Pseudospectral Fourier reconstruction with IPRM

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Summary of results

The Inverse Polynomial Reconstruction Method (IPRM) has been recently introduced by J.-H. Jung and B. Shizgal in order to remedy the Gibbs phenomenon, see [2], [3], [4], [5]. Their main idea is to reconstruct a given function from its n Fourier coefficients as an algebraic polynomial of degree $n-1$. This leads to an $n \times n$ system of linear equations, which is solved to find the Legendre coefficients of the polynomial. This approach is motivated by the classical observation that a smooth, function allows an efficient representation through its Legendre series. In particular, if a function has an analytic extension to a larger domain, its Legendre coefficients decay exponentially. In principle, the function can be efficiently reconstructed from its Fourier data indirectly by first computing its Legendre coefficients.

Several fundamental aspects of IPRM are still investigated. A rigorous proof of existence of the reconstruction was published only recently by Michel Krebs, see [1]. A major drawback of IPRM is that the condition number of the underlying $n \times n$ matrix grows approximately like $O(e^{0.4n})$, which quickly leads to ill-conditioning. For this reason, IPRM fails to converge in the case of a meromorphic function, whose singularities are located sufficiently close to the interval where the function is defined. This happens because the function's Legendre series does not converge fast enough to mitigate the exponential growth of the condition numbers.

To resolve this problem, we propose a modified version of IPRM, which achieves pseudospectral convergence even for restrictions of meromorphic functions. We reconstruct a function as an algebraic polynomial of degree $n-1$ from the function's m lowest Fourier coefficients, as long as $m \geqslant n$. We

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compute approximate Legendre coefficients of the function by solving a linear least squares problem dimensioned $m \times n$. If m is large in comparison with n, this procedure dramatically improves conditioning of the problem. Specifically, we show that the computation of the first n Legendre coefficients from the lowest αn^2 Fourier coefficients has condition number arbitrarily close to one, if α is taken large enough. Consequently, for an analytic function the convergence rate of the modified IPRM is root exponential on the whole interval of definition, even if the analytic extension has singularities.
Our main contributions are as follows.

1. We show that a piecewise polynomial function with uniformly bounded polynomial degrees can be reconstructed from its consecutive Fourier coefficients, if the number of the Fourier coefficients is greater or equal to the number of unknown polynomial coefficients.

2. We prove that if $m > \alpha_0 n^2$, the condition number of the underlying least squares problem does not exceed $\sqrt{\frac{m}{m-\alpha_0 n^2}}$ ($\alpha_0 = \frac{4\sqrt{2}}{\pi^2} = 0.573\ldots$).

3. We demonstrate that for analytic functions, the resulting algorithm has a root-exponential convergence rate, even if the function's analytic continuation has singularities.

Numerical experiments indicate that the exponent two in the inequality $m > \alpha_0 n^2$ is the lowest possible, i.e. any lower exponent gives rise to rectangular matrices with unbounded condition numbers as n approaches infinity.

Three aspects are important for practical applications. First, the modified IPRM allows us to incorporate all available Fourier coefficients for reconstruction. Second, thanks to low condition numbers, we can reduce – practically avoid – an amplification of any noise that is typically present in the measurements. Third, the method tells us how many Fourier coefficients are sufficient to describe a smooth, but non-periodic function. For smooth and periodic functions, it is routine to truncate the Fourier series when the coefficients have decreased below a given threshold. A similar procedure is now available for non-periodic functions via the modified IPRM, even though their Fourier series decay slowly.

The following two theorems are representative for our investigations. For a fixed sequence $\mathbf{a} = (a_0, \ldots, a_L)$ with the property

$$-1 = a_0 < a_1 < \ldots < a_{L-1} < a_L = 1, \tag{1}$$

we consider all piecewise polynomial functions defined on the interval $[-1, 1]$, with possible discontinuities occurring only at the points $a_1, \ldots, a_{L-1}$. For every nonnegative integer M, we denote by $\mathcal{P}_{M,\mathbf{a}}$ the linear space of all such functions, whose restrictions to every interval (a_{j-1}, a_j), $(j = 1, \ldots, L)$, are polynomials of degree not exceeding M. Since the sequence $\mathbf{a}$ defines a partition of $[-1, 1]$ into L subintervals, the dimension of $\mathcal{P}_{M,\mathbf{a}}$ equals $L(M+1)$.

Theorem 1 *Let d and D be integers such that $d \leqslant 0 \leqslant D$, and let $p \in \mathcal{P}_{M,\mathbf{a}}$ have vanishing $D - d + 1$ consecutive Fourier coefficients*

$$\widehat{p}(d) = \widehat{p}(d+1) = \ldots = \widehat{p}(D-1) = \widehat{p}(D) = 0. \tag{2}$$

If $D - d + 1 \geqslant L(M + 1)$, then $p = 0$ identically.

For positive integers m and n, we consider the $(2m + 1) \times n$ matrix $A_{m,n}$ of the Fourier coefficients of the normalized Legendre polynomials $\widetilde{P}_l$ with the entries

$$a_{kl} = \widehat{\widetilde{P}_l}(k) = \sqrt{2}\,(-\imath)^l \sqrt{l + \tfrac{1}{2}}\; j_l(k\pi), \tag{3}$$

where $k = -m, \ldots, m$, $l = 0, \ldots, n-1$, and j_l is the spherical Bessel function of order l.

Theorem 2 *Let $\alpha_0 = \frac{4\sqrt{2}}{\pi^2} = 0.573\ldots$. For every $n = 1, 2, \ldots$, and every integer $m > \alpha_0 n^2$, the condition number of the matrix $A_{m,n}$ does not exceed $\sqrt{\frac{m}{m - \alpha_0 n^2}}$.*

References

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