DECOMPOSITION AND COORDINATION FOR MULTIOBJECTIVE COMPLEX SYSTEMS

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Abstract. Complex systems are modeled as collections of multiobjective programs each representing a subproblem (a subsystem or component) of the overall system. The subproblems interact with each other in various ways adding to the complexity of the overall problem. Since the calculation of efficient sets of these complex systems presents a challenging problem, it is desirable to decompose the overall system into component multiobjective programs that are more easily solvable and then construct the efficient set of the overall system. Selected cases of complex system are presented and relationships between their efficient sets and the efficient sets of their subproblems are given.

KEYWORDS: multiobjective programs, complex systems, efficient set, decomposition, coordination

1. Introduction

Many entities of interest to humans are complex systems. For example, in business, an enterprise may involve several operative sectors within a national economy having different interests that usually are in conflict with each other not only within each sector but also in conflict with the goals of the other sectors. Similarly, a corporation may have several departments having different goals being in conflict with each other and with the goals of the other departments. In engineering, the design of a vehicle or airplane leads to a complex system involving design with respect to several disciplines such as aerodynamics, electrical systems, control systems, etc. However, a preferred design decision for one discipline may not be preferred for another. Additionally, a preferred design decision within a discipline may have to be made with respect to multiple and conflicting design criteria for that discipline.

A complex system is understood to be a natural or engineered system that is difficult to understand and analyze because of one or more of the following properties: (1) The system may involve interactions among many phenomena; (2) The system may have multiple and dissimilar subproblems (components or subsystems) that may be connected in a variety of ways; (3) The system may be characterized by noncomparable and conflicting criteria or objectives such as cost, performance, reliability, safety, productivity, affordability, and others. In the presence of multiple and interacting components and criteria, a unique decision optimal for the system does not exist but rather many or even infinitely many decisions are preferrable.

Studies on complex systems with multiple criteria propose (1) decomposition of the original problem modeled as a single multiobjective program (MOP) into a collection of smaller-sized

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subproblems, for which the development of a solution procedure becomes a more manageable task, and (2) coordination of the solutions of the subproblems to obtain the solution of the original problem. A large number of such approaches exists for specific applications in management science, engineering, and multidisciplinary optimization (see [14, 18, 19] among many others). Other papers deal with decomposition and coordination due to a large number of criteria in the original problem [13, 12, 1, 11, 2]. Finally some authors study objective decompositions from a predominantly mathematical perspective [20, 16, 3, 17]. However, there is a lack of mathematical studies to model complex systems not only as a single large-scale MOP but as a system of MOPs interacting with each other. The research work briefly reported in this paper intends to undertake this very modeling approach.

Complex systems are modeled as collections of multiobjective programs each representing a subproblem (a subsystem or component) of the overall system. The subproblems interact with each other in various ways adding to the complexity of the overall problem. Since the calculation of efficient sets of these complex systems presents a challenging problem, it is desirable to decompose the overall system into component MOPs that are more easily solvable and then construct the efficient set of the overall system.

The systems studied include a variety of configurations with composite objective functions as well as local, global, and/or linking variables. Relationships between the efficient set of the overall system and the efficient sets of subproblems are derived. These relationships indicate when efficient solutions of subproblems are also efficient for the system and reveal how an efficient solution of the system can be built from efficient solutions of the subproblems. In effect, decomposition and coordination approaches may be developed to find the efficient set of a complex system without ever dealing with this system in its entirety.

In Section 2, the overall problem formulation is introduced with the accompanying notation. In Section 3, selected cases of complex system are presented and the relationships between their efficient sets and the efficient sets of their subproblems are given. The paper is concluded in Section 4.

2. All at Once (AAO) Problem

MOPs are optimization problems with multiple and conflicting criteria. Generally we search for a solution that provides an acceptable optimization of all the objective functions, with solutions being acceptable when the only improvement in any of the objective functions causes a deterioration in at least one of the other objective functions.

We model complex systems as mathematical programs that can be written as a single optimization problem with a vector-valued objective function and an appropriate, nonempty feasible set. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ denote the vector-valued objective function to be minimized. Let $x \in \mathbb{R}^n$ denote the decision variables and let the feasible set for the decision variables be represented by $X \subseteq \mathbb{R}^n$. In general, these optimization problems can be written as multiobjective programs (MOPs) in the form

$$\min f(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n. \tag{2.1}$$

A functional representation of the general MOP (2.1) is depicted in Figure 2.1. The box represents the function operator. Inputs for this operator are elements from the function's domain, including decision variables and/or output from another function, while outputs for this function are the elements of the function's range generated by the given input(s). We could also think of this as the input-output representation.

FIGURE 2.1. Functional representation of the system

An efficient solution is a solution for which there does not exist another feasible solution that yields improvement in at least one of the objective functions without degrading any other objective function. The efficiency operator for MOP (2.1) is denoted $\mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right)$ and if $x \in \mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right)$, we say that x is efficient in X for f.

Definition 2.1. $\mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right) = \{x \in X :$ *there does not exist* $\hat{x} \in X$ *such that* $f\left(\hat{x}\right) \leq f\left(x\right)\}$.

The efficiency operator requires three inputs. The first is X, the feasible set of the MOP, the second is f, the objective function of the MOP, and the third is $\mathbb{R}^p_{\geqq}$, the domination cone with respect to the p objective functions (as developed by Yu [21]).

In general, we allow the system to be composed of N subproblems, sometimes referred to as subsystem or component problems. Let the global variable, also called the shared variable, be a subset of the decision variables, x, denoted $x_0 \in \mathbb{R}^{n_0}$, that is a necessary input for at least two of the subproblems. Let the local variable be a subset of the decision variables, denoted by $x_i \in \mathbb{R}^{n_i}$, that is an input for only subproblem i. We denote the decision variables $x = [x_0, x_1, \ldots, x_N]$ where $x_i \in \mathbb{R}^{n_i}$ for $i = 0, 1, \ldots, N$ and let the objective function f be written as $f\left(x\right) = \left[f_1\left(x\right), \ldots, f_N\left(x\right)\right]$ where $f_i : \mathbb{R}^{n_0+n_i} \longrightarrow \mathbb{R}^{p_i}$ for $i = 1, \ldots, N$. At times it may be of interest to consider the scalar components of the objective function which will be denoted $f_{ij} : \mathbb{R}^{n_0+n_i} \longrightarrow \mathbb{R}$ for $j = 1, \ldots, p_i$ and for $i = 1, \ldots, N$.

In the next section we present theoretical results for determining the efficient set of complex systems for several formulations. We assume that the efficient sets for subproblems of a system are available and build a relationship between the available efficient sets and the efficient set of the entire system. The structure of the system and the manner of the decomposition applied to form the subproblems governs the strength and the form of the relationship between the efficient sets of the subproblems and the efficient set of the system. For each type of system we formulate the specific AAO problem for that system, the feasible set, and the objective function. We then build the pertinent efficient sets from Definition 2.1 and present necessary auxiliary concepts. Finally, we state propositions describing relationships between the efficient sets of the subproblems and the efficient set of the system.

3. Decomposition of Selected Complex Systems

In the subsequent subsections we present four types of complex systems. In particular, we address systems with two types of independent variables, two subproblems in serial connection with independent and dependent variables, and cases of global and local independent variables.

3.1. One Subproblem with Two Independent Variables. We first consider systems with one subproblem and distinct decision variables, which we denote as x_1 and x_2, $x = [x_1, x_2]$. These independent variables allow us to write the feasible set, X, as the cross product of X_1 and X_2, where $x_1 \in X_1 \subseteq \mathbb{R}^{n_1}$ and $x_2 \in X_2 \subseteq \mathbb{R}^{n_2}$. This is equivalent to independence between x_1 and x_2, where feasible solutions for one variable are not affected by the feasible solution chosen for the other variable. This case has also been investigated by Li and Haimes [15].

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $X = X_1 \times X_2 = \{x = (x_1, x_2) : x_1 \in X_1 \text{ and } x_2 \in X_2\} \subseteq \mathbb{R}^n$, where $X_i \subseteq \mathbb{R}^{n_i}$ for $i = 1, 2$ and $n = n_1 + n_2$. This system yields the following AAO problem

$$\min f(x) = f(x_1, x_2) \text{ subject to } x \in X = X_1 \times X_2. \tag{3.2}$$

Figure 3.2 provides the physical representation of the system.

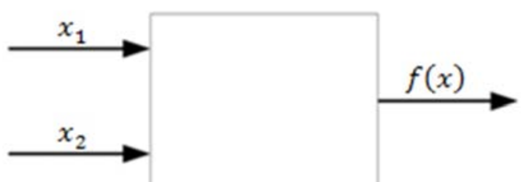

Figure 3.2. Physical representation of one subproblem and two independent variables

Without loss of generality, we fix the value of x_1 and then optimize over x_2. Fix $\bar{x}_1 \in X_1$ and let

$$X(\bar{x}_1) = \{(x_1, x_2) : x_1 = \bar{x}_1 \text{ and } x_2 \in X_2\} = \{\bar{x}_1\} \times X_2. \tag{3.3}$$

The decomposition scheme yields the following subproblem and leads us to the following proposition.

$$\min f(x) \text{ subject to } x \in X(\bar{x}_1) \tag{3.4}$$

Proposition 3.1. $\mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right) = \mathcal{E}\left(\bigcup_{\bar{x}_1 \in X_1} \mathcal{E}\left(X(\bar{x}_1), f, \mathbb{R}^p_{\geqq}\right), f, \mathbb{R}^p_{\geqq}\right)$.

Note that this decomposition yields a relationship that lends itself to problems with one independent variable being discrete or assuming a finite number of feasible values.

3.2. **Two Subproblems in Serial Connection with One Independent Variable.** We move to systems that have two subproblems and a single set of decision variables. However, the objective function is a function not only of the intermediate function, but also of the decision variable.

Let $r : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ and $f : \mathbb{R}^{n+q} \longrightarrow \mathbb{R}^p$ and $X \subseteq \mathbb{R}^n$. This system yields the following AAO problem

$$\min f\left(x, r\left(x\right)\right) \text{subject to } x \in X. \tag{3.5}$$

Figure 3.3 provides the physical representation of the system.

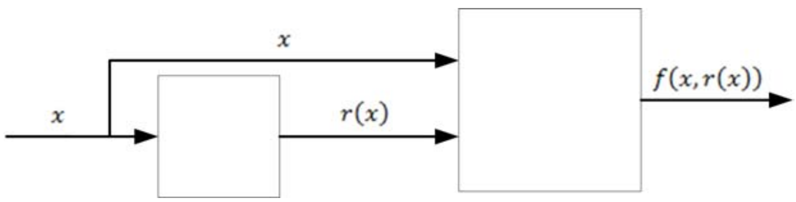

Figure 3.3. Physical representation of two subproblems and a single variable where f is a function of r and x

The decomposition scheme yields the following subproblem and leads us to the following propositions.

$$\min r\left(x\right) \text{subject to } x \in X \tag{3.6}$$

Proposition 3.2. *Let f be strongly increasing, f^{-1} and r^{-1} exist, and f^{-1} and r^{-1} be strongly increasing, then* $\mathcal{E}\left(X \times r\left(X\right), f, \mathbb{R}^p_{\geqq}\right) = \left\{\left(x, r\left(x\right)\right) : x \in \mathcal{E}\left(X, r, \mathbb{R}^q_{\geqq}\right)\right\}$.

This result shows when the vector-valued objective function f can be ignored when calculating the efficient set of the AAO problem.

3.3. N **Subproblems with a Global Variable.** We continue with a system that is composed of N independent subproblems where each subproblem is a function of a single global variable. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ and $f_i : \mathbb{R}^n \to \mathbb{R}^{p_i}$ for $i = 1, \ldots, N$ where $p = p_1 + \ldots + p_N$. Let $X \subseteq \mathbb{R}^n$ with the AAO problem formulation

$$\min f\left(x\right) = \left(f_1\left(x\right), \ldots, f_N\left(x\right)\right) \text{ subject to } x \in X. \tag{3.7}$$

Figure 3.4 presents the physical representation of the system.

Figure 3.4. Physical representation of a system with a single global variable

The subproblems for this system are of the following form for $i = 1, \ldots, N$.

$$\min f_i(x) \text{ subject to } x \in X \tag{3.8}$$

Figure 3.5 present the physical representation viewed at the subproblem level.

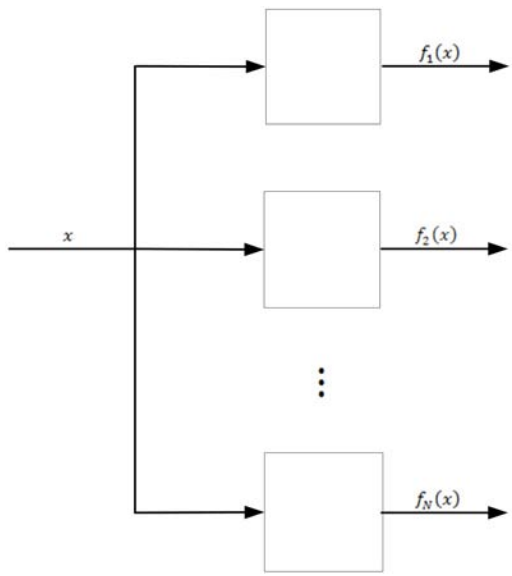

FIGURE 3.5. Physical representation of a system with a global variable decomposed into N subproblems

We first define weakly efficient sets and injectivity of a vector-valued function and then relate the weakly efficient set of X and f_i and the efficient set of X and f.

Definition 3.2. *A point $x \in X$ is said to be weakly efficient in X for f if there does not exist a point $\hat{x} \in X$ such that $f(\hat{x}) < f(x)$.*

The set of weakly efficient points in X for f is denoted $\mathcal{E}_w\left(X, f, \mathbb{R}^p_{\geqq}\right)$.

Definition 3.3. *A function $f_i : \mathbb{R}^n \to \mathbb{R}^{p_i}$ where $f_i = (f_{i1}, \ldots, f_{ip_i})$ is said to be an injective function on X if f_{ij} is an injective function on X for all $j = 1, \ldots, p_i$.*

Definition 3.4. *A function $f_{ij} : \mathbb{R}^n \to \mathbb{R}$ is said to be an injective function on X if $f_{ij}(x) = f_{ij}(\hat{x})$ implies that $x = \hat{x}$ for all $x, \hat{x} \in X$.*

The next two propositions were investigated by Engau [4].

Proposition 3.3. *Let $i \in \{1, \ldots, N\}$, then $\mathcal{E}_w\left(X, f_i, \mathbb{R}^{p_i}_{\geqq}\right) \subseteq \mathcal{E}_w\left(X, f, \mathbb{R}^p_{\geqq}\right)$.*

Proposition 3.4. *Let f_i be injective for $i = 1, \ldots, N$, then $\mathcal{E}\left(X, f_i, \mathbb{R}^{p_i}_{\geqq}\right) \subseteq \mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right)$.*

The relationships above indicate that while an efficient solution of a subproblem remains at least weakly efficient for the overall problem, the converse may not hold for any subproblem. In other words, there are efficient solutions of the overall problem that are not reachable while solving the subproblems for efficiency. These solutions, however, become reachable when the subproblems are solved for ε-efficiency [5, 6, 7, 8]. Using this relaxed efficiency, equivalence between solving the overall problem and a family of smaller-sized subproblems has been established and interactive procedures have been proposed to consider only subsets (or even pairs) of criteria at a time, thereby making decision-making the simplest possible [9].

3.4. N **Subproblems with a Global Variable and** N **Local Variables.** We conclude our presentation of selected complex systems with a system that is composed of N independent subproblems where each subproblem is a function of a single global variable and a local variable. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $f_i : \mathbb{R}^{n_0+n_i} \rightarrow \mathbb{R}^{p_i}$ for $i = 1, \ldots, N$ where $n = n_0 + n_1 + \ldots + n_N$ and $p = p_1 + \ldots + p_N$. Let $X_i \subseteq \mathbb{R}^{n_i}$ for $i = 0, 1, \ldots, N$ and $X = X_0 \times X_1 \times \ldots \times X_N \subseteq \mathbb{R}^n$ with the AAO problem formulation

$$\min f(x) = (f_1(x_0, x_1), \ldots, f_N(x_0, x_N)) \text{ subject to } x \in X = X_0 \times X_1 \times \ldots X_N. \tag{3.9}$$

Figure 3.6 presents the physical representation of the system.

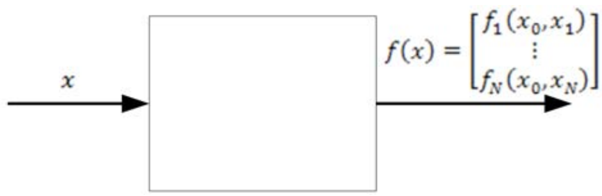

FIGURE 3.6. Physical representation of a system with global and local variables

Let $\bar{x}_0 \in X_0$ be fixed and define the set $X(\bar{x}_0)$ as

$$\begin{aligned} X(\bar{x}_0) &= \{(x_0, x_1, \ldots, x_N) \in X = X_0 \times \ldots \times X_N : x_0 = \bar{x}_0, x_i \in X_i, i = 1, \ldots, N\} \\ &= \{\bar{x}_0\} \times X_1 \times \ldots X_N. \end{aligned} \tag{3.10}$$

The decomposition scheme yields the following subproblem for this system.

$$\min f(x) \text{ subject to } x \in X(\bar{x}_0) \tag{3.11}$$

and leads us to the following proposition.

Figure 3.7 present the physical representation viewed at the subproblem level.

Proposition 3.5. $\mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right) = \mathcal{E}\left(\bigcup_{\bar{x}_0 \in X_0} \mathcal{E}\left(X(\bar{x}_0), f, \mathbb{R}^p_{\geqq}\right), f, \mathbb{R}^p_{\geqq}\right)$.

This result shows that the global variable may be treated as an independent variable as in Section 3.1. However, we take this proposition one step further by considering the independent local variables and fomulating another subproblem. For $i = 1, \ldots, N$ subproblem i for this system is

$$\min f_i(\bar{x}_0, x_i) \text{ subject to } x_i \in X_i. \tag{3.12}$$

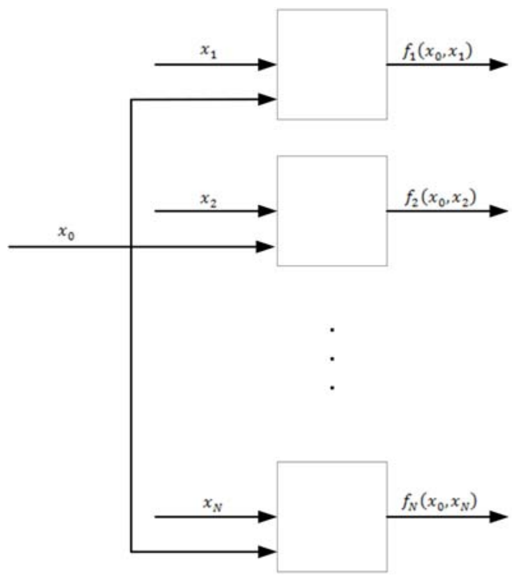

FIGURE 3.7. Physical representation of a system with global and local variables decomposed into N subproblems

Corollary 3.1. $\mathcal{E}\left(X, f, \mathbb{R}^p_{\geqq}\right)$
$$= \mathcal{E}\left(\bigcup_{\bar{x}_0 \in X_0} \{\bar{x}_0\} \times \mathcal{E}\left(X_1, f_1\left(\bar{x}_0, x_1\right), \mathbb{R}^{p_1}_{\geqq}\right) \times \ldots \times \mathcal{E}\left(X_N, f_N\left(\bar{x}_0, x_N\right), \mathbb{R}^{p_N}_{\geqq}\right), f, \mathbb{R}^p_{\geqq}\right)$$

This corollary indicates the possibility of finding the efficient set of the AAO problem by means of finding the efficient sets of the subproblems.

4. Conclusion

In this paper, some recent developments in the calculation of efficient sets for multiobjective complex systems are highlighted. The revealed relationship between the efficient set of the MOP problem associated with the entire system and the efficient sets of its subproblems indicate a type of method for coordination of the efficient sets into a set which is efficient for the original problem.

Other cases of multiobjective complex systems with all detailed proofs are presented in [10]. In particular, it is shown that multiple decomposition schemes, with varying assumptions regarding the system, can be applied to the same initial system yielding different subproblems and hence different relationships.

Among many future directions of research, robustness issues and interactive aspects for multiobjective complex system optimization seem to be especially relevant in the context of real-life applications of this modeling and optimization approach.

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