PP-DEFINABILITY IS CO-NEXPTIME-COMPLETE

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Abstract. PP-Def is the problem which takes as input a relation r and a finite set Γ of relations on the same finite domain A, and asks whether r is definable by a conjunctive query over (A, Γ), i.e., by a formula of the form $\exists \vec{y} \varphi(\vec{x}, \vec{y})$ where $\varphi(\vec{x}, \vec{y})$ is a conjunction of atomic formulas built using the relations in $\Gamma \cup \{=\}$, and where the variables range over A. (Such formulas $\exists \vec{y} \varphi(\vec{x}, \vec{y})$ are called *primitive positive formulas*.) PP-Def is known to be in co-*NEXPTIME*, and has been shown to be tractable on the boolean domain.

We show that there exists $k > 2$ such that PP-Def is co-*NEXPTIME*-complete on k-element domains, answering a question of Creignou, Kolaitis and Zanuttini. We also show that two related problems are *NEXPTIME*-complete.

1. The problems

Let Γ be a finite set of relations on a finite domain A. By a *pp-formula over* Γ we mean a first-order formula of the form $\exists \vec{y} \bigwedge_{i=1}^{t} \alpha_i(\vec{x}, \vec{y})$ where each $\alpha_i(\vec{x}, \vec{y})$ is an atomic formula naming a relation from $\Gamma \cup \{=\}$ applied to a tuple of variables from $\vec{x} \cup \vec{y}$. The *pp-definability problem* (or PP-Def) is:

Input:
A finite nonempty domain A;
A finite set Γ of relations on A;
Another relation r on A.
Question:
Is r definable by a pp-formula over Γ?

This problem is also known as $\exists$-InvSat in the theoretical computer science literature [4, 3]. The uniform version is known to be in co-*NEXPTIME* (folkore?), while the boolean ($|A| = 2$) case was shown to be locally in P by Dalmau [4] and to be globally in P by Creignou, Kolaitis and Zanuttini [3]. At a workshop at the American Institute of Mathematics in April 2008, a working group conjectured that PP-Def is co-*NEXPTIME* complete, even on 3-element domains, and speculated that the lower bound can be proved by interpreting a tiling problem [2].

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Given A, Γ, and some $m \geq 1$, the set of m-ary relations on A pp-definable over Γ includes A^m and is closed under intersections; hence given a relation r on A we can define the *pp-closure of r over Γ* to be the smallest relation of the same arity as r which contains r and is pp-definable over Γ. We denote the pp-closure of r over Γ by $[r]_\Gamma$. Thus r is pp-definable over Γ iff $[r]_\Gamma = r$.

The pp-closure of a relation over Γ may be conveniently described via polymorphisms, or equivalently, by homomorphisms of relational structures. Given A, Γ as above, let $\mathbf{A} = (A; \Gamma)$ be the corresponding relational structure. The m-ary polymorphisms of Γ are precisely the homomorphisms from $\mathbf{A}^m$ to $\mathbf{A}$. These include the so-called *dictator* functions p_i^m, $1 \leq i \leq m$, where $p_i^m(x_1, \ldots, x_m) = x_i$ for all inputs $x_1, \ldots, x_m \in A$. Let $\mathrm{Hom}_{\mathbf{A}^m, \mathbf{A}}$ denote the set of all homomorphisms from $\mathbf{A}^m$ to $\mathbf{A}$. Suppose now that $\vec{c}_1, \ldots, \vec{c}_n$ are chosen from A^m and let $\mathbf{c}$ denote $(\vec{c}_1, \ldots, \vec{c}_n)$. Define

$$\begin{aligned} H(\mathbf{c}) &= \{(h(\vec{c}_1), \ldots, h(\vec{c}_n)) \in A^n \,:\, h \in \mathrm{Hom}_{\mathbf{A}^m, \mathbf{A}}\} \\ P(\mathbf{c}) &= \{(p_i^m(\vec{c}_1), \ldots, p_i^m(\vec{c}_n)) \in A^n \,:\, 1 \leq i \leq m\}. \end{aligned}$$

Lemma 1.1. *Let $A, \Gamma, m, \mathbf{c}$ be as above. Suppose r is an n-ary relation satisfying $P(\mathbf{c}) \subseteq r \subseteq H(\mathbf{c})$. Then*

(1) $[r]_\Gamma = H(\mathbf{c})$.
(2) *Hence r is pp-definable over Γ iff $r = H(\mathbf{c})$.*

We can now describe two related problems which we will show are *NEXPTIME*-complete.

Related problem #1: the *pp-closure problem* (PP-CLS).

Input:
A finite relational structure $\mathbf{A} = (A; \Gamma)$ with Γ finite;
An n-ary relation r on A (for some $n \geq 1$);
An n-tuple $\vec{a} \in A^n$.

Question:
Is $\vec{a} \in [r]_\Gamma$?

Related problem #2: the *homomorphism extension problem* (HOM-EXT).

Input:
A finite relational structure $\mathbf{A} = (A; \Gamma)$ with Γ finite;
A subset $S \subseteq A^m$ (for some $m \geq 1$);
A function $h_0 : S \to A$.

Question:
Can h_0 be extended to a homomorphism $\mathbf{A}^m \to \mathbf{A}$ (i.e., a polymorphism of Γ)?

It is not hard to show that both PP-CLS and HOM-EXT are in *NEXPTIME*. In fact, PP-CLS and HOM-EXT are essentially the same problem[1], since:

[1]Provided relations are represented as lists, not as $\{0, 1\}$-valued tables.

(1) Given an instance $(A, \Gamma, r, \vec{a})$ to PP-CLS with $r \cup \{\vec{a}\} \subseteq A^n$, let $m = |r|$, choose an enumeration $\{\vec{b}_1, \dots, \vec{b}_m\}$ of r, let M be the $n \times m$ matrix whose jth column is $\vec{b}_j$, let $\vec{c}_i$ denote the ith row of this matrix, and put $\mathbf{c} = (\vec{c}_1, \dots, \vec{c}_n)$. Observe that $P(\mathbf{c}) = r$ and hence $H(\mathbf{c}) = [r]_\Gamma$ by Lemma 1.1. If there exist $i \neq j$ such that $\vec{c}_i = \vec{c}_j$ but $a_i \neq a_j$, then automatically $\vec{a} \notin [r]_{\mathbf{A}}$. Otherwise, define $S = \{\vec{c}_1, \dots, \vec{c}_n\} \subseteq A^m$ and $h_0 : S \to A$ by $h_0(\vec{c}_i) = a_i$. Then (A, Γ, S, h_0) is an equivalent instance of HOM-EXT.

(2) Conversely, given an instance (A, Γ, S, h_0) of HOM-EXT with $S \subseteq A^m$, let $n = |S|$, enumerate S as $\{\vec{c}_1, \dots, \vec{c}_n\}$, and let M be the $n \times m$ matrix whose ith row is $\vec{c}_i$. If we let r be the n-ary relation on A whose members are the columns of this matrix, and put $\vec{a} = (h_0(\vec{c}_1), \dots, h_0(\vec{c}_n))$, then by a similar argument as in the previous paragraph, $(A, \Gamma, r, \vec{a})$ is an equivalent instance of PP-CLS.

PP-CLS and PP-DEF^{co} appear to be closely related. Given an input (A, Γ, r) to PP-DEF where r is n-ary, we have that r is *not* pp-definable from Γ iff there exists $\vec{a} \in A^k \setminus r$ such that $\vec{a} \in [r]_\Gamma$. (Incidentally, this observation, together with the fact that PP-CLS is in *NEXPTIME*, gives a proof that PP-DEF is in co-*NEXPTIME*.) Conversely, given an input $(A, \Gamma, r, \vec{a})$ to PP-CLS, we have that $\vec{a} \notin [r]_\Gamma$ iff there exists a relation s of the same arity such that $r \subseteq s$, s is pp-definable from Γ, and $\vec{a} \notin s$. Despite these relationships, we do not see any straightforward polynomial-time reductions of either of PP-CLS, PP-DEF^{co} to the other.[2]

Nevertheless, to resolve the complexity of PP-DEF, we find it fruitful to first study PP-CLS via HOM-EXT. In section 3 we will show that there is a fixed finite relational structure $\mathbf{A} = (A; \Gamma)$ such that the local problems PP-CLS($\mathbf{A}$) and HOM-EXT($\mathbf{A}$) are *NEXPTIME*-complete. In section 4 we will give a more complicated construction which shows that there exists an integer $k \geq 3$ such that the restriction of PP-DEF to k-element domains is co-*NEXPTIME*-complete.

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2. An *NEXPTIME*-complete tiling problem

In this section we define two tiling-of-tori problems that are *NEXPTIME*-complete. Our presentation is inspired by and uses [1].

Definition 2.1.

(1) A *domino system* is a triple $\mathcal{D} = (D, H, V)$ where D is a finite non-empty set and $H, V \subseteq D \times D$.

(2) If $n \geq 2$, then $U(n)$ denotes the torus $\mathbb{Z}_n \times \mathbb{Z}_n$.

[2]We will see that they are polynomial-time equivalent, since they are both *NEXPTIME*-complete for polynomial-time reductions.

Definition 2.2. Suppose $\mathcal{D} = (D, H, V)$ is a domino system, $1 \le k \le n$, and $w = (w_0, w_1, \ldots, w_{k-1}) \in D^k$ is a word over D of length k. We say that $\mathcal{D}$ *tiles* $U(n)$ *with initial condition* w if there exists a mapping $\tau : U(n) \to D$ such that for all $(i,j) \in U(n)$:

(1) If $\tau(i,j) = d$ and $\tau(i+1,j) = e$, then $(d,e) \in H$;
(2) If $\tau(i,j) = d$ and $\tau(i,j+1) = e$, then $(d,e) \in V$;
(3) $\tau(i,0) = w_i$ for $0 \le i < k$.

The most general tiling problem we consider (call it ExpTile) is:

Input:
A domino system $\mathcal{D}$;
An integer $m \ge 2$ given in *unary* notation;
A nonempty word w over D of length $k \le m$.
Question:
Does $\mathcal{D}$ tile $U(2^m)$ with initial condition w?

The second tiling problem we want is a restriction of ExpTile. Say that a domino system $\mathcal{D} = (D, H, V)$ is *full* if $D = \mathrm{pr}_1(H) = \mathrm{pr}_2(H) = \mathrm{pr}_1(V) = \mathrm{pr}_2(V)$.

Definition 2.3. ExpTile$^{\circ *}$ is the restriction of ExpTile to instances $(\mathcal{D}, m, w)$ where $\mathcal{D}$ is full and m is a power of 2.

Definition 2.4. Let $\mathcal{D}$ be a full domino system. ExpTile$^{\circ *}(\mathcal{D})$ is the local version of ExpTile$^{\circ *}$ in which the inputs are restricted to those whose domino system is $\mathcal{D}$.

Proposition 2.5. ExpTile *and* ExpTile$^{\circ *}$ *are NEXPTIME-complete with respect to polynomial-time reductions. Moreover, there exists a full domino system* $\mathcal{D}$ *such that* ExpTile$^{\circ *}(\mathcal{D})$ *is NEXPTIME-complete.*

Proof. See the Appendix. □

3. The first construction

Let $\mathcal{D}$ be a full domino system such that ExpTile$^{\circ *}(\mathcal{D})$ is *NEXPTIME*-complete (as promised by Proposition 2.5). In this section we construct a relational structure $\mathbf{A}$ and give polynomial-time reductions of ExpTile$^{\circ *}(\mathcal{D})$ to Hom-Ext$(\mathbf{A})$ and PP-Cls$(\mathbf{A})$.

Let $(\mathcal{D}, m, w)$ be an input to ExpTile$^{\circ *}(\mathcal{D})$. Since $m \ge 2$ and m is a power of 2, we can write $m = 2^{t+1}$ for some $t \ge 0$. We will use binary strings of length m to address elements of $\{0, 1, \ldots, 2^m - 1\}$ in the usual way. We will need relations which, when interpreted coordinate-wise on a pair of such binary strings of length m, determine whether they address adjacent $x, x+1$ in $\mathbb{Z}_{2^m}$. This can easily be done via a ternary relation which models the action of "adding 1" to the first binary string to get the second binary string; the third argument takes a special "parameter" string over a three-character alphabet; the role of this string is to indicate how far down the first string the "carrying of ones" should proceed.

More precisely, for $n > 0$ we use $I(n)$ to denote the set of integers $\{0, 1, \ldots, n-1\}$, which we also identify with $\mathbb{Z}_n$ in the obvious way. For each $x \in I(2^m)$ we let $\lg(x)$ denote the largest integer $k \leq m$ such that 2^k divides x. Note in particular that $\lg(0) = m$. For $x \in I(2^m)$, we use $\widehat{x}$ to denote the reverse m-bit binary representation of x. That is, if the usual binary representation of x is $c_p \cdots c_2 c_1 c_0$ $(p < m)$, then

$$\widehat{x} = (c_0, c_1, c_2 \ldots, c_p, \underbrace{0, \ldots, 0}_{m-p-1}) \in \{0,1\}^m.$$

Define $\beta_0, \beta_1, \ldots, \beta_m \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^m$ by

$$\begin{aligned} \beta_i &= (\underbrace{\mathtt{a}, \ldots, \mathtt{a}}_{i}, \mathtt{b}, \underbrace{\mathtt{c}, \ldots, \mathtt{c}}_{m-i-1}), \qquad (0 \leq i < m) \\ \beta_m &= (\underbrace{\mathtt{a}, \mathtt{a}, \ldots, \mathtt{a}}_{m}). \end{aligned}$$

Define a ternary relation $\prec \subseteq \{0,1\} \times \{0,1\} \times \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ as follows:

$$\prec = \{(1, 0, \mathtt{a}),\ (0, 1, \mathtt{b}),\ (0, 0, \mathtt{c}),\ (1, 1, \mathtt{c})\}.$$

Lemma 3.1. *For $x, y \in I(2^m)$ and $0 \leq k \leq m$, the following are equivalent:*

(1) *The triple $(\widehat{x}, \widehat{y}, \beta_k)$ is coordinatewise in $\prec$.*
(2) $\lg(y) = k$ *and* $x = y - 1 \pmod{2^m}$.

We now define the relational structure we wish to associate with $(\mathcal{D}, m, w)$. Define $B = \{0,1\}^2$, $C = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, $X = \{\top, \bot\}$, write $\mathcal{D} = (D, H, V)$, and put

$$A \;=\; B \cup C \cup D \cup X \cup \{\infty\}.$$

Define the following ternary relations on A:

$$\begin{aligned} \prec_H &= \{((x_1, y), (x_2, y), t) \in B \times B \times C \,:\, (x_1, x_2, t) \in \prec,\ y \in \{0,1\}\} \\ &\quad \cup \{(d, e, f) \in D \times D \times X \,:\, f = \bot \text{ or } (d, e) \in H\}, \\ \prec_V &= \{((x, y_1), (x, y_2), t) \in B^2 \times B^2 \times C \,:\, (y_1, y_2, t) \in \prec,\ x \in \{0,1\}\} \\ &\quad \cup \{(d, e, f) \in D \times D \times X \,:\, f = \bot \text{ or } (d, e) \in V\}. \end{aligned}$$

Definition 3.2. $\mathbf{A} = (A; \Gamma)$ where $\Gamma = \{\prec_H, \prec_V\}$.

Recall that $U(2^m)$ denotes the torus $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$. For $(x, y) \in U(2^m)$ we define $[x, y] \in B^m$ as follows: if $\widehat{x} = (x_0, x_1, \ldots, x_{m-1})$ and $\widehat{y} = (y_0, y_1, \ldots, y_{m-1})$, then

$$[x, y] \;=\; ((x_0, y_0),\ (x_1, y_1), \ldots, (x_{m-1}, y_{m-1})).$$

Theorem 3.3. *$\mathcal{D}$ tiles the torus $U(2^m)$ with initial condition w iff there exists a homomorphism $h : \mathbf{A}^m \to \mathbf{A}$ satisfying $h(\beta_i) = \top$ for all $i \leq m$ and $h([i, 0]) = w_i$ for all $i < |w|$.*

Proof. Suppose first that such a homomorphism $h : \mathbf{A}^m \to \mathbf{A}$ exists. Given $(i,j) \in U(2^m)$, set $\vec{x} = [i{-}1, j]$, $\vec{y} = [i,j]$, and $k = \lg(j)$. Then $(\vec{x}, \vec{y}, \beta_k)$ is coordinate-wise in $\prec_H$. As h is a homomorphism and $h(\beta_k) = \top$, we get $(h([i{-}1,j]), h([i,j]), \top) \in \prec_H$. By definition this implies $h([i{-}1,j]), h([i,j]) \in D$ and $(h([i{-}1,j]), h([i,j])) \in H$. A similar argument shows that $(h([i,j{-}1]), h([i,j])) \in V$ for all $(i,j) \in U(2^m)$. Thus we can define $\tau : U(2^m) \to D$ by $\tau(i,j) = h([i,j])$, and τ is a tiling of $U(2^m)$ by $\mathcal{D}$ with initial condition w.

Conversely, if τ is a tiling of $U(2^m)$ by $\mathcal{D}$ with initial condition w, we can define $h : A^m \to A$ by

$$h(\vec{u}) = \begin{cases} \tau(i,j) & \text{if } \vec{u} = [i,j] \in B^m \\ \top & \text{if } \vec{u} \in \{\beta_0, \dots, \beta_m\} \\ \bot & \text{if } \vec{u} \in C^m \setminus \{\beta_0, \dots, \beta_m\} \\ \infty & \text{otherwise.} \end{cases}$$

Then h is a homomorphism $\mathbf{A}^m \to \mathbf{A}$ having the required properties. □

Since $\mathbf{A}$ is determined by $\mathcal{D}$ only (that is, not by m, w), and the construction from (m,w) of the set $S := \{\beta_i : i \le m\} \cup \{[i,0] : i < |w|\}$ and the map $S \to A$ given by $\beta_i \mapsto \top$, $[i,0] \mapsto w_i$ can be done in polynomial-time as a function of the size of (m,w), we have a polynomial-time reduction of $\text{ExpTile}^{\circ *}(\mathcal{D})$ to $\text{Hom-Ext}(\mathbf{A})$. Hence:

Corollary 3.4. $\text{Hom-Ext}(\mathbf{A})$ *and* $\text{PP-Cls}(\mathbf{A})$ *are NEXPTIME-complete with respect to polynomial-time reductions.*

Proof. By Theorem 2.5, the previous construction gives an explicit polynomial-time reduction of $\text{ExpTile}^{\circ *}(\mathcal{D})$ to $\text{Hom-Ext}(\mathbf{A})$. For completeness, we describe the corresponding reduction of $\text{ExpTile}^{\circ *}(\mathcal{D})$ to $\text{PP-Cls}(\mathbf{A})$. Given (m,w) with $|w| = k \le m$, let $\ell = \ulcorner \log k \urcorner$ and define $\vec{c}_i \in C^{m+1} \times B^k$ $(0 \le i < m)$ by

$$\vec{c}_i \;=\; \begin{cases} (\underbrace{\mathtt{c}, \dots, \mathtt{c}}_{i}, \mathtt{b}, \underbrace{\mathtt{a}, \dots, \mathtt{a}}_{m-i}, \underbrace{(0,0), \dots}_{2^i}, \underbrace{(1,0), \dots}_{2^i}, \underbrace{(0,0), \dots}_{2^i}, \text{etc}), & (i < \ell), \\ (\underbrace{\mathtt{c}, \dots, \mathtt{c}}_{i}, \mathtt{b}, \underbrace{\mathtt{a}, \dots, \mathtt{a}}_{m-i}, \underbrace{(0,0), \dots, (0,0)}_{k}), & (\ell \le i < m), \end{cases}$$

and let $r = \{\vec{c}_i : 0 \le i < m\} \subseteq A^{m+k+1}$. Finally, define $\vec{a} \in X^{m+1} \times D^k$ by

$$\vec{a} = (\underbrace{\top, \dots, \top}_{m+1}, w_0, w_1, \dots, w_{k-1}).$$

If M is the $m \times (m{+}k{+}1)$-matrix whose rows are $\vec{c}_0, \dots, \vec{c}_{m-1}$, then the columns of M are $\beta_0, \beta_1, \dots, \beta_m, [0,0], [1,0], \dots, [k-1,0]$. Thus by Theorem 3.3 and the connection described between Hom-Ext and PP-Cls in section 1, $\vec{a} \in [r]_\Gamma$ iff $\mathcal{D}$ tiles $U(2^m)$ with initial condition w. As $|r| = m$, the space needed to represent r is polynomial in the size of the original input to $\text{ExpTile}^{\circ *}(\mathcal{D})$. Hence the map $(m,w) \mapsto (r, \vec{a})$ is a polynomial-time reduction of $\text{ExpTile}^{\circ *}(\mathcal{D})$ to $\text{PP-Cls}(\mathbf{A})$. □

4. The second construction

The construction in the previous section does not seem to lead to a proof that PP-Def is co-*NEXPTIME*-complete, in part because the relation r constructed in the proof of Corollary 3.4 is such that the cardinality of $[r]_\Gamma$ is always exponential in the size of input to the tiling problem being encoded, so is too large to coincide with any relation we might care to test for pp-definability (in the context of the argument in the previous section). In this section we describe a variant of the construction from the previous section which avoids this problem and simultaneously gives polynomial-time reductions of ExpTile$^{\circ *}$ to PP-Def^{co}, PP-Cls and Hom-Ext.

Let $(\mathcal{D}, m, w)$ be an input to ExpTile$^{\circ *}$; write $m = 2^{t+1}$ with $t \geq 0$. Again, addresses in $U(2^m)$ will be represented by double-binary strings in B^m. The main new idea is to revise the means by which adjacent addresses are recognized. In place of the 3-ary relation $\prec$ and the $m+1$ "parameter" strings $\beta_0, \ldots, \beta_m \in C^m$ which were used in the previous section, we will use $m+1$ relations each of arity $t+3$, which will jointly require only $t+1$ "parameter" strings $\gamma_0, \ldots, \gamma_t \in \{0,1\}^m$. Since $t+1 = \log m$, the number of parameters we will need is now logarithmic in the size of the original input, a crucial fact in ensuring that the relation we will ultimately test for pp-definability is not too large.

More precisely, if $k \in I(m)$ then we'll use $\langle\!\langle k \rangle\!\rangle$ to denote the reverse $(t+1)$-bit binary representation of k. Define the following $t+1$ elements of $\{0,1\}^m$:

$$\begin{aligned}
\gamma_0 &= (0,1,0,1,0,1,0,1,\ldots,0,1,0,1,0,1,0,1) \\
\gamma_1 &= (0,0,1,1,0,0,1,1,\ldots,0,0,1,1,0,0,1,1) \\
\gamma_2 &= (0,0,0,0,1,1,1,1,\ldots,0,0,0,0,1,1,1,1) \\
&\vdots \\
\gamma_t &= (0,0,0,0,0,0,0,0,\ldots,1,1,1,1,1,1,1,1).
\end{aligned}$$

In other words, if $\gamma_i = (c^i_0, c^i_1 \ldots, c^i_{m-1})$, then c^i_k is the ith bit in $\langle\!\langle k \rangle\!\rangle$. Note that if M is the $(t+1) \times m$ matrix whose rows are $\gamma_0, \ldots, \gamma_t$, then the columns of M are $\langle\!\langle 0 \rangle\!\rangle, \langle\!\langle 1 \rangle\!\rangle, \ldots, \langle\!\langle m-1 \rangle\!\rangle$, and the set of columns of M is $\{0,1\}^{t+1}$.

For each $q \in I(2^m)$ write $\widehat{q} = (c_0, c_1, \ldots, c_{m-1})$ and define the $(t+2)$-ary relation $P_q \subseteq \{0,1\}^{t+2}$ as follows:

$$P_q = \{(c_j, \langle\!\langle j \rangle\!\rangle) : 0 \leq j < m\}.$$

For each $0 \leq k < m$ define the $(t+3)$-ary relation $L_k \subseteq \{0,1\}^{t+3}$ as follows:

$$\begin{aligned}
L_k = {} & \{(1,0,\langle\!\langle j \rangle\!\rangle) : 0 \leq j < k\} \cup \{(0,1,\langle\!\langle k \rangle\!\rangle)\} \cup \\
& \{(x,x,\langle\!\langle j \rangle\!\rangle) : x \in \{0,1\} \text{ and } k < j < m\}.
\end{aligned}$$

Also define the $(t+3)$-ary relation $L_m \subseteq \{0,1\}^{t+3}$ by

$$L_m \;=\; \{(1,0,\langle\!\langle j\rangle\!\rangle) \,:\, 0 \le j < m\}.$$

Lemma 4.1. *Suppose $x, y, q \in I(2^m)$ and $0 \le k \le m$.*

(1) *$(\widehat{x}, \gamma_0, \gamma_1, \ldots, \gamma_t)$ is coordinate-wise in P_q iff $x = q$.*
(2) *$(\widehat{x}, \widehat{y}, \gamma_0, \gamma_1, \ldots, \gamma_t)$ is coordinate-wise in L_k iff $\lg(y) = k$ and $x = y - 1 \pmod{2^m}$.*

Proof. By construction. □

We now begin the definition of the relational structure we wish to associate with the given input $(\mathcal{D}, m, w)$ to ExpTile$^{\circ *}$. Define $B = \{0,1\}^2$, $C = \{0,1\}$, $E = \{\mathtt{a}, \mathtt{b}\}$, $X = \{\top, \bot\}$, write $\mathcal{D} = (D, H, V)$, and put

$$A \;=\; B \cup C \cup D \cup E \cup X \cup \{\infty\}.$$

Define $\phi : C^{t+1} \to E$ by

$$\phi(\vec{u}) = \begin{cases} \mathtt{b} & \text{if } \vec{u} = 1^i 0^{t+1-i} \text{ for some } 0 \le i \le t+1 \\ \mathtt{a} & \text{otherwise.} \end{cases}$$

Note that if we write $\vec{u} = \langle\!\langle k\rangle\!\rangle$ with $k \in I(m)$, then $\vec{u} = 1^i 0^{t+1-i}$ iff $k = 2^i - 1$.

For each $q \in I(2^m)$ we define the $(t+3)$-ary relation $P_q^0 \subseteq B \times C^{t+1} \times E$ ("row zero" analogue of P_q) as follows:

$$P_q^0 \;=\; \{((x,0), \vec{u}, \phi(\vec{u})) \,:\, (x, \vec{u}) \in P_q\}.$$

For each $0 \le k \le m$ we define $(t+4)$-ary relations $L_k^H, L_k^V \subseteq B^2 \times C^{t+1} \times E$ ("horizontal" and "vertical" analogues of L_k) as follows:

$$L_k^H \;=\; \{((x_1,y),(x_2,y),\vec{u},\phi(\vec{u})) \in B^2 \times C^{t+1} \times E \,:\, (x_1,x_2,\vec{u}) \in L_k,\ y \in \{0,1\}\}$$
$$L_k^V \;=\; \{((x,y_1),(x,y_2),\vec{u},\phi(\vec{u})) \in B^2 \times C^{t+1} \times E \,:\, (y_1,y_2,\vec{u}) \in L_k,\ x \in \{0,1\}\}$$

For each $d \in D$ define the $(t+3)$-ary relation $T_d^+ \subseteq D \times X^{t+2}$ by

$$T_d^+ \;=\; \{(x, \vec{v}) \in D \times X^{t+2} \,:\, x = d \text{ or } \bot \in \{v_0, \ldots, v_{t+1}\}\}.$$

Similarly, define the $(t+4)$-ary relations $H^+, V^+ \subseteq D^2 \times X^{t+2}$ by

$$H^+ \;=\; \{(x, y, \vec{v}) \in D^2 \times X^{t+2} \,:\, (x,y) \in H \text{ or } \bot \in \{v_0, \ldots, v_{t+1}\}\},$$
$$V^+ \;=\; \{(x, y, \vec{v}) \in D^2 \times X^{t+2} \,:\, (x,y) \in V \text{ or } \bot \in \{v_0, \ldots, v_{t+1}\}\}.$$

We now assemble the relations for our relational structure. For $0 \le k \le m$ define the $(t+4)$-ary relations

$$H_k \;=\; L_k^H \,\cup\, H^+ \,\cup\, \{(\infty, \infty, \ldots, \infty)\}$$
$$V_k \;=\; L_k^V \,\cup\, V^+ \,\cup\, \{(\infty, \infty, \ldots, \infty)\}$$

Recall that our input to $\textsc{ExpTile}^{\circ *}$ is $(\mathcal{D}, m, w)$. Write $w = w_0 w_1 \cdots w_{\ell-1}$ with $|w| = \ell \le m$, and for each $q < \ell$ define the $(t+3)$-ary relation

$$T_q \;=\; P^0_q \;\cup\; T^+_{w_q} \;\cup\; \{(\infty, \infty, \ldots, \infty)\}.$$

Definition 4.2. $\mathbf{A} = (A; \Gamma)$ where $\Gamma = \{H_0, \ldots, H_m, V_0, \ldots, V_m, T_0, \ldots, T_{\ell-1}\}$.

Define $\beta \in \{\mathtt{a}, \mathtt{b}\}^m$ by

$$\beta \;=\; (\phi(\langle\!\langle 0 \rangle\!\rangle), \phi(\langle\!\langle 1 \rangle\!\rangle), \phi(\langle\!\langle 2 \rangle\!\rangle), \ldots, \phi(\langle\!\langle m-1 \rangle\!\rangle)).$$

Claim 4.3. *Suppose $\vec{x}, \vec{y}, \vec{z}_0, \vec{z}_1, \ldots, \vec{z}_t, \vec{u} \in A^m$, $0 \le q < \ell$, and $0 \le k \le m$, and let σ be a self-map from $\{0, 1, \ldots, t\}$ to itself.*

(1) *If $(\vec{x}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is coordinate-wise in T_q, then either $(\vec{x}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is in $B^m \times C^m \times \cdots \times C^m \times E^m$ or $\{\vec{x}, \vec{z}_0, \ldots, \vec{z}_t\}$ is disjoint from $B^m \cup C^m \cup E^m$.*

(2) *If $(\vec{x}, \vec{y}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is coordinate-wise in H_k, then either $(\vec{x}, \vec{y}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is in $B^m \times B^m \times C^m \times \cdots \times C^m \times E^m$ or $\{\vec{x}, \vec{y}, \vec{z}_0, \ldots, \vec{z}_t\}$ is disjoint from $B^m \cup C^m \cup E^m$. The same is true for V_k.*

(3) *$(\vec{x}, \gamma_0, \ldots, \gamma_t, \beta)$ is coordinate-wise in T_q iff $\vec{x} = [q, 0]$.*

(4) *$(\vec{x}, \vec{y}, \gamma_0, \ldots, \gamma_t, \beta)$ is coordinate-wise in H_k iff there exist $i, j \in I(2^m)$ such that $\lg(i) = k$, $\vec{x} = [i-1, j]$ and $\vec{y} = [i, j]$.*

(5) *$(\vec{x}, \vec{y}, \gamma_0, \ldots, \gamma_t, \beta)$ is coordinate-wise in V_k iff there exist $i, j \in I(2^m)$ such that $\lg(j) = k$, $\vec{x} = [i, j-1]$ and $\vec{y} = [i, j]$.*

(6) *If $(\vec{x}, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in T_q, then $\sigma(i) = i$ for all i.*

(7) *If $(\vec{x}, \vec{y}, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in H_k, then $\sigma(i) = i$ for all i. The same is true for V_k.*

Proof. Items (1)–(5) follow easily from Lemma 4.1 and the definitions. To prove item (6), assume that $(x, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in T_q. We first prove that σ must be a permutation. Suppose $i < t$ and $i \notin \operatorname{ran}(\sigma)$. Let $k = m - 1 - 2^i$. Then $\langle\!\langle k \rangle\!\rangle = 1^i 0 1^{t-i}$. Since $i \ne t$, we have $\phi(\langle\!\langle k \rangle\!\rangle) = \mathtt{a}$. However, γ_j has a 1 at coordinate k for all $j \in \operatorname{ran}(\sigma)$, so $(\gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ at coordinate k is $(1, \ldots, 1, \mathtt{a})$, which is not in $\operatorname{graph}(\phi)$. This contradicts the assumption that $(x, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in T_q and hence proves $\{0, 1, \ldots, t-1\} \subseteq \operatorname{ran}(\sigma)$. Finally, suppose $t \notin \operatorname{ran}(\sigma)$. Let $k = 2^t$. Then $\langle\!\langle k \rangle\!\rangle = 0^t 1$ and $\phi(\langle\!\langle k \rangle\!\rangle) = \mathtt{a}$, so $(\gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ at coordinate k equals $(0, \ldots, 0, \mathtt{a})$, which again is not in $\operatorname{graph}(\phi)$, contradicting the assumption that $(x, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in T_q. Hence $t \in \operatorname{ran}(\sigma)$, so σ is a permutation.

To prove that σ is the identity map, it now suffices to prove that $\sigma(0) \le \sigma(1) \le \cdots \le \sigma(t+1)$. Suppose instead there exists $j < t$ with $\sigma(j) > \sigma(j+1)$. Let $r = \sigma(j)$ and $k = 2^r - 1$. Then $(\gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ at coordinate k has the form $(*, \ldots, *, 0, 1, *, \ldots, *, \mathtt{b})$, which is not in $\operatorname{graph}(\phi)$, again contradicting the assumption that $(x, \gamma_{\sigma(0)}, \ldots, \gamma_{\sigma(t)}, \beta)$ is coordinate-wise in T_q. $\square$

Definition 4.4.

(1) $\widehat{\top} = (\top, \top, \dots, \top) \in A^{t+2}$.
(2) $\widehat{\infty} = (\infty, \infty, \dots, \infty) \in A^{t+2}$.
(3) $r = \text{graph}(\phi) = \{(\vec{u}, \phi(\vec{u})) : \vec{u} \in C^{t+1}\} \subseteq A^{t+2}$.
(4) $s = r \cup (X^{t+2} \setminus \{\widehat{\top}\}) \cup \{\widehat{\infty}\}$.

Lemma 4.5.

(1) $[r]_\Gamma = \{(a_0, \dots, a_t, b) \in A^{t+2}$: *there exists a homomorphism* $h : \mathbf{A}^m \to \mathbf{A}$ *with* $h(\gamma_i) = a_i$ *for* $0 \le i \le t$ *and* $h(\beta) = b\}$.
(2) $s \subseteq [r]_\Gamma \subseteq s \cup \{\widehat{\top}\}$.

Proof. Let M be the $(t+2) \times m$ matrix whose columns in order are $(\langle\!\langle k \rangle\!\rangle, \phi(\langle\!\langle k \rangle\!\rangle))$, $0 \le k < m$. Then the columns of M enumerate r, and the rows of M are precisely $\gamma_0, \gamma_1, \dots, \gamma_t, \beta$. (1) then follows immediately from the connection between PP-Cls and Hom-Ext described on page 2.

Next, we'll show $s \subseteq [r]_{\mathbf{A}}$. Obviously $r \subseteq [r]_{\mathbf{A}}$. It is easy to check that the constant function $A^m \to \{\infty\}$ is a homomorphism $\mathbf{A}^m \to \mathbf{A}$ which, with item (1), proves $\widehat{\infty} \in [r]_{\mathbf{A}}$. Finally, assume $\mathbf{f} = (f_0, f_1, \dots, f_t, f_{t+1}) \in X^{t+2} \setminus \{\widehat{\top}\}$. Pick any $d_0 \in D$ and define $h_{\mathbf{f}} : A^m \to A$ by

$$h_{\mathbf{f}}(\vec{u}) = \begin{cases} d_0 & \text{if } \vec{u} \in B^m \\ f_i & \text{if } \vec{u} = \gamma_i \text{ for some } 0 \le i \le t \\ f_{t+1} & \text{if } \vec{u} = \beta \\ \bot & \text{if } \vec{u} \in C^m \cup E^m \setminus \{\gamma_0, \dots, \gamma_t, \beta\} \\ \infty & \text{otherwise.} \end{cases}$$

Clearly $\mathbf{f} = (h_{\mathbf{f}}(\gamma_0), \dots, h_{\mathbf{f}}(\gamma_t), h_{\mathbf{f}}(\beta))$, so to prove $\mathbf{f} \in [r]_{\mathbf{A}}$ it suffices in light of item (1) to show that $h_{\mathbf{f}}$ is a homomorphism $\mathbf{A}^m \to \mathbf{A}$. Suppose first that $q < \ell$; we verify that $h_{\mathbf{f}}$ preserves T_q. Assume $\vec{x}, \vec{z}_0, \dots, \vec{z}_t, \vec{u} \in A^m$ and $(\vec{x}, \vec{z}_0, \dots, \vec{z}_t, \vec{u})$ is coordinate-wise in T_q, yet $\mathbf{g} = (h_{\mathbf{f}}(\vec{x}), h_{\mathbf{f}}(\vec{z}_0), \dots, h_{\mathbf{f}}(\vec{z}_t), h_{\mathbf{f}}(\vec{u})) \notin T_q$. Then at least one of $h_{\mathbf{f}}(\vec{x}), h_{\mathbf{f}}(\vec{z}_0), \dots, h_{\mathbf{f}}(\vec{z}_t), h_{\mathbf{f}}(\vec{u})$ is not equal to ∞. Hence by the definition of $h_{\mathbf{f}}$, $\{\vec{x}, \vec{z}_0, \dots, \vec{z}_t, \vec{u}\}$ is not disjoint from $B^m \cup C^m \cup E^m$, from which it follows that $(\vec{x}, \vec{z}_0, \dots, \vec{z}_t, \vec{u}) \in B^m \times C^m \times \cdots \times C^m \times E^m$ by Claim 4.3(1). Hence $\mathbf{g} = (d_0, f'_0, \dots, f'_t, f'_{t+1})$ for some $f'_i \in X$, by definition of $h_{\mathbf{f}}$. The only way that $\mathbf{g}$ can fail to be in T_q is if $d_0 \ne w_q$ and $f'_i = \top$ for all i. Since $\vec{z}_0, \dots, \vec{z}_t \in C^m$ and $\vec{u} \in E^m$, and using the definition of $h_{\mathbf{f}}$, we get that $\vec{u} = \beta$ and $f_{t+1} = \top$, and $\{\vec{z}_0, \dots, \vec{z}_t\} \subseteq \{\gamma_i : 0 \le i \le t, f_i = \top\}$. Since $\mathbf{f} \ne \widehat{\top}$, there exists $\lambda \le t$ such that $f_\lambda = \bot$, so $\{\vec{z}_0, \dots, \vec{z}_t\}$ is a proper subset of $\{\gamma_0, \dots, \gamma_t\}$. But this and the fact that $(\vec{x}, \vec{z}_0, \dots, \vec{z}_t, \vec{u})$ is coordinate-wise in T_q contradicts Claim 4.3(6). Hence $h_{\mathbf{f}}$ preserves T_q. The proofs for H_k and V_k are similar. Hence $h_{\mathbf{f}}$ is a homomorphism.

The remarks in the preceding paragraph show that $s \subseteq [r]_\Gamma$. To finish the proof of (2), note that the pp-formula $\exists z T_0(z, x_0, \ldots, x_t, y)$ defines the relation $s \cup \{\widehat{\top}\}$. Hence $[r]_\Gamma \subseteq s \cup \{\widehat{\top}\}$. □

Theorem 4.6. *The following are equivalent:*

(1) *$\mathcal{D}$ tiles the torus $U(2^m)$ with initial condition w.*
(2) *There exists a homomorphism $h : \mathbf{A}^m \to \mathbf{A}$ with $h(\gamma_i) = \top_i$ for $0 \le i \le t$ and $h(\beta) = \top$.*
(3) *$\widehat{\top} \in [r]_\Gamma$.*
(4) *s is not pp-definable over Γ.*

Proof. (2) ⇔ (3) ⇔ (4) follows from Lemma 4.5. Thus it suffices to prove (1) ⇔ (2).

(1) ⇒ (2). Assume $\tau : U(2^m) \to D$ is a tiling of the torus $U(2^m)$ by $\mathcal{D}$ with initial condition w. Define $h_\tau : A^m \to A$ by

$$h_\tau(\vec{u}) = \begin{cases} \tau(i,j) & \text{if } \vec{u} = [i,j] \in B^m \\ \top & \text{if } \vec{u} \in \{\gamma_0, \ldots, \gamma_t, \beta\} \\ \bot & \text{if } \vec{u} \in C^m \cup E^m \setminus \{\gamma_0, \ldots, \gamma_t, \beta\} \\ \infty & \text{otherwise.} \end{cases}$$

Clearly $\widehat{\top} = (h_\tau(\gamma_0), \ldots, h_\tau(\gamma_t), h_\tau(\beta))$, so to prove (2) it suffices to show that h_τ is a homomorphism $\mathbf{A}^m \to \mathbf{A}$.

Suppose $q < \ell$; we'll verify that h_τ preserves T_q. Assume $\vec{x}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u} \in A^m$ and $(\vec{x}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is coordinate-wise in T_q, yet $\mathbf{g} = (h_\tau(\vec{x}), h_\tau(\vec{z}_0), \ldots, h_\tau(\vec{z}_t), h_\tau(\vec{u})) \notin T_q$. Then as in the proof of Lemma 4.5, we get $\vec{x} \in B^m$, $\vec{z}_0, \ldots, \vec{z}_t \in C^m$, $\vec{u} \in E^m$, and $\mathbf{g} = (h_\tau(\vec{x}), \top, \ldots, \top)$ with $h_\tau(\vec{x}) \neq w_q$. This forces $\vec{u} = \beta$ and $\{\vec{z}_0, \ldots, \vec{z}_t\} \subseteq \{\gamma_0, \ldots, \gamma_t\}$. Since $(\vec{x}, \vec{z}_0, \ldots, \vec{z}_t, \vec{u})$ is coordinate-wise in T_q, Claim 4.3(6) yields $\vec{z}_i = \gamma_i$ for each $i \le t$. Claim 4.3(3) then yields $\vec{x} = [q, 0]$. But the assumption that τ is a tiling of $U(2^m)$ with initial condition w implies $\tau(q, 0) = w_q$, contradicting the fact that $\tau(q, 0) = h_\tau([q, 0]) = h_\tau(\vec{x}) \neq w_q$. Hence h_τ preserves each relation T_q. The proof for the relations H_k and V_k is similar. Hence h_τ is a homomorphism, proving (2).

(2) ⇒ (1). Assume $h : \mathbf{A}^m \to \mathbf{A}$ is a homomorphism satisfying $h(\gamma_i) = \top$ for all $i \le t$ and $h(\beta) = \top$. Given $(i, j) \in U(2^m)$, set $\vec{x} = [i-1, j] \in B^m$, $\vec{y} = [i, j] \in B^m$, and $k = \lg(i)$. Then $(\vec{x}, \vec{y}, \gamma_0, \ldots, \gamma_t, \beta)$ is coordinate-wise in H_k by Claim 4.3(4). Since h is a homomorphism, we get $(h(\vec{x}), h(\vec{y}), \top, \ldots, \top) \in H_k$, which implies $h(\vec{x}), h(\vec{y}) \in D$ and $(h(\vec{x}), h(\vec{y})) \in H$. Thus we can define a function $\tau_h : U(2^m) \to D$ by $\tau_h(i, j) = h([i, j])$. The above argument and its analogue for the relations V_k show that τ_h is a tiling of $U(2^m)$ by $\mathcal{D}$. The analagous argument for the relations T_q prove that τ_h satisfies the initial condition w. □

Theorem 4.7. PP-DEF^{co}*,* PP-CLS *and* HOM-EXT *are NEXPTIME-complete for polynomial-time reductions.*

Proof. The preceding construction takes an instance $(\mathcal{D}, m, w)$ of $\text{ExpTile}^{\circ *}$ and produces equivalent instances $(\mathbf{A}, \{\gamma_0, \dots, \gamma_t, \beta\}, \gamma_i \mapsto \top,\ \beta \mapsto \top)$ of Hom-Ext, $(\mathbf{A}, r, \widehat{\top})$ of PP-Cls, and $(\mathbf{A}, s)$ of PP-Def^{co}. It suffices to show that the reductions are polynomial-time computable; the only issue is whether the sizes of the constructed instances are polynomially bounded in the size of $(\mathcal{D}, m, w)$.

Because $\mathcal{D}$ is full we certainly have $d + m \le ||(\mathcal{D}, m, w)||$, where $||(\mathcal{D}, m, w)||$ denotes the size of a standard encoding of $(\mathcal{D}, m, w)$ and $d = |D|$. Analyzing the above construction, we see that

$$\begin{aligned} |A| &= d + 11 \\ |H_k|, |V_k| &\le 4m + d^2 \cdot 2^{t+2} + 1 \ \le\ 2m(d+1)^2, \\ |T_q| &\le m + d \cdot 2^{t+2} + 1 \ \le\ 2m(d+1) \\ |r| &= 2^{t+1} \ =\ m, \\ |s| &= m + (2^{t+2} - 1) + 1 \ =\ 3m. \end{aligned}$$

Hence

$$\begin{aligned} ||(\mathbf{A}, s)|| \ \le\ \log|A| \Big(& 1 + 2(m+1) \cdot 2m(d+1)^2 \cdot (t+4) + \\ & \ell \cdot 2m(d+1) \cdot (t+3) + 3m \cdot (t+2) \Big). \end{aligned}$$

Since $t+1 = \log m$ and $\ell \le m$, the above upper bound is $O((d+m)^5)$, proving $||(\mathbf{A}, s)||$ is polynomial in $||(\mathcal{D}, m, w)||$. The analysis for $||(\mathbf{A}, r, \widehat{\top})||$ is just as easy, and the size of $(\mathbf{A}, \{\gamma_0, \dots, \gamma_t, \beta\}, \gamma_i \mapsto \top_i,\ \beta \mapsto \top)$ is essentially the same as $||(\mathbf{A}, r, \widehat{\top})||$. □

Corollary 4.8. *There exists $k \ge 3$ such that the restrictions of* PP-Def^{co}*,* PP-Cls *and* Hom-Ext *to k-element domains are NEXPTIME-complete.*

Proof. Fix a full domino system $\mathcal{D}$ for which $\text{ExpTile}^{\circ *}(\mathcal{D})$ is *NEXPTIME*-complete. (Such a $\mathcal{D}$ is promised by Proposition 2.5.) If $\mathcal{D} = (D, H, V)$, then the above argument shows that we can take $k = |D| + 11$. □

5. Miscellaneous remarks and open questions

Remark 5.1. PP-Def is the relational "dual" of Gen-Clo, the algebraic "clone generation" problem. Kozik [5] proves that there exists a fixed, finite algebra $\mathbb{A}$ for which the local problem Gen-Clo($\mathbb{A}$) is *EXPTIME*-complete.

Question 1: By analogy, is there a fixed finite relational structure $\mathbf{A}$ for which the local problem PP-Def($\mathbf{A}$) is co-*NEXPTIME*-complete?

Question 2: Can the parameter k in Corollary 4.8 be reduced to $k = 3$ (as conjectured by the working group at AIM)?

Remark 5.2. PP-Def^{co} and PP-Cls are each polynomial-time reducible to the other, since they are both *NEXPTIME*-complete with respect to polynomial-time reductions.

Question 3: is there a relatively simple, *direct* polynomial-time reduction of either problem to the other?

Remark 5.3. The construction in section 4, incorporating the proof of Corollary 4.8, shows that PP-Def^{co} and PP-Cls remain *NEXPTIME*-complete even if the input relations are represented (generally less efficiently) by their characteristic functions (hence a k-ary relation on a d-element set has size d^k).

Remark 5.4. The construction in section 4 also shows more specifically that the following variant of PP-Def is co-*NEXPTIME*-complete:

Input:
A finite relational structure $\mathbf{A} = \langle A; \Gamma \rangle$ of finite signature;
A k-ary relation $p \in \Gamma$.
A k-tuple $\vec{a} \in A^k$.

Question:
Is $p \setminus \{\vec{a}\}$ primitive-positive definable over Γ?

6. Appendix: *NEXPTIME*-completeness for ExpTile and ExpTile$^{\circ *}$

We first prove that both ExpTile and ExpTile$^{\circ *}$ are *NEXPTIME*-complete. Later we will also prove that there exists a fixed $\mathcal{D}$ such that ExpTile$^{\circ *}(\mathcal{D})$ is *NEXPTIME*-complete, completing the proof of Proposition 2.5.

Define two problems intermediate to ExpTile and ExpTile$^{\circ *}$ as follows:

Definition 6.1.

(1) ExpTile* is the restriction of ExpTile to instances $(\mathcal{D}, m, w)$ where m is a power of 2.
(2) ExpTile$^\circ$ is the restriction of ExpTile to instances $(\mathcal{D}, m, w)$ where $\mathcal{D}$ is *full* (see section 2).

Note first that ExpTile and ExpTile* can be polynomial-time reduced to their full restrictions ExpTile$^\circ$ and ExpTile$^{\circ *}$ respectively, by repeatedly deleting dominoes not mentioned in $\mathrm{pr}_1(H) \cap \mathrm{pr}_2(H) \cap \mathrm{pr}_1(V) \cap \mathrm{pr}_2(V)$ and re-indexing the remaining input data, immediately answering "no" if the initial condition mentions a deleted domino.

Thus it will suffice to show:

(1) ExpTile$^\circ$ is in *NEXPTIME*.
(2) ExpTile* is *NEXPTIME*-hard.

Lemma 6.2. ExpTile$^\circ$ *is in NEXPTIME.*

Proof. Suppose $(\mathcal{D}, m, w)$ is an input to ExpTile$^\circ$, with $\mathcal{D} = (D, H, V)$, $|D| = d$, and $|w| = k \leq m$. Because $\mathcal{D}$ is full, we can assume $d + m \leq ||(\mathcal{D}, m, w)||$.

If the answer for this input is "yes," a tiling τ witnessing this can be presented in $2^{2m} \log d$ space and its correctness can be checked on a multi-tape deterministic Turing machine with additional input tape for τ in time bounded by a polynomial in $2^m d$. Since $\log(2^m d) = m + \log d \leq ||(\mathcal{D}, n, w)||$, this proves ExpTile$^\circ \in$ *NEXPTIME*. □

Lemma 6.3. ExpTile* *is NEXPTIME-hard for polynomial-time reductions.*

Proof. Assume L is a language in *NEXPTIME*. Let Σ be the alphabet of L, and fix a nondeterministic Turing machine M which accepts the language L in time $f(n) \leq 2^{Cn^k}$. We can assume (at the expense of increasing C, k) that M works on a single semi-infinite tape, that the alphabet of M contains Σ and at least one other symbol $\Box$ (blank), that M never tries to move left from the left-most tape cell, and that at every stage of a computation of M there is never a blank symbol to the left of a non-blank symbol.

Under these assumptions, Börger, Grädel and Gurevich [1] describe a domino system $\mathcal{D}_L = (D, H, V)$ and a linear-time reduction which takes any input word $x \in \Sigma^+$ to a word $\varphi(x) \in D^+$ of the same length $k > 0$, such that

- If some computation of M accepts x in time less than or equal to $t_0 \geq k$, then $\mathcal{D}_L$ tiles $U(n)$ with initial condition $\varphi(x)$ for all $n \geq t_0 + 2$.
- If M does not accept x, then $\mathcal{D}_L$ does not tile $U(n)$ with initial condition $\varphi(x)$ for any $n \geq k + 2$.

Thus we can reduce L to ExpTile* be sending $x \mapsto (\mathcal{D}_L, m(x), \varphi(x))$ where $m(x)$ is the least power of 2 greater than $C|x|^k + 1$. Since $\mathcal{D}_L$, C and k are fixed (for L), this is clearly a polynomial-time reduction. □

Corollary 6.4. ExpTile *and* ExpTile$^{\circ *}$ *are NEXPTIME-complete for polynomial-time reductions.*

Corollary 6.5. *There exists a full domino system* $\mathcal{D}$ *such that* ExpTile$^{\circ *}(\mathcal{D})$ *is NEXPTIME-complete for polynomial-time reductions.*

Proof. Fix a standard encoding $(\mathcal{D}, m, w) \mapsto \ulcorner(\mathcal{D}, m, w)\urcorner$ of inputs to ExpTile as strings over a finite alphabet Σ, and define

$$L = \{\ulcorner(\mathcal{D}, m, w)\urcorner : \mathcal{D} \text{ tiles } U(2^m) \text{ with initial condition } w\}.$$

L is *NEXPTIME*-complete by Corollary 6.4. The proof of Lemma 6.3 produces a domino system $\mathcal{D}_L$ and a polynomial-time reduction of L to ExpTile$^*(\mathcal{D}_L)$. By the comments following Definition 6.1, we can find a full domino system $\mathcal{D}$ and a polynomial-time reduction of ExpTile$^*(\mathcal{D}_L)$ to ExpTile$^{\circ *}(\mathcal{D})$, so ExpTile$^{\circ *}(\mathcal{D})$ is *NEXPTIME*–complete. □

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