A Software Library for Reliable Online-Arithmetic with Rational Numbers

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Numerical codification formats for digital computers are basically conditioned by the features of the numeric sets to be represented. There exist conventional formats related to the representation of natural numbers, integers or real numbers [21] [15] [20]. The positional fractional codification offers a direct way to express the number which consists on an integer and fractional parts. One of the most representative codification formats is the floating point specification IEEE754/854 [11] [12], which has being adopted as the standard in most of conventional computer systems.

Advances in most of the scientific fields have become more and more dependent on those of computer technologies. Particularly, scientific and engineering computing demand more and more reliability according to the complexity growth of the models to be evaluated [22]. In this context, advances in Computable Analysis provide formal paradigms which allow for dealing with the computability and complexity issues and therefore guarantee reliability in software development [27]. In this context, some work concerned with the development of specialized software libraries for reliable scientific computing can be found [8]. The IRRAM C++, developed by N. Müller, has shown to be one of the most successful approaches [2]. The interest of these software libraries is motivated by the lack of reliability of the IEEE754/854 floating point standard hardware support for scientific computing applications [16] [24].

Other attempts to introduce a newer standard based on Interval Arithmetic did not success due to both commercial and theoretical reasons [9]. There exist other approaches to symbolic computing which aim for an exact representation of mathematical expressions [17]. Considering this criteria, a design of an arithmetic unit in which rational numbers are represented symbolically, that is to say, by means of fractions [13] [14] has being developed. This arithmetic unit implements the basic operations of addition, subtraction, multiplication and division by using integer arithmetic. However, this approach is computationally expensive, particularly because there are no easy means to obtain irreducible fractions [4].

The continuous fractions approach also offers an exact representation method for rational numbers. In this case, the codification of numbers is performed by successive fractions [3] [19]. However, the hardware designs developed [17] [18] [25] [23] outline a high complexity of the arithmetic operations involved [26]. As it happened in symbolic computing approaches, if a numeric result in positional factional notation is required, some additional operations need to be performed and then, imprecisions in expression translations may appear.

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An interesting proposal for error-free codification of rational numbers is based on the explicit representation of the periodic development of fractional numbers [10]. However, as this research was limited to the theoretical formulation and no suitable procedures and architectures where developed, no interest was shown by the scientific community.

Considering all the theoretical and applied approaches mentioned, we claim that signed digit arithmetic resembles an interesting approach [1] as a conceptual convergence between two paradigms which belong to two different Computer Science fields can be realized: Type-2 Theory of Effectivity (TTE) [27], in Computable Analysis, and online arithmetic [7], in Computer Arithmetic. The former, developed by Klaus Weihrauch, proposes computable representations of real numbers based on signed digit representations of rational and real numbers which, at the same time, establish the basis for higher abstraction level representations such as common computable spaces of functions. The latter, developed by Trivedi and Ercegovac, deals with the hardware implementation of digit-serial left-to-right (online or Most Significant Bit First) arithmetic operators for signed digit numbers, whose operation dynamics resembles that of the Turing Machine model.

The research proposed extends [5] and [6] by presenting the implementation issues of a software library for basic exact rational arithmetic operations (addition, subtraction, multiplication and division) using a signed digit representation. Some issues related to the implementation criteria for the software library are developed, according to a feasible implementation in hardware of on-line arithmetic operators.

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