Sparse Reconstructions for Inverse PDE Problems

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Abstract. We are concerned with the numerical solution of linear parameter identification problems for parabolic PDE, written as an operator equation $Ku = f$. The target object u is assumed to have a sparse expansion with respect to a wavelet system $\Psi = \{\psi_\lambda\}$ in space-time. For the recovery of the unknown coefficient array, we use Tikhonov regularization with ℓ_p coefficient penalties and the associated iterative shrinkage algorithms. Since any application of K and K^* involves the numerical solution of a PDE, perturbed versions of the iteration have to be studied. In particular, for reasons of efficiency, adaptive operator applications are indispensable. By a suitable choice of the respective tolerances and stopping criteria, also the adaptive iteration converges and it has regularizing properties. We illustrate the performance of the resulting method by numerical computations for one- and two-dimensional inverse heat conduction problems.

Keywords. inverse problems, sparse regularization, adaptive wavelet methods, inverse heat conduction

1 Introduction

The analysis and the numerical treatment of inverse problems has become a field of increasing importance, due to its relevance in practical applications like medical imaging (computer tomography), geophysical problems (analysis of seismic data), quality control (nondestructive testing) or process monitoring (detection of corrosive effects in machinery components). The common feature of these applications is that the quantities of interest cannot be accessed directly, but their values have to be deduced indirectly from the effect on observable data.

In a mathematical formulation, the unknown quantity u and the observable f in an inverse problem are linked via a model operator K, the *forward operator*, such that

$$Ku = f. \tag{1}$$

Here we assume that K is a bounded injective linear mapping between Hilbert spaces X, Y. Moreover, only noisy data f^δ are available, with $\|f - f^\delta\|_Y \le \delta$. The problem is ill-posed in the sense that the operator K is not boundedly

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invertible. In particular, (1) does not have to possess a solution for each right–hand side; noisy measurements are typically not in the range of K. Therefore, regularization methods are needed to recover u in a stable way.

Recently, increasing interest has been drawn to the analysis of regularization schemes that exploit the *sparse* expansibility of u in a given ansatz system. Under that a priori knowledge, good or even perfect reconstructions are possible with only a few significant expansion coefficients. Among other sparsity-promoting regularization schemes, Tikhonov regularization with ℓ_p coefficient penalties is a common approach, where $1 \leq p < 2$. The minimizer of the Tikhonov functional can be computed by iterative soft shrinkage, see [1].

In the class of problems we have in mind, the target quantity u is closely related to unknown boundary data, coefficients, or source terms in an underlying PDE. We are looking for reconstructions that are globally smooth functions with few discontinuities along lower-dimensional curves or surfaces. Such functions are known to have sparse expansions with respect to suitable wavelet systems on the computational domain, so that sparsity assumptions are justified.

However, when considering parameter identification problems for partial differential equations, another issue has to be addressed. In this case each application of the forward operator K typically involves the numerical solution of an associated boundary value problem. This not only makes the reconstruction procedure computationally intensive, we also have to keep track of numerical errors propagating through the iteration steps. In order to end up with an efficient recovery algorithm, *adaptive* discretization methods for K are therefore indispensable. Recently the convergence properties of iterative shrinkage algorithms with inexact operator applications have been analyzed [2] for the case $p > 1$. Numerical experiments from an application to inverse heat conduction problems can be found in [3].

In Section 2, we briefly review the key ingredients of ℓ_p-sparse Tikhonov regularization and the associated shrinkage algorithms, and the need for adaptive discretization methods is discussed in Section 3. Numerical experiments from the application of sparse regularization methods to one- and two-dimensional inverse heat conduction problems are presented in Sectino 4. We finish with concluding remarks in Section 5.

2 Sparse Reconstructions

In order to stabilize the reconstruction problem, we shall assume that the unknown solution u has an ℓ_p-*sparse* expansion with respect to a stable ansatz system $\Psi = \{\psi_\lambda\}_\lambda \subset X$. By this we mean that there exists $\mathbf{u} = (u_\lambda) \in \ell_p$, $p < 2$, such that $u = F^*\mathbf{u} := \sum_\lambda u_\lambda \psi_\lambda$. Abbreviating $A := K \circ F^*$, such a coefficient array may be reconstructed by nonquadratic Tikhonov regularization, minimizing the functional

$$J(\mathbf{u}) = \|A\mathbf{u} - f^\delta\|_Y^2 + \sum_\lambda w_\lambda |u_\lambda|^p,$$

where $w_\lambda > 0$ are some weights. It is well-known, see [1], that the minimizer $\mathbf{u}^*$ of J can be computed as a fixed point of iterative shrinkage

$$\mathbf{u}^{n+1} = \mathbf{S}_{\mathbf{w},p}\big(\mathbf{u}^n + A^*(f^\delta - A\mathbf{u}^n)\big), \quad n = 0, 1, \ldots \tag{2}$$

where $S_{\mathbf{w},p}$ is a vector-valued shrinkage operator. In the special case $p = 1$, the minimizer $\mathbf{u}^*$ is finitely supported, and $\mathbf{S}_{\mathbf{w},1}(\mathbf{v}) = (\operatorname{sign}(v_\lambda)(|v_\lambda| - w_\lambda/2))_\lambda$ denotes *soft thresholding*.

It was shown in [1,4] that (2) strongly converges for $1 \le p \le 2$, $\|A^*A\| < 2$ and $w_\lambda \ge w > 0$. Moreover, the convergence is linear in the sense that

$$\|\mathbf{u}^{n+1} - \mathbf{u}^*\|_{\ell_2} \le \theta \|\mathbf{u}^n - \mathbf{u}^*\|_{\ell_2}$$

for some $0 < \theta < 1$. However, in practice the contraction constant may be close to 1, which is illustrated in Figure 1 by a simple but instructive example.

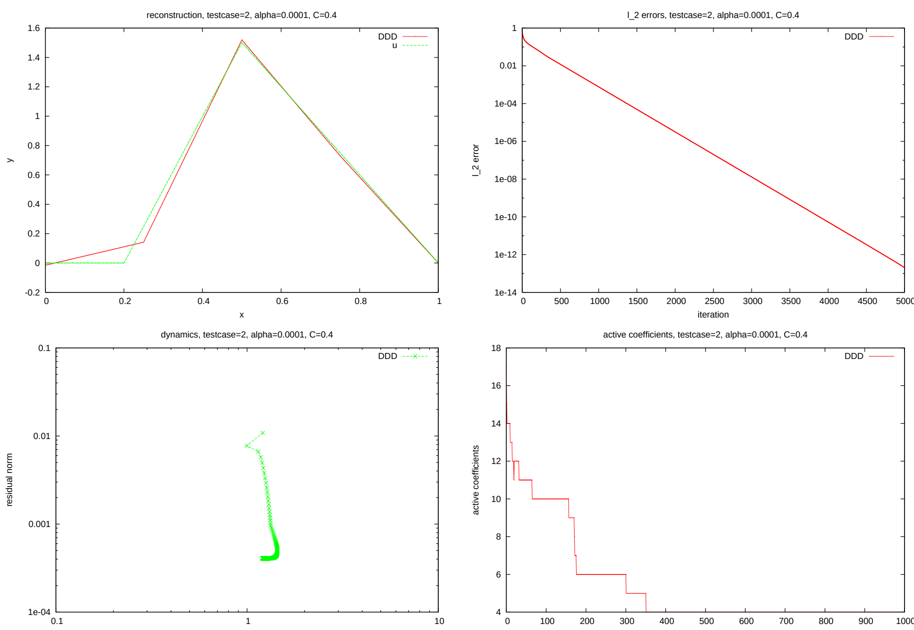

Fig. 1. Recovery of piecewise linear functions by a sparsity regularization of $Ku(t) = \int_0^t u(s)\,\mathrm{d}s$, using spline wavelet bases and iterative soft thresholding. *Top left*: reconstruction $\mathbf{u}^n$ after $n = 5000$ iterations, *top right*: error $\|\mathbf{u}^n - \mathbf{u}^*\|_{\ell_2}$ versus iteration depth n ($\theta \approx 0.9945$), *bottom left*: residual error $\|A\mathbf{u}^n - f^\delta\|_Y$ versus penalty term $\|\mathbf{u}^n\|_{\ell_1}$, *bottom right*: degrees of freedom $\#\operatorname{supp}\mathbf{u}^n$ versus iteration depth n

3 Adaptivity

A concrete realization of the iteration step (2) will involve the application of the forward operator K and its adjoint K^* to the current iterate. However, in case that K is related to a PDE problem, this cannot be done without additional discretization errors per iteration step.

As a way out, one may consider inexact, pointwise operator applications

$$\|A\mathbf{v} - [A\mathbf{v}]_\epsilon\|_Y \le \epsilon, \quad \|A^* v - [A^* v]_\epsilon\|_{\ell_2} \le \epsilon$$

up to a prescribed target accuracy $\epsilon > 0$. These operations can be realized whenever adaptive numerical approximations of the forward operator are available. In case that K involves the solution of a boundary value problem, one may think of adaptive finite element or wavelet discretizations with a posteriori error estimators and suitable refinement strategies.

Inserting inexact operator evaluations into the ideal iteration (2), one obtains a nonlinearly perturbed variant

$$\tilde{\mathbf{u}}^{n+1} = \mathbf{S}_{\mathbf{w},p}\big(\tilde{\mathbf{u}}^n + [A^* f^\delta]_{\epsilon_n} - [A^* A \tilde{\mathbf{u}}^n]_{\epsilon_n}\big) \tag{3}$$

with suitable target accuracies ϵ_n. The main issue is now how to choose ϵ_n in order to preserve the convergence properties of the original thresholding iteration.

For $p \ge 1$, it is known that (3) strongly converges to $\mathbf{u}^*$ whenever $\sum_n \epsilon_n < \infty$, see [4]. Moreover, with a judicious parameter choice $\epsilon_n = \epsilon_n(\delta, \alpha, \mathbf{w}, p, n)$ and for $p > 1$, it was shown in [2] that $\|\tilde{\mathbf{u}}^n - \mathbf{u}^*\|_{\ell_2} \le C\delta$ after a finite, controllable number of iteration steps. Using the techniques of [4] and the injectivity of K, these results can be transferred to the limit case $p = 1$, which can also be illustrated by the numerical experiments in Section 4 and in [3].

4 Application to Inverse Heat Conduction Problems

Our aim is to apply sparsity regularization methods to an inverse parabolic problem that stems from monitoring the industrial process of steel production in a blast furnace. The life span of such a steel furnace is determined by some critical thickness of its outer wall. However, due to the high temperature in the interior, this piece of information can be determined only by indirect measurements on the outside of the furnace wall.

The mathematical modelling of this process leads to an inverse heat conduction problem on a domain $\Omega \subset \mathbb{R}^n$, where $\Gamma_1 \subset \partial\Omega$ is an inaccessible boundary part. In two space dimensions, we think of a ring-shaped domain $\Omega = \{x \in \mathbb{R}^2 \mid 0 < r_1 < \|x\| < r_2\}$, with inner boundary $\Gamma_1 = \{x \in \mathbb{R}^2 \mid \|x\| = r_1\}$ and outer boundary $\Gamma_2 = \{x \in \mathbb{R}^2 \mid \|x\| = r_2\}$.

The task is then to determine temperature data g on Γ_1 by indirect measurements on $\Gamma_2 = \partial\Omega \setminus \partial\Omega$. In the interior of the domain, the temperature distribution u fulfills the parabolic boundary value problem

$$\begin{cases} u_t = \Delta u & \text{in } (0,T) \times \Omega \\ u = g & \text{on } (0,T) \times \Gamma_1 \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } (0,T) \times \Gamma_2 \\ u(0,\cdot) = 0 & \text{in } \Omega \end{cases}$$

The linear forward operator $K : g \mapsto u(\cdot,\cdot;g)u|_{(0,T)\times\Gamma_2}$ falls into the aforementioned category of operators, see [3], and K is well-known to be infinitely smoothing. In space dimension $n = 1$, the problem is called the "sideways heat equation" and K has an explicit integral representation with smooth kernel.

In the following, we briefly discuss the outcome of numerical experiments in one and two spatial dimensions, see [3] for details. For different noise levels δ and a variety of regularization parameters α, the iterative thresholding algorithm with inexact operator evaluations is executed until convergence. The underlying ansatz system Ψ for the representation of the boundary data on $(0,T) \times \Gamma_i$ is chosen to be a tensor product spline wavelet basis in space-time. The parabolic subproblems K and K^* are discretized by an adaptive wavelet-Rothe method, see [5] for details. Furthermore, we choose the parameters $w_\lambda = p = 1$. Via the wavelet characterization of Besov spaces, this corresponds to a $B^{n/2}_{n/2}(L_1)$ constraint.

4.1 1D Reconstructions

On the unit interval $\Omega = (0,1)$, we try to recover a piecewise linear function by a sparsity regularization of the sideways heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } (0,T) \times \Omega \\ u = g & \text{on } (0,T) \times \{0\} \\ u_x = 0 & \text{on } (0,T) \times \{1\} \\ u(0,\cdot) = 0 & \text{in } \Omega \end{cases}$$

Figure 2 shows the unknown function g and the observed right-hand side Kg. g has a finite representation in the underlying piecewise-linear spline wavelet basis.

Reconstructions for the noise level $\delta = 0.01$ and $p \in \{1,2\}$ can be found in Figure 3. It is remarkable that for $p = 1$, sparse regularization is able to reconstruct the target quantity almost perfectly, even under the presence of moderate measurement noise. This corresponds to the fact the fact that ℓ_1 regularization is an *exact* regularization method, see also [6]. For the same level of data error, however, quadratic Tikhonov regularization is not able to recover the unknown temperature distribution as accurately.

4.2 2D Reconstructions

On the ring-shaped domain $\Omega = \{x \in \mathbb{R}^2 | 0.5 < \|x\| < 2\}$, we try to recover a piecewise smooth function on $(0,T)\times\Gamma_1$, where $\Gamma_1 = \{x \in \mathbb{R}^2 \mid \|x\| = r_1\}$. Figure

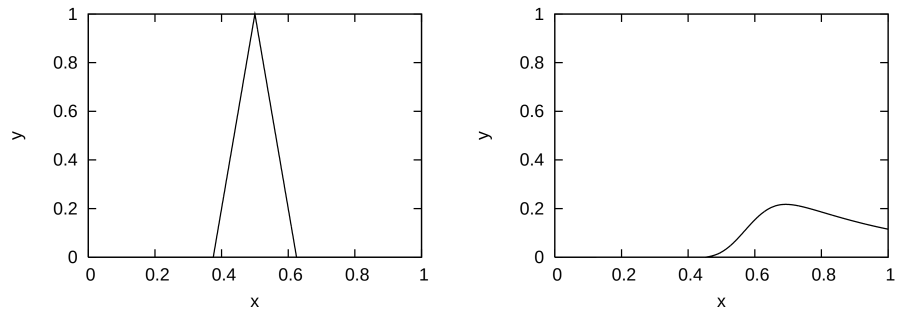

Fig. 2. Temperature data in the one-dimensional example. *Left*: function g to be recovered, *right*: observed data at $x = 1$

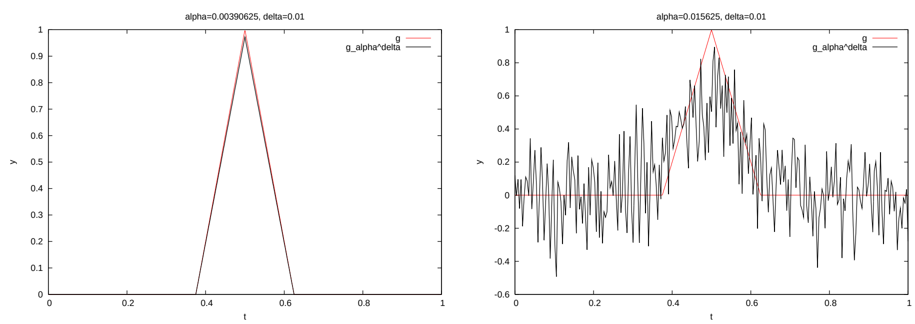

Fig. 3. Recovered temperature data in the one-dimensional example. *Left*: reconstruction for $p = 1$ under 6% relative noise ($\delta = 0.01$), *right*: best possible reconstruction for $p = 2$ under the same conditions

4 shows the unknown target quantity g and several reconstructions, plotted in polar coordinates.

For moderate noise levels δ, sparse regularization with $p = 1$ and $p = 1.1$ is able to localize the unknown peak almost perfectly. It becomes obvious that in the case $p = 1$, the reconstructions exhibit significantly fewer active degrees of freedom at the same degree of accuracy.

For more figures and details, we refer the reader to [3].

5 Conclusion

Iterative thresholding algorithms with adaptive operator evaluations are capable of reconstructing local features in inverse PDE problems, even under moderate data noise, by exploiting sparse wavelet expansions of the target quantity. However, the application of sparsity-promoting regularization methods to parameter identification problems still poses challenging analytical and computational tasks.

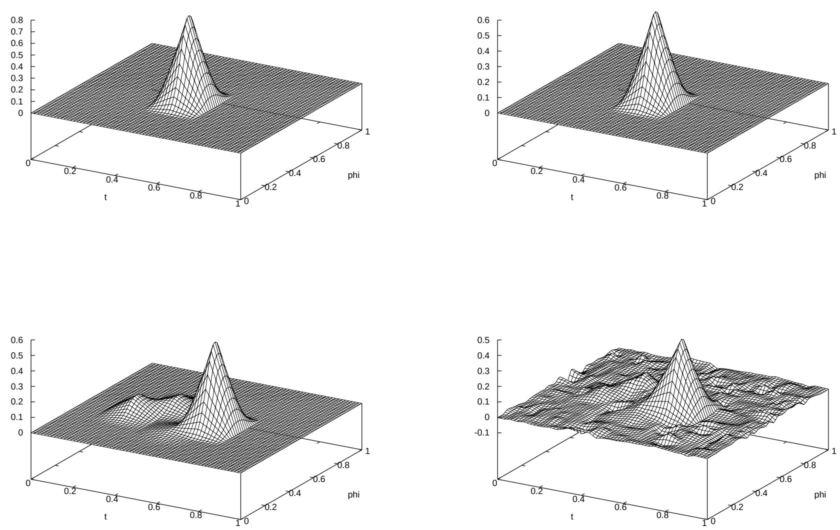

Fig. 4. Unknown temperature data and reconstructions in the two-dimensional example. *Top left*: function g to be reconstructed, *top right*: best reconstruction for $p = 1$ under 0.5% relative noise ($\delta = 0.001$), *bottom left*: best reconstruction for $p = 1$ under 5% relative noise ($\delta = 0.01$), *bottom right*: best possible reconstruction for $p = 1.1$ under the same conditions

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References

1. Daubechies, I., Defrise, M., Mol, C.D.: An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. Commun. Pure Appl. Math. **57** (2004) 1413–1457
2. Bonesky, T., Maass, P.: Iterated soft shrinkage with adaptive operator evaluations. preprint (2008) To appear in *Inverse and Ill-Posed Problems*.
3. Bonesky, T., Dahlke, S., Maass, P., Raasch, T.: Adaptive wavelet methods and sparsity reconstruction for inverse heat conduction problems. Technical Report 5, Preprint Series DFG-SPP 1324 (2009)
4. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4** (2005) 1168–1200
5. Raasch, T.: Adaptive wavelet and frame schemes for elliptic and parabolic equations. Dissertation, Philipps-Universität Marburg (2007)

6. Grasmair, M., Haltmeier, M., Scherzer, O.: Sparse regularization with l^q penalty term. Inverse Probl. **24** (2008)