the case when $\lambda = 4$. Figure 3 shows the tetrahedron T_4 (which has vertices $F_{1234}, F_{234}, F_{34}$, and F_4) intersected with Ω_3 (embedded in $\mathbb{R}^3$). The intersection is the quadrilateral with vertices F_{123}, F_{23}, F_3, and X. Figure 4 shows the intersection projected back onto Ω_3 - the triangle above T_3 contains all points that correspond to Coxian distributions with poles $-1, -2, -3$ whose triangular order is 4 that has a Coxian representation with eigenvalues $-1, -2, -3, -4$. Here, we have labeled the fourth vertex of the quadrilateral X despite the two points in Figures 3 and 4 being different, because they represent the same distribution.

If we allow λ to vary so that $\lambda \geq 3$ and take the union of all sets formed in the manner described above, we get the set of all points corresponding to Coxian distributions of algebraic degree 3 and triangular order 4. This set, denoted by S_4, is shown in Figure 5.

If we repeat the process and embed Ω_3 in $\mathbb{R}^4$, intersect it with the hypertetrahedron with vertices $F_{12345}, F_{2345}, F_{345}, F_{45}$, and F_5, and project the intersection back onto Ω_3 we get the set of all points that correspond to Coxian distributions of algebraic degree 3

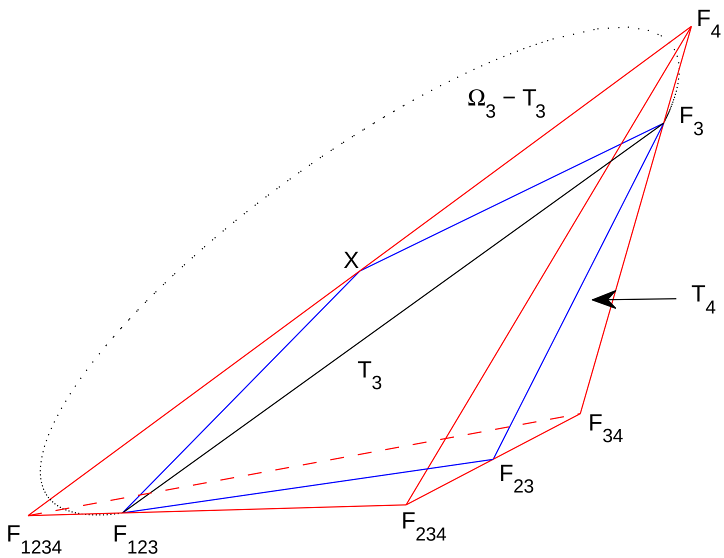

Figure 3: The intersection of Ω_3 (embedded in $\mathbb{R}^3$) and T_4.

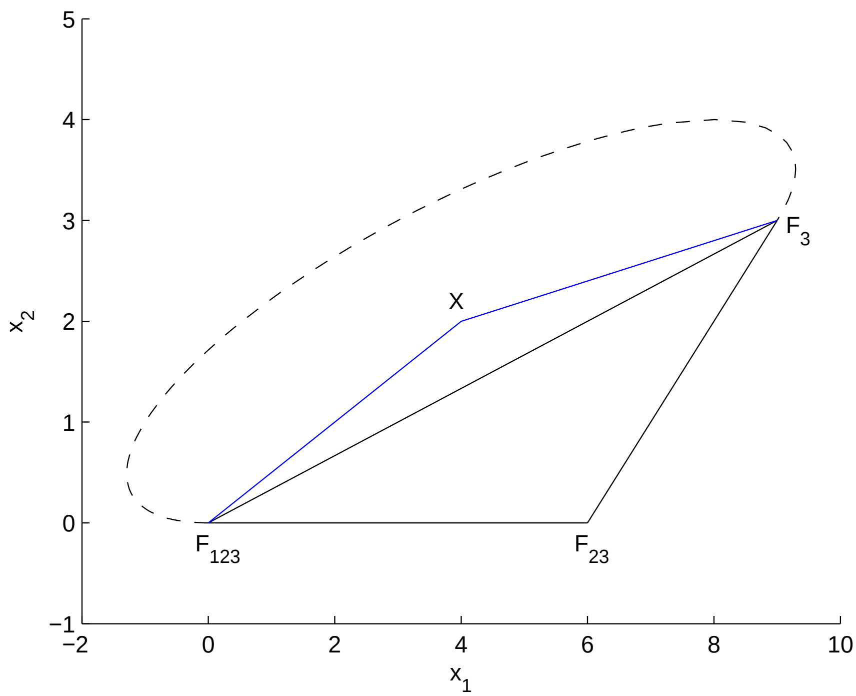

Figure 4: The projection of the intersection of Ω_3 (embedded in $\mathbb{R}^3$) and T_4 onto Ω_3. The triangle above T_3 contains all points (x_1, x_2) that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$, that have triangular order 4 whose Coxian representations have eigenvalues $-1, -2, -3, -4$.

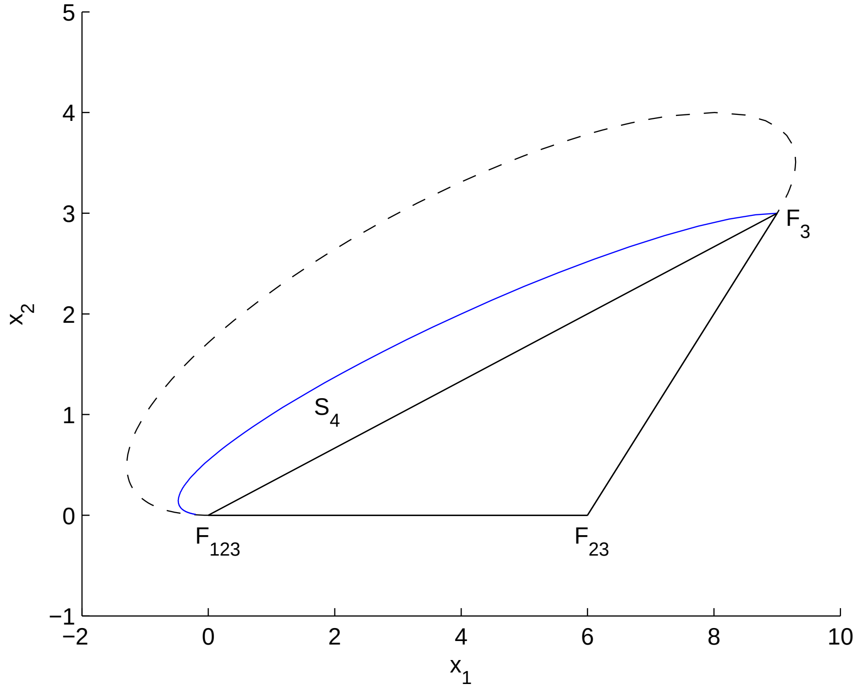

Figure 5: The set of all points that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$, that have triangular order 4.

whose poles are $-1, -2, -3$ that have triangular order 5 whose Coxian representations have eigenvalues $-1, -2, -3, -4, -5$. This set is the trapezium above T_3 depicted in Figure 6. If we now consider Coxian representations with $-1, -2, -3, -\lambda, -\mu$, where $\mu \geq \lambda \geq 3$, and intersect all the resultant intersecting sets, we get the set S_5 depicted in Figure 7. Since the boundary of S_5 depends on the two parametrs λ and μ with $\lambda \leq \mu$ it is difficult to plot it as a single curve. The "shaded" and the thin "unshaded" regions above the boundary of S_4 constitute S_5.

To further illustrate the process we find the set of points in Ω_4 that correspond to Coxian distributions of algebraic degree 4 with poles $-1, -1, -1, -1$ that have triangular order 5 with eigenvalues $-1, -1, -1, -1, -2$. Figure 8 shows this set. Here, the set of points that correspond to all Coxian distributions of algebraic degree 4 with poles $-1, -1, -1, -1$ that have triangular order 5 with eigenvalues $-1, -1, -1, -1, -2$, consists of two tetrahedra on top of T_4. The vertices of the first tetrahedron are $F_{11}, F_{111}, F_{1111}$, and A_1, and the vertices of the second tetrahedron are F_1, F_{11}, F_{1111}, and A_2. Also, since A_1 and A_2 and the line $\overline{F_1F_{1111}}$ are in the same plane, the set is convex. When we allow $\lambda \geq 1$ to vary, the points A_1 and A_2 trace out two curves which are labeled C_1 and C_2 in Figure 9. The set that contains all points that corresponds to all Coxian distributions with poles $-1, -1, -1, -1$ that have triangular order 5 consists of two conical shapes, one with F_{1111} as its vertex and $\overline{F_1F_{11}}$, $\overline{F_{11}F_{111}}$, and the curve C_1 forming its base, the other with F_1 as its vertex and $\overline{F_{11}F_{111}}$, $\overline{F_{111}F_{1111}}$, and the curve C_2 forming its base. We note here, that for $\lambda \geq 1$, each set is convex (as previously remarked), but since the line segment $\overline{F_1F_{1111}}$ is common to each set, the union of all the (convex) sets is *not* convex. Thus, the set of all *PH* distributions of algebraic degree 3, with poles $-1, -1, -1, -1$ that have triangular order 5 is not convex. This phenomena was observed by He and Zhang [7].

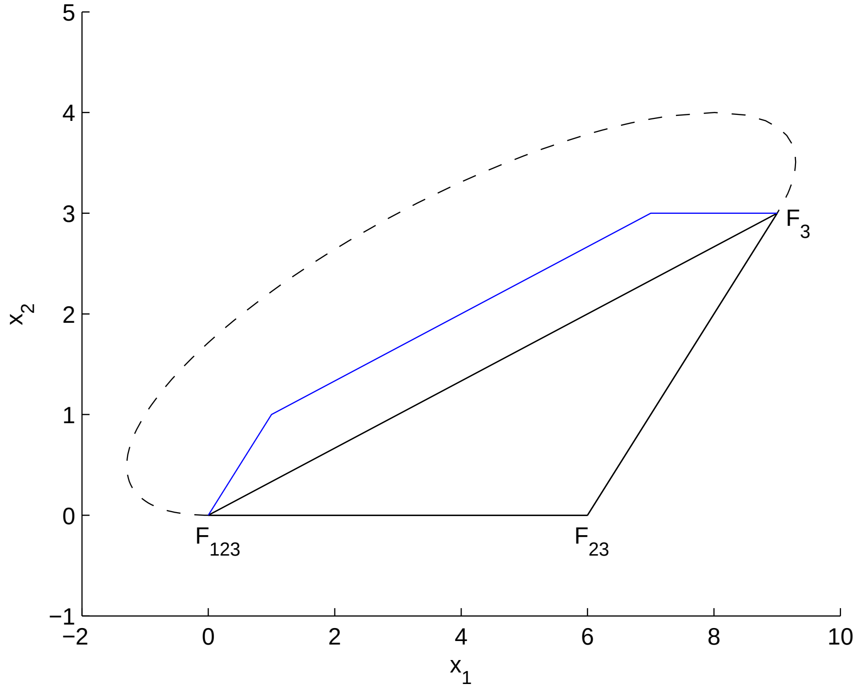

Figure 6: The set of all points that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$, that have triangular order 5 whose representations have eigenvalues $-1, -2, -3, -4, -5$.

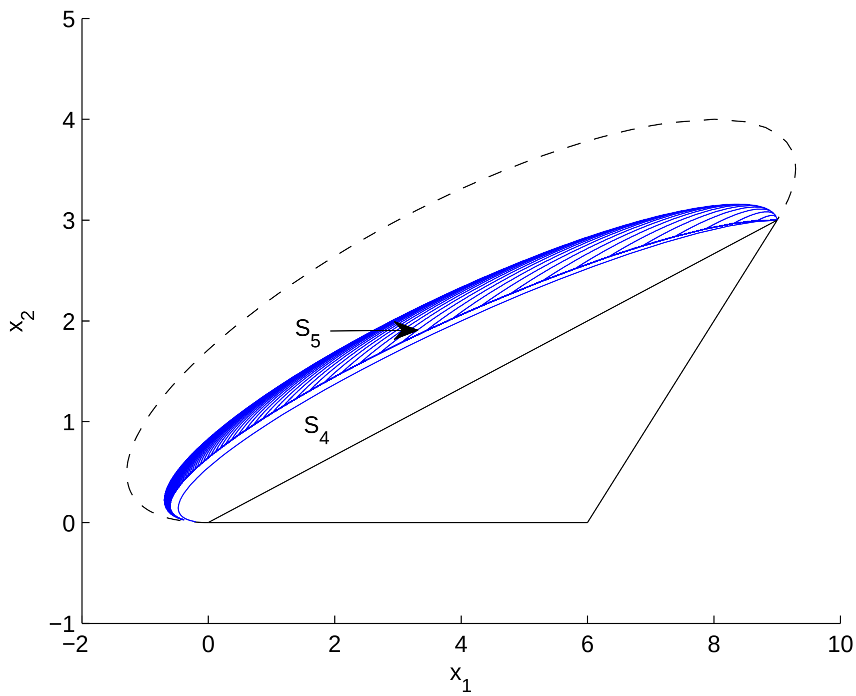

Figure 7: The set of all points that correspond to Coxian distributions of algebraic degree 3 with poles $-1, -2, -3$, that have triangular order 5.

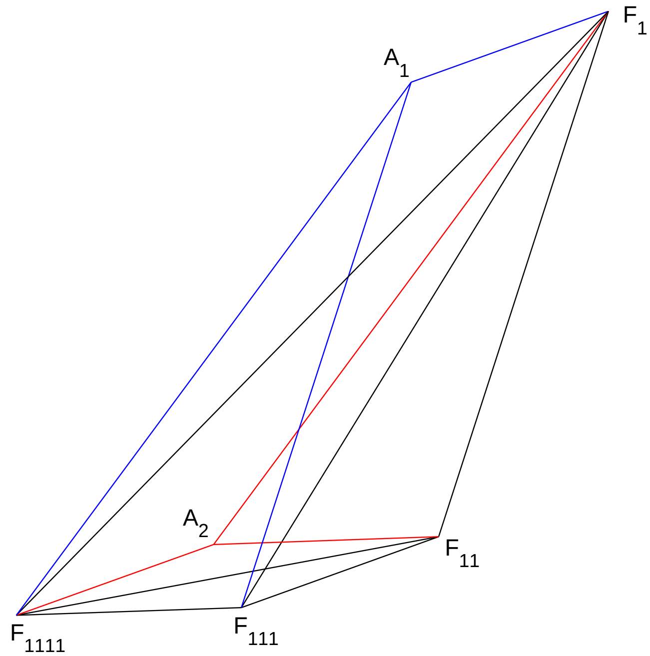

Figure 8: The set of all points that correspond to Coxian distributions of algebraic degree 4 with poles $-1, -1, -1, -1$, that have triangular order 5 with eigenvalues $-1, -1, -1, -1, -2$.

The procedure described in this research note enables us to describe the set of all Coxian distributions of algebraic degree q with real poles that have triangular order p with Coxian generators that have real eigenvalues. For some cases of low algebraic degree and low triangular order we are able to characterize the whole set of Coxian distributions with algebraic degree q and triangular order p. However, a general description eludes us because of the complexity of the algebra involved. However, given that the procedure only requires operations from linear algebra it seems plausible that such a description could be achieved entirely using the products and sums of matrices. This will be left to later work.

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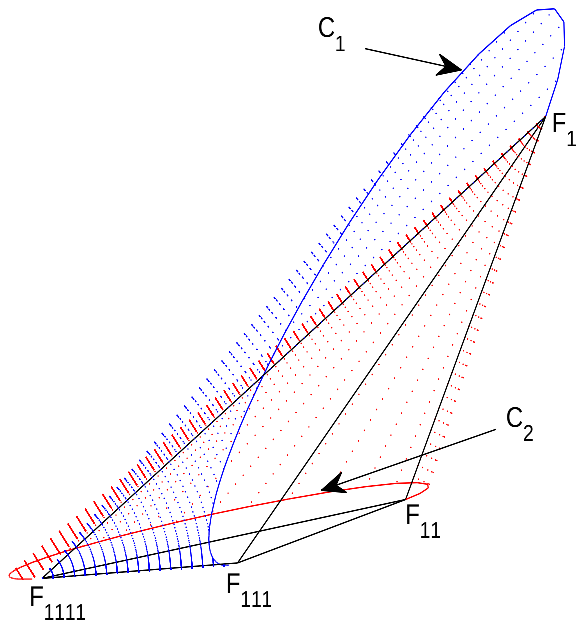

Figure 9: The set of all points that correspond to Coxian distributions of algebraic degree 4 with poles $-1, -1, -1, -1$, that have triangular order 5.

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