Finding Connected Secluded Subgraphs*

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Abstract

Problems related to finding induced subgraphs satisfying given properties form one of the most studied areas within graph algorithms. Such problems have given rise to breakthrough results and led to development of new techniques both within the traditional P vs NP dichotomy and within parameterized complexity. The Π-Subgraph problem asks whether an input graph contains an induced subgraph on at least k vertices satisfying graph property Π. For many applications, it is desirable that the found subgraph has as few connections to the rest of the graph as possible, which gives rise to the Secluded Π-Subgraph problem. Here, input k is the size of the desired subgraph, and input t is a limit on the number of neighbors this subgraph has in the rest of the graph. This problem has been studied from a parameterized perspective, and unfortunately it turns out to be W[1]-hard for many graph properties Π, even when parameterized by $k + t$. We show that the situation changes when we are looking for a connected induced subgraph satisfying Π. In particular, we show that the Connected Secluded Π-Subgraph problem is FPT when parameterized by just t for many important graph properties Π.

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1 Introduction

Vertex deletion problems are central in parameterized algorithms and complexity, and they have contributed hugely to the development of new algorithmic methods. The Π-Deletion problem, with input a graph G and an integer ℓ, asks whether at most ℓ vertices can be deleted from G so that the resulting graph satisfies graph property Π. Its dual, the Π-Subgraph problem, with input G and k, asks whether G contains an induced subgraph on at least k vertices satisfying Π. The problems were introduced already in 1980 by Yannakakis and Lewis [11], who showed their NP-completeness for almost all interesting graph properties Π. During the last couple of decades, these problems have been studied extensively with respect to parameterized complexity and kernelization, which has resulted in numerous new techniques and methods in these fields [4, 5].

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In many network problems, the size of the *boundary* between the subgraph that we are looking for and the rest of the graph makes a difference. A small boundary limits the exposure of the found subgraph, and notions like isolated cliques have been studied in this respect [7, 8, 10]. Several measures for the boundary have been proposed; in this work we use the open neighborhood of the returned induced subgraph. For a set of vertices U of a graph G and a positive integer t, we say that U is *t-secluded* if $|N_G(U)| \leq t$. Analogously, an induced subgraph H of G is *t-secluded* if the vertex set of H is t-secluded. For a given graph property Π, we get the following formal definition of the problem SECLUDED Π-SUBGRAPH.

SECLUDED Π-SUBGRAPH
Input: A graph G and nonnegative integers k and t.
Task: Decide whether G contains a t-secluded induced subgraph H on at least k vertices, satisfying Π.

Lewis and Yannakakis [11] showed that Π-SUBGRAPH is NP-complete for every hereditary nontrivial graph property Π. This immediately implies that SECLUDED Π-SUBGRAPH is NP-complete for every such Π. As a consequence, the interest has shifted towards the parameterized complexity of the problem, which has been studied by van Bevern et al. [14] for several classes Π. Unfortunately, in most cases SECLUDED Π-SUBGRAPH proves to be W[1]-hard, even when parameterized by $k+t$. In particular, it is W[1]-hard to decide whether a graph G has a t-secluded independent set of size k when the problem is parameterized by $k + t$ [14]. In this extended abstract, we show that the situation changes when the secluded subgraph we are looking for is required to be connected, in which case we are able to obtain positive results that apply to many properties Π. In fact, connectivity is central in recently studied variants of secluded subgraphs, like SECLUDED PATH [2, 9] and SECLUDED STEINER TREE [6]. However, in these problems the boundary measure is the closed neighborhood of the desired path or the steiner tree, connecting a given set of vertices. The following formal definition describes the problem that we study in this extended abstract, CONNECTED SECLUDED Π-SUBGRAPH. For generality, we define a weighted problem.

CONNECTED SECLUDED Π-SUBGRAPH
Input: A graph G, a weight function $\omega\colon V(G) \to \mathbb{Z}_{>0}$, a nonnegative integer t and a positive integer w.
Task: Decide whether G contains a connected t-secluded induced subgraph H with $\omega(V(H)) \geq w$, satisfying Π.

Observe that CONNECTED SECLUDED Π-SUBGRAPH remains NP-complete for all hereditary nontrivial graph properties Π, following the results of Yannakakis [15]. It can be also seen that CONNECTED SECLUDED Π-SUBGRAPH parameterized by w is W[1]-hard even for unit weights, if it is W[1]-hard with parameter k to decide whether G has a connected induced subgraph on at least k vertices, satisfying Π (see, e.g., [5, 13]).

It is thus more interesting to consider parameterization by t. We consider CONNECTED SECLUDED Π-SUBGRAPH for all graph properties Π that are characterized by finite sets $\mathcal{F}$ of forbidden induced subgraphs and refer to this variant of the problem as CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH. We show that the problem is fixed parameter tractable when parameterized by t by proving the following theorem.

▶ **Theorem 1.** CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH *can be solved in time* $2^{2^{2^{\mathcal{O}(t \log t)}}} \cdot n^{\mathcal{O}(1)}$.

In this extended abstract, we only sketch the proofs and omit some of them due to space constraints.

2 Preliminaries

We consider only finite undirected graphs. We use n to denote the number of vertices and m the number of edges of the considered graphs unless it creates confusion. A graph G is identified by its vertex set $V(G)$ and edge set $E(G)$. For $U \subseteq V(G)$, we write $G[U]$ to denote the subgraph of G induced by U. We write $G - U$ to denote the graph $G[V(G) \setminus U]$; for a single-element $U = \{u\}$, we write $G - u$. A set of vertices U is *connected* if $G[U]$ is a connected graph. For a vertex v, we denote by $N_G(v)$ the *(open) neighborhood* of v in G, i.e., the set of vertices that are adjacent to v in G. For a set $U \subseteq V(G)$, $N_G(U) = (\cup_{v \in U} N_G(v)) \setminus U$. We denote by $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v; respectively, $N_G[U] = \cup_{v \in U} N_G[v])$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. A set of vertices $S \subset V(G)$ of a connected graph G is a *separator* if $G - S$ is disconnected. A vertex v is a *cut vertex* if $\{v\}$ is a separator.

A graph property is *hereditary* if it is preserved under vertex deletion, or equivalently, under taking induced subgraphs. A graph property is *trivial* if either the set of graphs satisfying it, or the set of graphs that do not satisfy it, is finite. Let F be a graph. We say that a graph G is *F-free* if G has no induced subgraph isomorphic to F. For a set of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. Let Π be the property of being $\mathcal{F}$-free. Then, depending on whether $\mathcal{F}$ is a finite or an infinite set, we say that Π is *characterized by a finite / infinite set of forbidden induced subgraphs.*

We use the *recursive understanding* technique introduced by Chitnis et al. [3] for graph problems to solve Connected Secluded Π-Subgraph when Π is defined by forbidden induced subgraphs or Π is the property to be a forest. This powerful technique is based on the following idea. Suppose that the input graph has a vertex separator of bounded size that separates the graph into two sufficiently big parts. Then we solve the problem recursively for one of the parts and replace this part by an equivalent graph such that the replacement keeps all essential (partial) solutions of the original part. By such a replacement we obtain a graph of smaller size. Otherwise, if there is no separator of bounded size separating graphs into two big parts, then either the graph has bounded size or it is highly connected, and we exploit these properties. We need the following notions and results from Chitnis et al. [3].

Let G be a graph. A pair (A, B), where $A, B \subseteq V(G)$ and $A \cup B = V(G)$, is a *separation of G of order $|A \cap B|$* if G has no edge uv with $u \in A \setminus B$ and $v \in B \setminus A$, i.e., $A \cap B$ is an (A, B)-separator. Let q and k be nonnegative integers. A graph G is *(q,k)-unbreakable* if for every separation (A, B) of G of order at most k, $|A \setminus B| \leq q$ or $|B \setminus A| \leq q$. Combining Lemmas 19, 20 and 21 of [3], we obtain the following.

▶ **Lemma 2** ([3]). *Let q and k be nonnegative integers. There is an algorithm with running time $2^{\mathcal{O}(\min\{q,k\} \log(q+k))} \cdot n^3 \log n$ that, for a graph G, either finds a separation (A, B) of order at most k such that $|A \setminus B| > q$ and $|B \setminus A| > q$, or correctly reports that G is $((2q + 1)q \cdot 2^k, k)$-unbreakable.*

We conclude this section by noting that the following variant of Connected Secluded Π-Subgraph is FPT when parameterized by $k + t$. We will rely on this result in the subsequent sections, however we believe that it is also of interest on its own.

Connected Secluded Colored Π-Subgraph of Exact Size

Input: A graph G, coloring $c\colon V(G) \to \mathbb{N}$, a weight function $\omega\colon V(G) \to \mathbb{Z}_{\geq 0}$ and nonnegative integers k, t and w.

Task: Decide whether G contains a connected t-secluded induced subgraph H such that (H, c'), where $c'(v) = c|_{V(H)}(v)$, satisfies Π, $|V(H)| = k$ and $\omega(V(H)) \geq w$.

We say that a mapping $c\colon V(G) \to \mathbb{N}$ is a *coloring* of G; note that we do not demand a coloring to be proper. Analogously, we say that Π is a *property of colored graphs* if Π is a property on pairs (G, c), where G is a graph and c is a coloring. Notice that if some vertices of the input graph have labels, then we can assign to each label (or a combination of labels if a vertex can have several labels) a specific color and assign some color to unlabeled vertices. Then we can redefine a considered graph property with the conditions imposed by labels as a property of colored graphs. Observe that we allow zero weights. Our next theorem presents two possible running times for the mentioned cases. The latter running times will be useful when $k \gg t$.

▶ **Theorem 3.** *If property Π can be recognized in time $f(n)$, then* Connected Secluded Colored Π-Subgraph of Exact Size *can be solved both in time $2^{k+t} \cdot f(k) \cdot n^{\mathcal{O}(1)}$, and in time $2^{\mathcal{O}(\min\{k,t\} \log(k+t))} \cdot f(k) \cdot n^{\mathcal{O}(1)}$.*

In particular, the theorem implies that if Π can be recognized in polynomial time, then Connected Secluded Colored Π-Subgraph of Exact Size can be solved both in time $2^{k+t} \cdot n^{\mathcal{O}(1)}$, and in time $2^{\mathcal{O}(\min\{k,t\} \log(k+t))} \cdot n^{\mathcal{O}(1)}$.

3 Solving Connected Secluded $\mathcal{F}$-Free Subgraph

In this section we prove Theorem 1. Throughout this section, we assume that we are given a fixed finite set $\mathcal{F}$ of graphs.

Recall that to apply the recursive understanding technique introduced by Chitnis et al. [3], we should be able to recurse when the input graph contains a separator of bounded size that separates the graph into two sufficiently big parts. To do this, we have to combine partial solutions in both parts. A danger in our case is that a partial solution in one part might contain a subgraph of a graph in $\mathcal{F}$. We have to avoid creating subgraphs belonging to $\mathcal{F}$ when we combine partial solutions. To achieve this goal, we need some definitions and auxiliary combinatorial results.

Let p be a nonnegative integer. A pair (G, x), where G is a graph and $x = (x_1, \ldots, x_p)$ is a p-tuple of distinct vertices of G, is called a *p-boundaried graph* or simply a *boundaried graph*. Respectively, $x = (x_1, \ldots, x_p)$ is a *boundary*. Note that a boundary is an ordered set. Hence, two p-boundaried graphs that differ only by the order of the vertices in theirs boundaries are distinct. Observe also that a boundary could be empty. We say that (G, x) is a *properly p-boundaried graph* if each component of G has at least one vertex of the boundary. Slightly abusing notation, we may say that G is a (p-) boundaried graph assuming that a boundary is given.

Two p-boundaried graphs $(G_1, x^{(1)})$ and $(G_2, x^{(2)})$, where $x^{(h)} = (x_1^{(h)}, \ldots, x_p^{(h)})$ for $h = 1, 2$, are *isomorphic* if there is an isomorphism of G_1 to G_2 that maps each $x_i^{(1)}$ to $x_i^{(2)}$ for $i \in \{1, \ldots, p\}$. We say that $(G_1, x^{(1)})$ and $(G_2, x^{(2)})$ are *boundary-compatible* if for any distinct $i, j \in \{1, \ldots, p\}$, $x_i^{(1)} x_j^{(1)} \in E(G_1)$ if and only if $x_i^{(2)} x_j^{(2)} \in E(G_2)$.

Let $(G_1, x^{(1)})$ and $(G_2, x^{(2)})$ be boundary-compatible p-boundaried graphs and let $x^{(h)} = (x_1^{(h)}, \ldots, x_p^{(h)})$ for $h = 1, 2$. We define the *boundary sum* $(G_1, x^{(1)}) \oplus_b (G_2, x^{(2)})$ (or simply $G_1 \oplus_b G_2$) as the (non-boundaried) graph obtained by taking vertex disjoint copies of G_1 and G_2 and identifying $x_i^{(1)}$ and $x_i^{(2)}$ for each $i \in \{1, \ldots, p\}$.

Let G be a graph and let $y = (y_1, \ldots, y_p)$ be a p-tuple of vertices of G. For an s-boundaried graph (H, x) with the boundary $x = (x_1, \ldots, x_s)$ and pairwise distinct $i_1, \ldots, i_s \in \{1, \ldots, p\}$, we say that H is an *induced boundaried subgraph of G with respect to* $(y_{i_1}, \ldots, y_{i_s})$ if G contains an induced subgraph H' isomorphic to H such that the corresponding isomorphism of H to H' maps x_j to y_{i_j} for $j \in \{1, \ldots, s\}$ and $V(H') \cap \{y_1, \ldots, y_p\} = \{y_{i_1}, \ldots, y_{i_s}\}$.

We construct the set of boundaried graphs $\mathcal{F}_b$ as follows. For each $F \in \mathcal{F}$, each separation (A, B) of F and each $p = |A \cap B|$-tuple x of the vertices of $(A \cap B)$, we include $(F[A], x)$ in $\mathcal{F}_b$ unless it already contains an isomorphic boundaried graph. We say that two properly p-boundaried graphs $(G_1, x^{(1)})$ and $(G_2, x^{(2)})$, where $x^{(h)} = (x_1^{(h)}, \dots, x_p^{(h)})$ for $h = 1, 2$, are *equivalent (with respect to $\mathcal{F}_b$)* if

(i) $(G_1, x^{(1)})$ and $(G_2, x^{(2)})$ are boundary-compatible,

(ii) for any $i, j \in \{1, \dots, p\}$, $x_i^{(1)}$ and $x_j^{(1)}$ are in the same component of G_1 if and only if $x_i^{(2)}$ and $x_j^{(2)}$ are in the same component of G_2,

(iii) for any pairwise distinct $i_1, \dots, i_s \in \{1, \dots, p\}$, G_1 contains an s-boundaried induced subgraph $H \in \mathcal{F}_b$ with respect to the s-tuple $(x_{i_1}^{(1)}, \dots, x_{i_s}^{(1)})$ if and only if H is an induced subgraph of G_2 with respect to the s-tuple $(x_{i_1}^{(2)}, \dots, x_{i_s}^{(2)})$.

It is straightforward to verify that the introduced relation is indeed an equivalence relation on the set of properly p-boundaried graphs. The following property of the equivalence with respect to $\mathcal{F}_b$ is crucial for our algorithm.

▶ **Lemma 4.** *Let (G, x), $(H_1, y^{(1)})$ and $(H_2, y^{(2)})$ be boundary-compatible p-boundaried graphs, $x = (x_1, \dots, x_p)$ and $y^{(h)} = (y_1^{(h)}, \dots, y_p^{(h)})$ for $h = 1, 2$. If $(H_1, y^{(1)})$ and $(H_2, y^{(2)})$ are equivalent with respect to $\mathcal{F}_b$, then $(G, x) \oplus_b (H_1, y^{(1)})$ is $\mathcal{F}$-free if and only if $(G, x) \oplus_b (H_2, y^{(2)})$ is $\mathcal{F}$-free.*

It also should be noted that the equivalence of two properly p-boundaried graphs can be checked in polynomial time.

For each nonnegative integer p, we consider a set $\mathcal{G}_p$ of properly p-boundaried graphs obtained by picking a graph with minimum number of vertices in each equivalence class. We show that the size of $\mathcal{G}_p$ and the size of each graph in the set is upper bounded by some functions of p, and this set can be constructed in time that depends only on p assuming that $\mathcal{F}_b$ is fixed.

▶ **Lemma 5.** *For every positive integer p, $|\mathcal{G}_p| = 2^{\mathcal{O}(p^2)}$, and for every $H \in \mathcal{G}'_p$, $|V(H)| = p^{\mathcal{O}(1)}$, where the constants hidden in the $\mathcal{O}$-notations depend on $\mathcal{F}$ only. Moreover, for every p-boundaried graph G, the number of p-boundaried graphs in $\mathcal{G}_p$ that are compatible with G is $2^{\mathcal{O}(p \log p)}$.*

Consider now the class $\mathcal{C}$ of p-boundaried graphs, such that a p-boundaried graph $(G, (x_1, \dots, x_p\}) \in \mathcal{C}$ if and only if it holds that for every component H of $G - \{x_1, \dots, x_p\}$, $N_G(V(H)) = \{x_1, \dots, x_p\}$. We consider our equivalence relation with respect to $\mathcal{F}_b$ on $\mathcal{C}$ and define $\mathcal{G}'_p$ as follows. In each equivalence class, we select a graph $(G, (x_1, \dots, x_p\}) \in \mathcal{C}$ such that both the number of components of $G - \{x_1, \dots, x_p\}$ is minimum and the number of vertices of G is minimum subject to the first condition, and then include it in $\mathcal{G}'_p$. Similarly to Lemma 5 we show the following.

▶ **Lemma 6.** *For every positive integer p, $|\mathcal{G}'_p| = 2^{\mathcal{O}(p^2)}$, and for each $H \in \mathcal{G}_p$, $|V(H)| = p^{\mathcal{O}(1)}$, and the constants hidden in the $\mathcal{O}$-notations depend on $\mathcal{F}$ only. Moreover, for any p-boundaried graph G, the number of p-boundaried graphs in $\mathcal{G}'_p$ that are compatible with G is $p^{\mathcal{O}(1)}$.*

Lemmas 5 and 6 immediately imply that $\mathcal{G}_p$ and $\mathcal{G}'_p$ can be constructed by brute force.

▶ **Lemma 7.** *The sets $\mathcal{G}_p$ and $\mathcal{G}'_p$ can be constructed in time $2^{p^{\mathcal{O}(1)}}$.*

To apply the recursive understanding technique, we also have to solve a special variant of Connected Secluded Π-Subgraph tailored for recursion. First, we define the following auxiliary problem for a positive integer w.

MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH

Input: A graph G, sets $I, O, B \subseteq V(G)$ such that $I \cap O = \emptyset$ and $I \cap B = \emptyset$, a weight function $\omega\colon V(G) \to \mathbb{Z}_{\geq 0}$ and a nonnegative integer t.

Task: Find a t-secluded $\mathcal{F}$-free induced connected subgraph H of G of maximum weight or weight at least w such that $I \subseteq V(H)$, $O \subseteq V(G) \setminus V(H)$ and $N_G(V(H)) \subseteq B$ and output $\emptyset$ if such a subgraph does not exist.

Notice that MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH is an optimization problem and a *solution* is either an induced subgraph H of maximum weight or of weight at least w, or $\emptyset$. Observe also that we allow zero weights for technical reasons.

We recurse if we can separate graphs by a separator of bounded size into two big parts and we use the vertices of the separator to combine partial solutions in both parts. This leads us to the following problem. Let (G, I, O, B, ω, t) be an instance of MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH and let $T \subseteq V(G)$ be a set of *border terminals*. We say that an instance $(G', I', O', B', \omega', t')$ is obtained by a *border complementation* if there is a partition (X, Y, Z) of T (some sets could be empty), where $X = \{x_1, \ldots, x_p\}$, such that $Y = \emptyset$ if $X = \emptyset$, $I \cap T \subseteq X$, $O \cap T \subseteq Y \cup Z$ and $Y \subseteq B$, and there is a p-boundaried graph $(H, y) \in \mathcal{G}_p$ such that (H, y) and $(G, (x_1, \ldots, x_p))$ are boundary-compatible, and the following holds:

(i) G' is obtained from $(G, (x_1, \ldots, x_p)) \oplus_b (H, y)$ (we keep the notation $X = \{x_1, \ldots, x_p\}$ for the set of vertices obtained by the identification in the boundary sum) by adding edges joining every vertex of $V(H)$ with every vertex of Y,

(ii) $I' = I \cup V(H)$,

(iii) $O' = O \cup Y \cup Z$,

(iv) $B' = B \setminus X$,

(v) $\omega'(v) = \omega(v)$ for $v \in V(G)$ and $\omega'(v) = 0$ for $v \in V(H) \setminus X$,

(vi) $t' \leq t$.

We also say that (G', I', O', B', w', t') is a *border complementation of* (G, I, O, B, w, t) *with respect to* $(X = \{x_1, \ldots, x_p\}, Y, Z, H)$. We say that $(X = \{x_1, \ldots, x_p\}, Y, Z, H)$ is *feasible* if it holds that $Y = \emptyset$ if $X = \emptyset$, $I \cap T \subseteq X$, $O \cap T \subseteq Y \cup Z$ and $Y \subseteq B$, and the p-boundaried graph $H \in \mathcal{G}_p$ and $(G, (x_1, \ldots, x_p))$ are boundary-compatible.

BORDERED MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH

Input: A graph G, sets $I, O, B \subseteq V(G)$ such that $I \cap O = \emptyset$ and $I \cap B = \emptyset$, a weight function $\omega\colon V(G) \to \mathbb{Z}_{\geq 0}$, a nonnegative integer t, and a set $T \subseteq V(G)$ of border terminals of size at most $2t$.

Task: Output a solution for each instance (G', I', O', B', w', t') of MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH that can be obtained from (G, I, O, B, w, t) by a border complementation distinct from the border complementation with respect to $(\emptyset, \emptyset, T, \emptyset)$, and for the border complementation with respect to $(\emptyset, \emptyset, T, \emptyset)$ output a nonempty solution if it has weight at least w and output $\emptyset$ otherwise.

Two instances $(G_1, I_1, O_1, B_1, \omega_1, t, T)$ and $(G_2, I_2, O_2, B_2, \omega_2, t, T)$ of BORDERED MAXIMUM OR w-WEIGHTED CONNECTED SECLUDED $\mathcal{F}$-FREE SUBGRAPH (note that t and T are the same) are said to be *equivalent* if

(i) $T \cap I_1 = T \cap I_2$, $T \cap O_1 = T \cap O_2$ and $T \cap B_1 = T \cap B_2$,

(ii) for the border complementations $(G_1', I_1', O_1', B_1', \omega_1', t')$ and $(G_2', I_2', O_2', B_2', \omega_2', t')$ of the instances $(G_1, I_1, O_1, B_1, \omega_1, t')$ and $(G_2, I_2, O_2, B_2, \omega_2, t')$ respectively of MAXIMUM OR

w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph with respect to every feasible $(X = \{x_1, \ldots, x_p\}, Y, Z, H)$ and $t' \leq t$, it holds that

(a) if $(G'_1, I'_1, O'_1, B'_1, \omega'_1, t')$ has a nonempty solution R_1, then $(G'_2, I'_2, O'_2, B'_2, \omega'_2, t')$ has a nonempty solution R_2 with $w'_2(V(R_2)) \geq \min\{\omega'_1(V(R_1)), w\}$ and, vice versa,

(b) if $(G'_2, I'_2, O'_2, B'_2, \omega'_2, t')$ has a nonempty solution R_2, then $(G'_1, I'_1, O'_1, B'_1, \omega'_1, t')$ has a nonempty solution R_1 with $\omega'_1(V(R_1)) \geq \min\{\omega'_2(V(R_2)), w\}$.

Strictly speaking, if $(G_1, I_1, O_1, B_1, \omega_1, t, T)$ and $(G_2, I_2, O_2, B_2, \omega_2, t, T)$ are equivalent, then a solution of the first problem is not necessarily a solution of the second. Nevertheless, Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph is an auxiliary problem and in the end we use it to solve an instance (G, ω, t, w) of Connected Secluded $\mathcal{F}$-Free Subgraph by calling the algorithm for Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph for $(G, \emptyset, \emptyset, V(G), \omega, t, \emptyset)$. Clearly, (G, ω, t, w) is a yes-instance if and only if a solution for the corresponding instance of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph contains a connected subgraph R with $\omega(V(R)) \geq w$. It allows us to not distinguish equivalent instances of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph and their solutions.

3.1 High connectivity phase

In this section we solve Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph for (q, t)-unbreakable graphs. The following lemma shows that we can separately lists all graphs R in a solution of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph with $|V(R) \cap V(G)| \leq q$ and all graphs R with $|V(G) \setminus V(R)| \leq q + t$.

▶ **Lemma 8.** *Let $(G, I, O, B, \omega, t, T)$ be an instance of* Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph *where G is a (q, t)-unbreakable graph for a positive integer q. Then for each nonempty graph R in a solution of* Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph*, either $|V(R) \cap V(G)| \leq q$ or $|V(G) \setminus V(R)| \leq q + t$.*

To list R with $|V(R) \cap V(G)| \leq q$, we use Theorem 3. To list R with $|V(G) \setminus V(R)| \leq q+t$, we use *important separators* defined by Marx in [12]. The main observation in this second case is that if $|V(G) \setminus V(R)| \leq q + t$, then there is a hitting set S of size at most $q + t$ for all copies of graphs of $\mathcal{F}$ that lies outside R in the corresponding graph. Moreover, hitting sets of size at most $q + t$ can be enumerated in FPT time. Then we can use important separators between the closed neighborhood of I and $S \cup O$ to find R. It gives us the following crucial lemma.

▶ **Lemma 9.** Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph *for (q, t)-unbreakable graphs can be solved in time $2^{(q+t\log(q+t)))} \cdot n^{\mathcal{O}(1)}$ if the sets $\mathcal{G}_p$ for $p \leq 2t$ are given.*

3.2 The FPT algorithm for Connected Secluded $\mathcal{F}$-Free Subgraph

In this section we construct an FPT algorithm for Connected Secluded $\mathcal{F}$-Free Subgraph parameterized by t. We do this by solving Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph in FPT-time for general case.

▶ **Lemma 10.** Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph *can be solved in time $2^{2^{2^{\mathcal{O}(t\log t)}}} \cdot n^{\mathcal{O}(1)}$.*

Sketch of the Proof. Given $\mathcal{F}$, we construct the set $\mathcal{F}_b$. Then we use Lemma 7 to construct the sets $\mathcal{G}_p$ for $p \in \{0,\ldots,t\}$ in time $2^{t^{\mathcal{O}(1)}}$.

By Lemma 5, there is a constant c that depends only on $\mathcal{F}$ such that for every nonnegative p and for any p-boundaried graph G, there are at most $2^{cp\log p}$ p-boundaried graphs in $\mathcal{G}_p$ that are compatible with G and there are at most p^c p-boundaried graphs in $\mathcal{G}'_p$ that are compatible with G. We define

$$q = 2^{((t+1)t3^{2t}2^{c2t\log(2t)}+2t)} \cdot 2((t+1)t3^{2t}2^{c2t\log(2t)}+2t)^c t^c + (t+1)t3^{2t}2^{c2t\log(2t)} + 2t. \quad (1)$$

The choice of q will become clear later in the proof. Notice that $q = 2^{2^{\mathcal{O}(t\log t)}}$.

Consider an instance $(G, I, O, B, \omega, t, T)$ of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph.

We use the algorithm from Lemma 2 for G. This algorithm in time $2^{2^{\mathcal{O}(t\log t)}} \cdot n^{\mathcal{O}(1)}$ either finds a separation (U, W) of G of order at most t such that $|U \setminus W| > q$ and $|W \setminus U| > q$ or correctly reports that G is $((2q+1)q \cdot 2^t, t)$-unbreakable. In the latter case we solve the problem using Lemma 9 in time $2^{2^{2^{\mathcal{O}(t\log t)}}} \cdot n^{\mathcal{O}(1)}$. Assume from now that there is a separation (U, W) of order at most t such that $|U \setminus W| > q$ and $|W \setminus U| > q$.

Recall that $|T| \le 2t$. Then $|T \cap (U \setminus W)| \le t$ or $|T \cap (W \setminus U)| \le t$. Assume without loss of generality that $|T \cap (W \setminus U)| \le t$. Let $\tilde{G} = G[W]$, $\tilde{I} = I \cap W$, $\tilde{O} = O \cap W$, $\tilde{\omega}$ is the restriction of ω to W, and define $\tilde{T} = (T \cap W) \cup (U \cap W)$. Since $|U \cap W| \le t$, $|\tilde{T}| \le 2t$.

If $|W| \le (2q+1)q \cdot 2^t$, then we solve Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph for the instance $(\tilde{G}, \tilde{I}, \tilde{O}, \tilde{B}, \tilde{\omega}, t, \tilde{T})$ by brute force in time $2^{2^{2^{\mathcal{O}(t\log t)}}}$ trying all possible subset of W and at most $t+1$ values of $0 \le t' \le t$. Otherwise, we solve $(\tilde{G}, \tilde{I}, \tilde{O}, \tilde{B}, \tilde{\omega}, t, \tilde{T})$ recursively. Let $\mathcal{R}$ be the set of nonempty induced subgraphs R that are included in the obtained solution for $(\tilde{G}, \tilde{I}, \tilde{O}, \tilde{B}, \tilde{\omega}, t, \tilde{T})$.

For $R \in \mathcal{R}$, define S_R to be the set of vertices of $W \setminus V(R)$ that are adjacent to the vertices of R in the graph obtained by the border complementation for which R is a solution of the corresponding instance of Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph. Note that $|S_R| \le t$. If $\mathcal{R} \neq \emptyset$, then let $S = \tilde{T} \cup_{R\in\mathcal{R}} S_R$, and $S = \tilde{T}$ if $\mathcal{R} = \emptyset$. Since Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph is solved for at most $t+1$ of values of $t' \le t$, at most 3^{2t} three-partitions (X, Y, Z) of $\tilde{T}$ and at most $2^{c2t\log(2t)}$ choices of a p-boundaried graph $H \in \mathcal{F}_b$ for $p = |X|$, we have that $|\mathcal{R}| \le (t+1)3^{2t}2^{c2t\log(2t)}$. Taking into account that $|T'| \le 2t$,

$$|S| \le (t+1)t3^{2t}2^{c2t\log(2t)} + 2t. \quad (2)$$

Let $\hat{B} = (B \cap U) \cup (B \cap S)$. We claim that the instances $(G, I, O, B, \omega, t, T)$ and $(G, I, O, \hat{B}, \omega, t, T)$ of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph are equivalent.

Since, $(G, I, O, B, \omega, t, T)$ and $(G, I, O, \hat{B}, \omega, t, T)$ of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph are equivalent, we can consider $(G, I, O, \hat{B}, \omega, t, T)$. Now we apply some reduction rules that produce equivalent instances of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph or report that we have no solution. The ultimate aim of these rules is to reduce the size of G.

Let Q be a component of $G[W] - S$. Notice that for every nonempty graph R in a solution of $(G, I, O, \hat{B}, w, t, T)$, either $V(Q) \subseteq V(R)$ or $V(Q) \cap V(R) = \emptyset$, because $N_{G[W]}(V(R)) \subseteq S$. Moreover, if $V(Q) \cap V(R) = \emptyset$, then $N_{G[W]}[V(Q)] \cap V(R) = \emptyset$. Notice also that if $v \in N_{G[W]}(V(Q))$ is a vertex of R, then $V(Q) \subseteq V(R)$. These observation are crucial for the following reduction rules.

▶ **Reduction Rule 3.1.** For a component Q of $G[W] - S$ do the following in the given order:

- if $N_{G[W]}[V(Q)] \cap I \neq \emptyset$ and $V(Q) \cap O \neq \emptyset$, then return $\emptyset$ and stop,
- if $N_{G[W]}[V(Q)] \cap I \neq \emptyset$, then set $I = I \cup V(Q)$,
- if $V(Q) \cap O \neq \emptyset$, then set $O = O \cup N_{G[W]}[V(Q)]$.

The rule is applied to each component Q exactly once. Notice that after application of the rule, for every component Q of $G[W] - S$, we have that either $V(Q) \subseteq I$ or $V(Q) \subseteq O$ or $V(Q) \cap (I \cup O \cup \hat{B}) = \emptyset$.

Suppose that Q_1 and Q_2 are components of $G[W] - S$ such that $N_{G[W]}(V(Q_1)) = N_{G[W]}(V(Q_2))$ and $|N_{G[W]}(V(Q_1))| = |N_{G[W]}(V(Q_2))| > t$. Then if $V(Q_1) \subseteq V(R)$ for a nonempty graph R in a solution of $(G, I, O, \hat{B}, \omega, t, T)$, then at least one vertex of $N_{G[W]}(V(Q_1))$ is in R as R have at most t neighbors outside R. This gives the next rule.

▶ **Reduction Rule 3.2.** For components Q_1 and Q_2 of $G[W] - S$ such that $N_{G[W]}(V(Q_1)) = N_{G[W]}(V(Q_2))$ and $|N_{G[W]}(V(Q_1))| = |N_{G[W]}(V(Q_2))| > t$ do the following in the given order:

- if $(V(Q_1) \cup V(Q_2)) \cap I \neq \emptyset$ and $(V(Q_1) \cup V(Q_2)) \cap O \neq \emptyset$, then return $\emptyset$ and stop,
- if $(V(Q_1) \cup V(Q_2)) \cap I \neq \emptyset$, then set $I = I \cup (V(Q_1) \cup V(Q_2))$,
- if $(V(Q_1) \cup V(Q_2)) \cap O \neq \emptyset$, then set $O = O \cup N_{G[W]}[V(Q_1) \cup V(Q_2)]$.

We apply the rule for all pairs of components Q_1 and Q_2 with $N_{G[W]}(V(Q_1)) = N_{G[W]}(V(Q_2))$ and $|N_{G[W]}(V(Q_1))| = |N_{G[W]}(V(Q_2))| > t$, and for each pair the rule is applied once.

If $V(Q) \subseteq O$ for a component Q of $G[W] - S$, then $N_{G[W]}(V(Q)) \subseteq O$. It immediately implies that the vertices of Q are irrelevant and can be removed.

▶ **Reduction Rule 3.3.** If there is a component Q of $G[W] - S$ such that $N_{G[W]}(V(Q)) \subseteq O$, then set $G = G - V(Q)$, $W = W \setminus V(Q)$ and $O = O \setminus V(Q)$.

Notice that for each component Q, we have that either $V(Q) \subseteq I$ or $V(Q) \subseteq W \setminus (I \cup O \cup \hat{B})$.

To define the remaining rules, we construct the sets $\mathcal{G}'_p$ for $p \in \{0, \ldots, |S|\}$ in time $2^{2^{\mathcal{O}(t \log t)}}$ using Lemma 7.

Let Q be a component of $G[W] - S$ and let $N_{G[W]}(V(Q)) = \{x_1, \ldots, x_p\}$. Let G' be the graph obtained from G by the deletion of the vertices of $V(Q)$ and let $x = (x_1, \ldots, x_p)$. Let (H, y) be a connected p-boundaried graph of the same weight as $G[N_{G[W]}[V(Q)]]$. Then by Lemma 4, we have that the instance of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph obtained from $(G, I, O, \hat{B}, \omega, t, T)$ by the replacement of G by $(G', x) \oplus_b (H, y)$ is equivalent to $(G, I, O, \hat{B}, \omega, t, T)$. We use it in the remaining reduction rules.

Suppose again that Q_1 and Q_2 are components of $G[W] - S$ such that $N_{G[W]}(V(Q_1)) = N_{G[W]}(V(Q_2))$ and $|N_{G[W]}(V(Q_1))| = |N_{G[W]}(V(Q_2))| > t$. Then, as we already noticed, if $V(Q_1) \cup V(Q_2) \subseteq V(R)$ for a nonempty graph R in a solution of $(G, I, O, \hat{B}, \omega, t, T)$, then at least one vertex of $N_{G[W]}(V(Q_1))$ is in R. It means that if we are constructing a solution R, then the restriction of the size of the neighborhood of R ensures the connectivity between Q_1 and Q_2 if we decide to include these components in R. Together with Lemma 4 this shows that the following rule is safe.

▶ **Reduction Rule 3.4.** Let $L = \{x_1, \ldots, x_p\} \subseteq S$, $p > t$, and let $x = (x_1, \ldots, x_p)$. Let also $Q_1, \ldots, Q_r, r \geq 1$, be the components of $G[W] - S$ with $N_{G[W]}(V(Q_i)) = L$ for all $i \in \{1, \ldots, r\}$. Let $Q = G[\cup_{i=1}^{r} N_{G[W]}[V(Q_i)]]$ and $w' = \sum_{i=1}^{r} \omega(V(Q_i))$. Find a p-boundaried graph $(H,y) \in \mathcal{G}'_p$ that is equivalent to (Q,x) with respect to $\mathcal{F}_b$ and denote by A the set of nonboundary vertices of H. Then do the following.

- Delete the vertices of $V(Q_1),\dots,V(Q_r)$ from G and denote the obtained graph G'.
- Set $G = (G',x) \oplus_b (H,y)$ and $W = (W \setminus \cup_{i=1}^r V(Q_i)) \cup A$.
- Select arbitrarily $u \in A$ and modify ω as follows:
  - keep the weight same for every $v \in V(G')$ including the boundary vertices $x_1,\dots,x_p$,
  - set $\omega(v) = 0$ for $v \in A \setminus \{u\}$,
  - set $\omega(u) = w'$.
- If $V(Q_1) \subseteq I$, then set $I = I \setminus (\cup_{i=1}^r V(Q_i)) \cup A$.

The rule is applied exactly once for each inclusion maximal sets of components $\{Q_1,\dots,Q_r\}$ having the same neighborhood of size at least $t+1$.

We cannot apply this trick if we have several components $Q_1,\dots,Q_r$ of $G[W]-S$ with the same neighborhood $N_{G[W]}(V(Q_i))$ if $|N_{G[W]}(V(Q_i))| \le t$. Now it can happen that there are $i,j \in \{1,\dots,r\}$ such that $V(Q_i) \subseteq V(R)$ and $N_{G[W]}[V(Q_j)] \cap V(R) = \emptyset$ for R in a solution of $(G,I,O,\hat{B},\omega,t,T)$. But if $N_{G[W]}[V(Q_j)] \cap V(R) = \emptyset$, then by the connectivity of R and the fact that $G[W]-S$ does not contain border terminals, we have that $R = Q_i$. Notice that $I = \emptyset$ in this case and, in particular, it means that R is a solution for an instance of Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph obtained by the border complementation with respect to $(\emptyset,\emptyset,T,\emptyset)$. Recall that we output R in this case only if its weight is at least w. Still, we can modify Reduction Rule 3.4 for the case when there are components Q of $G[W]-S$ such that $V(Q) \subseteq I$. Notice that if there are components $Q_0,\dots,Q_r$ of $G[W]-S$ with the same neighborhood and $V(Q_0) \subseteq I$, then for any nonempty R in a solution of $(G,I,O,\hat{B},\omega,t,T)$, either $R = Q_0$ or $\cup_{i=0}^r V(Q_i) \subseteq V(R)$. Applying Lemma 4, we obtain that the following rule is safe.

▶ **Reduction Rule 3.5.** Let $L = \{x_1,\dots,x_p\} \subseteq S$, $p \le t$, and let $x = (x_1,\dots,x_p)$. Let also $Q_0,\dots,Q_r$,$r \ge 0$, be the components of $G[W]-S$ with $N_{G[W]}(V(Q_i)) = L$ for all $i \in \{0,\dots,r\}$ such that $V(Q_0) \subseteq I$. Let $Q = G[\cup_{i=1}^r N_{G[W]}[V(Q_i)]]$ and $w' = \sum_{i=1}^r \omega(V(Q_i))$. Find a p-boundaried graph $(H_0,y) \in \mathcal{G}'_p$ that is equivalent to (Q_0,x) with respect to $\mathcal{F}_b$ and denote by A_0 the set of nonboundary vertices of H_0, and find a p-boundaried graph $(H,y) \in \mathcal{G}'_p$ that is equivalent to (Q,x) with respect to $\mathcal{F}_b$ and denote by A the set of nonboundary vertices of H. Then do the following.

- Delete the vertices of $V(Q_0),\dots,V(Q_r)$ from G and denote the obtained graph G'.
- Set $G = (((G',x) \oplus_b (H_0,y)),y) \oplus_b (H,y)$ and $W = (W \setminus \cup_{i=0}^r V(Q_i)) \cup A_0 \cup A$.
- Select arbitrarily $u \in A_0$ and $v \in A$ and modify ω as follows:
  - keep the weight same for every $z \in V(G')$ including the boundary vertices $x_1,\dots,x_p$,
  - set $\omega(z) = 0$ for $z \in (A_0 \setminus \{u\}) \cup (A \setminus \{v\})$,
  - set $\omega(u) = \omega(V(Q_0))$ and $\omega(v) = w'$.
- If $V(Q_i) \subseteq I$ for some $i \in \{1,\dots,r\}$, then set $I = I \setminus (\cup_{i=1}^r V(Q_i)) \cup A$.

Assume now that we have an inclusion maximal set of components $\{Q_1,\dots ,Q_r\}$ of $G[W]-S$ with the same neighborhoods $N_{G[W]} = \{x_1,\dots,x_p\}$ such that the p-boundaried graphs $(G[N_{G[W]}[V(Q_i)]],(x_1,\dots,x_p))$ and $(G[N_{G[W]}[V(Q_j)]],(x_1,\dots,x_p))$ are equivalent with respect to $\mathcal{F}_b$ for each $i,j \in \{1,\dots,p\}$. Suppose also that $V(Q_i) \cap I = \emptyset$ for $i \in \{1,\dots,r\}$. Let $\omega(V(Q_1)) \ge \omega(V(Q_i))$ for every $i \in \{1,\dots,r\}$. Recall that if R is a nonempty graph in a solution, then either $R = Q_i$ for some $i \in \{1,\dots,r\}$ or $\cup_{i=1}^r V(Q_i) \subseteq V(R)$. Recall also that R is a solution for the instance of Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph obtained by a border complementation with respect to $(\emptyset,\emptyset,T,\emptyset)$ and we output it only if $\omega(V(R)) \ge w$. Since all $(G[N_{G[W]}[V(Q_i)]],(x_1,\dots,x_p))$

are equivalent, we can assume that if $R = Q_i$, then $i = 1$, because Q_1 has maximum weight. Then by Lemma 4, our final reduction rule is safe.

▶ **Reduction Rule 3.6.** Let $L = \{x_1, \ldots, x_p\} \subseteq S$, $p \leq t$, and let $x = (x_1, \ldots, x_p)$. Let also $Q_0, \ldots, Q_r, r \geq 0$, be the components of $G[W] - S$ with $N_{G[W]}(V(Q_i)) = L$ for all $i \in \{0, \ldots, r\}$ such that $\omega(V(Q_0)) \geq \omega(V(Q_i))$ for every $i \in \{1, \ldots, r\}$ and the p-boundaried graphs $(G[N_{G[W]}[V(Q_i)]], (x_1, \ldots, x_p))$ are pairwise equivalent with respect to $\mathcal{F}_b$ for $i \in \{0, \ldots, r\}$. Let $Q = G[\cup_{i=1}^r N_{G[W]}[V(Q_i)]]$ and $w' = \min\{w - 1, \sum_{i=1}^r \omega(V(Q_i))\}$. Find a p-boundaried graph $(H_0, y) \in \mathcal{G}'_p$ that is equivalent to (Q_0, x) with respect to $\mathcal{F}_b$ and denote by A_0 the set of nonboundary vertices of H_0, and find a p-boundaried graph $(H, y) \in \mathcal{G}'_p$ that is equivalent to (Q, x) with respect to $\mathcal{F}_b$ and denote by A the set of nonboundary vertices of H. Then do the following.

- Delete the vertices of $V(Q_0), \ldots, V(Q_r)$ from G and denote the obtained graph G'.
- Set $G = (((G', x) \oplus_b (H_0, y)), y) \oplus_b (H, y)$ and $W = (W \setminus \cup_{i=0}^r V(Q_i)) \cup A_0 \cup A$.
- Select arbitrarily $u \in A_0$ and $v \in A$ and modify ω as follows:
  - keep the weight same for every $z \in V(G')$ including the boundary vertices $x_1, \ldots, x_p$,
  - set $\omega(z) = 0$ for $z \in (A_0 \setminus \{u\}) \cup (A \setminus \{v\})$,
  - set $\omega(u) = \omega(V(Q_0))$ and $\omega(v) = w'$.

The Reduction Rule 3.6 is applied for each inclusion maximal sets of components $\{Q_0, \ldots, Q_r\}$ satisfying the conditions of the rule such that Reduction Rule 3.5 was not applied to these components before.

Denote by $(G^*, I^*, O^*, B^*, \omega^*, t, T)$ the instance of Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph obtained from $(G, I, O, \hat{B}, \omega, t, T)$ by Reduction Rules 3.1-3.6. Notice that all modifications were made for $G[W]$. Denote by W^* the set of vertices of the graph obtained from the initial $G[W]$ by the rules. Observe that there are at most $2^{|S|}$ subsets L of S such that there is a component Q of $G[W] - S$ with $N_{G[W]}(V(Q)) = L$. If $|L| > t$, then all Q with $N_{G[W]}(V(Q)) = L$ are replaced by one graph by Reduction Rule 3.4 and the number of vertices of this graph is at most $|L|^c$ by Lemma 5 and the definition of c. If $|L| \leq t$, then we either apply Reduction Rule 3.5 for all Q with $N_{G[W]}(V(Q)) = L$ and replace these components by two graph with at most $|L|^c$ vertices or we apply Reduction Rule 3.6. For the latter case, observe that there are at most t^c partitions of the components Q with $N_{G[W]}(V(Q)) = L$ into equivalence classes with respect to $\mathcal{F}_b$ by Lemma 5. Then we replace each class by two graphs with at most $|L|^c$ vertices. Taking into account the vertices of S, we obtain the following upper bound for the size of W^*: $|W^*| \leq 2^{|S|} 2|S|^c t^c + |S|$. By (1) and (2), $|W^*| \leq q$. Recall that $|W \setminus U| > q$. Therefore, $|V(G^*)| < |V(G)|$. We use it and solve Bordered Maximum or w-Weighted Connected Secluded $\mathcal{F}$-Free Subgraph for $(G^*, I^*, O^*, B^*, \omega^*, t, T)$ recursively.

Following the general scheme from [3], we show that the total running time is $2^{2^{2^{\mathcal{O}(t \log t)}}} \cdot n^{\mathcal{O}(1)}$. ◀

It remains to observe that Lemma 10 immediately implies Theorem 1.

4 Concluding remarks

In addition to our general result from the previous section, we are also able to show that Connected Secluded Π-Subgraph is FPT parameterized by t, when Π is defined by an *infinite* set of forbidden induced subgraphs, namely, the set of all cycles. In other words, a graph has the property Π considered if it is a forest. Using the recursive understanding technique, we proved that the problem can be solved in time $2^{2^{\mathcal{O}(t \log t)}} \cdot n^{\mathcal{O}(1)}$. We believe that the same approach can be used for other graph properties Π as well. Nevertheless, the drawback of applying the recursive understanding technique is that we get double or even triple-exponential dependence on the parameter in our FPT algorithms. It is natural to ask whether we can do better for some properties Π. This can in fact be done when Π is the property of being a complete graph, a star, a path or a d-regular graph.

Finally, we conclude by briefly touching upon the kernelization question. For Connected Secluded Π-Subgraph, we hardly can hope to obtain polynomial kernels as it could be easily proved by applying the results of Bodlaender et al. [1] that, unless $NP \subseteq coNP/poly$, Connected Secluded Π-Subgraph has no polynomial kernel when parameterized by t if Connected Secluded Π-Subgraph is NP-complete. Nevertheless, Connected Secluded Π-Subgraph can have a polynomial *Turing kernel*. In particular, we are able to show that Connected Secluded Π-Subgraph has a polynomial Turing kernel if Π is the property of being a star.

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