Permutation Pattern Matching for Doubly Partially Ordered Patterns

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Abstract
We study in this paper the Doubly Partially Ordered Pattern Matching (or DPOP Matching) problem, a natural extension of the Permutation Pattern Matching problem. Permutation Pattern Matching takes as input two permutations \( \sigma \) and \( \pi \), and asks whether there exists an occurrence of \( \sigma \) in \( \pi \); whereas DPOP Matching takes two partial orders \( P_v \) and \( P_p \) defined on the same set \( X \) and a permutation \( \pi \), and asks whether there exist \( |X| \) elements in \( \pi \) whose values (resp., positions) are in accordance with \( P_v \) (resp., \( P_p \)). Posets \( P_v \) and \( P_p \) aim at relaxing the conditions formerly imposed by the permutation \( \sigma \), since \( \sigma \) yields a total order on both positions and values. Our problem being NP-hard in general (as Permutation Pattern Matching is), we consider restrictions on several parameters/properties of the input, e.g., bounding the size of the pattern, assuming symmetry of the posets (i.e., \( P_v \) and \( P_p \) are identical), assuming that one partial order is a total (resp., weak) order, bounding the length of the longest chain/anti-chain in the posets, or forbidding specific patterns in \( \pi \). For each such restriction, we provide results which together give an almost complete landscape for the algorithmic complexity of the problem.

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1 Preamble

Let us play the following little puzzle game. Among the selection of fifteen cities of the Czech Republic depicted in Figure 1 together with their geographic coordinates, find (if they exist) five cities, say \( A, B, C, D \) and \( E \), such that:
- \( A \) and \( C \) are west of \( D \) and north of \( B \),
- \( E \) is east of \( B \) and south of \( A \),
- \( D \) is west of \( B \) and north of \( A \) and \( C \).

It is assumed that no two cities have the same longitude or latitude. Notice that the game does not provide complete information as, for example, no information is provided about the relative positioning of \( A \) and \( C \) (and silence is tantamount to consent). We may assume that the information is minimal: requiring \( C \) is west of \( B \) is unnecessary since \( C \) is west of \( D \) and \( D \) is west of \( B \). One solution is \( A = \) Praha, \( B = \) Brno, \( C = \) Plzeň, \( D = \) Liberec and \( E = \) Olomouc. Note that the solution is not unique, as \( A = \) Plzeň, \( B = \) Jindřichův Hradec, \( C = \) Cheb, \( D = \) Ústí nad Labem and \( E = \) Brno is another solution.
We show that this puzzle game can be modeled as a permutation pattern matching problem for doubly partially ordered patterns. Let us first associate a permutation $\pi \in \mathcal{S}(15)$ with the problem (see Figure 2). We sort the fifteen cities of the Czech Republic depicted in Figure 1 both by increasing longitude (E) and by increasing latitude (N), so that $\pi(i) = j$ if the $i$-th city going west to east is also the $j$-th city going south to north. In our example, the “Czech Republic permutation” is $\pi = 11 \ 6 \ 14 \ 12 \ 1 \ 2 \ 15 \ 10 \ 13 \ 3 \ 5 \ 4 \ 9 \ 7 \ 8$. For example, $\pi(2) = 6$ since Plzeň is the second city going west to east, and the sixth city going south to north. What is left is to define our pattern $P$: $P$ is composed of two partially ordered sets on the variables $\{A, B, C, D, E\}$ (see Figure 3): one partially ordered set (denoted $P_v$ for value poset in the sequel) describes the south-to-north constraints and another partially ordered set (denoted $P_p$ for position poset in the sequel) describes the west-to-east constraints.

2 Introduction

We say that a permutation $\sigma$ occurs in another permutation $\pi$ (or that $\pi$ contains $\sigma$) if there exists a subsequence of elements of $\pi$ that has the same relative order as $\sigma$. Otherwise, we say that $\pi$ avoids $\sigma$. For example, $\pi$ contains the permutation $\sigma = 123$ (resp., $\sigma = 321$) if it has an increasing (resp., a decreasing) subsequence of size 3. Similarly, $\sigma = 4312$ occurs in $\pi = 6152347$, as shown in $\{6\}(15)23\{47\}$, but the same $\pi = 6152347$ avoids $\sigma' = 2341$.

Deciding whether a permutation $\sigma \in \mathcal{S}(k)$ occurs in some permutation $\pi \in \mathcal{S}(n)$ is NP-complete [7], but is fixed-parameter tractable for the parameter $k$ [15, 17]. Several exponential-time algorithms have been recently proposed [5, 16], improving upon [1, 10]. A vast literature is devoted to the case where both the pattern $\sigma$ and the target $\pi$ are restricted to a proper permutation class, e.g., 321-avoiding permutations [18, 2, 21], (213, 231)-avoiding permutations [26], (2413, 3412)-avoiding (a.k.a. separable) permutations [19, 25], and $(k \ldots 1)$-avoiding permutations [11]. For more background on permutation patterns and pattern avoidance, we refer to [6] and [24].
Increasing longitudes

Increasing latitudes

Cheb 1
Plzeň 2
Ústí nad Labem 3
Praha 4
ˇCeské Budějovice 5
Jindřichův Hradec 6
Liberec 7
Pardubice 8
Hradec Králové 9
Brno 10
Olomouc 11
Zlín 12
Ostrava 13
Karviná 14
ˇCeské Budějovice 15

Figure 2 Permutation $\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$ corresponding to the map from Figure 1. The solution of our puzzle, depicted Figure 1, is also represented.

Figure 3 A dpop $P = (P_v, P_p)$ representing the pattern for our puzzle, together with three distinct occurrences. Note that partial orders are represented by Hasse diagrams, i.e., a bottom-up path in $P_v$ (resp., $P_p$) implies a bottom-up (resp., left-right) relation in the occurrence of $P$ in $\pi$.

In the last years, the notion of pattern has been generalized in several ways. A vincular pattern is a permutation in which some entries must occur consecutively [4]. Consecutive patterns are a special case of vincular patterns in which all entries need to be adjacent [13]. Bivincular patterns generalize classical patterns even further than vincular patterns by requiring that not only positions but also values of elements involved in a matching may be forced to be adjacent [8]. Mesh patterns (a further generalization of bivincular patterns)
impose further restrictions on the relative positions of the entries in an occurrence of a pattern [9] and boxed mesh patterns are special cases of mesh patterns [3]. Strongly related to our approach are partially ordered patterns that are vincular patterns in which the relative order of some elements is not fixed [23]. The best general reference is [24].

In this paper, we consider a new generalization of classical patterns in which both the relative order and the relative positioning of some elements are not fixed. The idea is to allow the possibility for some elements to be incomparable in value (i.e., their relative order is unknown) and to go one step further by allowing the possibility for some elements to be incomparable in position (i.e., their relative positioning in the occurrence is unknown). Since the problem is clearly \( \text{NP}- \text{hard} \) (as it contains permutation pattern matching as a sub-problem), our goal is to identify tractable cases when restrictions apply to the pattern and/or to the permutation.

The restrictions we consider here apply to the following parameters of the problem: size of the pattern; symmetry (i.e., same partial order in positions and values); one partial order is a total (resp., weak) order; size of the longest chain (resp., anti-chain) in the partial orders (height and width); forbidden patterns in \( \pi \). On the positive side, we show that the \( \text{FPT} \) algorithm for permutation pattern matching can be generalized to our setting (with the pattern size as parameter). We further give polynomial-time algorithms when the pattern is a symmetric disjoint union of a constant number of weak orders. Finally, we also provide polynomial-time algorithms when the pattern is symmetric and the permutation belongs to some restricted classes, such as \( (123,132) \)-avoiding permutations. We complement these positive results with \( \text{NP} \)- or \( \text{W}[1] \)-hardness proofs in most of the remaining cases.

3 Definitions

Permutations and Patterns

A permutation \( \sigma \) is said to be contained in (or is a sub-permutation of) another permutation \( \pi \), which we denote by \( \sigma \preceq \pi \), if \( \pi \) has a (not necessarily contiguous) subsequence whose terms are order-isomorphic to \( \sigma \). We also say that \( \pi \) admits an occurrence of the pattern \( \sigma \). If no such subsequence exists, we say that \( \pi \) avoids \( \sigma \) (or is \( \sigma \)-avoiding). A permutation is separable if it avoids both 2413 and 3142. Permutation Pattern Matching is the problem of deciding whether a permutation is contained into another permutation.

For any non-negative integer \( n \), we denote by \([n]\) the set \( \{1,2,\ldots,n\} \). When \( n \geq 1 \), we also note \( \text{ip}(n) = 1 2 \ldots n \) the increasing permutation of length \( n \) and \( \text{dp}(n) = n (n-1) \ldots 1 \) the decreasing permutation of length \( n \). Let \( \pi \in \mathcal{S}(n) \). The reverse (resp., complement) of \( \pi \) is the permutation \( \pi^r = \pi(n)\pi(n-1)\ldots\pi(1) \) (resp., \( \pi^c = (n-\pi(1)+1)(n-\pi(2)+1)\ldots(n-\pi(n)+1) \)). The inverse of \( \pi \) is the permutation \( \pi^{-1} \in \mathcal{S}(n) \) defined by \( \pi^{-1}(j) = i \) if and only if \( \pi(i) = j \). Given a permutation \( \pi \) of size \( m \) and a permutation \( \sigma \) of size \( n \), the skew sum of \( \pi \) and \( \sigma \) is the permutation of size \( m+n \) defined by

\[
(\pi \ominus \sigma)(i) = \begin{cases} 
\pi(i) + n & \text{for } 1 \leq i \leq m, \\
\sigma(i-m) & \text{for } m+1 \leq i \leq m+n,
\end{cases}
\]

and the direct sum of \( \pi \) and \( \sigma \) is the permutation of size \( m+n \) defined by

\[
(\pi \oplus \sigma)(i) = \begin{cases} 
\pi(i) & \text{for } 1 \leq i \leq m, \\
\sigma(i-m) + m & \text{for } m+1 \leq i \leq m+n.
\end{cases}
\]
Orders

A relation $\leq$ is a partial order on a set $X$ if it has:

- **reflexivity**: for all $x \in X$, $x \leq x$ (i.e., every element is related to itself);
- **transitivity**: for all $x, x', x'' \in X$, if $x \leq x'$ and $x' \leq x''$, then $x \leq x''$;
- **antisymmetry**: for all $x, x' \in X$, if $x \leq x'$ and $x' \leq x$, then $x = x'$ (i.e., no two distinct elements precede each other).

If $\leq$ has the following additional property, we say that it is a weak order on $X$:

- **transitivity of incomparability**: for all pairwise distinct $x, x', x'' \in X$, if $x$ is incomparable with $x'$ (i.e., neither $x \leq x'$ nor $x' \leq x$ is true) and if $x'$ is incomparable with $x''$, then $x$ is incomparable with $x''$.

Two subsets $X_1, X_2$ are independent if there is no $x_1 \in X_1$, $x_2 \in X_2$ such that $x_1 \leq x_2$ or $x_2 \leq x_1$. We say that a partial order is $k$-weak if there exists a partition of $X$ into $k$ pairwise independent sets $X_1, \ldots, X_k$ such that, for each $i$, the restriction of $\leq$ to $X_i$ is a weak order (in other words, $\leq$ is the disjoint union of $k$ weak partial orders).

Let $\mathcal{P} = (X, \leq)$ be a finite partially ordered set. A chain in $\mathcal{P}$ is a set of pairwise comparable elements (i.e., a totally ordered subset) and an antichain in $\mathcal{P}$ is a set of pairwise incomparable elements. The **partial order height** of $\mathcal{P}$, denoted by $\text{height}(\mathcal{P})$, is defined as the maximum cardinality of a chain in $\mathcal{P}$, and the **partial order width** of $\mathcal{P}$, denoted by $\text{width}(\mathcal{P})$, is defined as the maximum cardinality of an antichain in $\mathcal{P}$. By Dilworth Theorem, $\text{width}(\mathcal{P})$ is also the minimum number of chains in any partition of $\mathcal{P}$ into chains. The dual of $\mathcal{P}$ is the partial order $\mathcal{P}^\partial = (X, \leq^\partial)$ defined by letting $\leq^\partial$ be the converse relation of $\leq$, i.e., $x \leq^\partial x'$ if and only if $x' \leq x$. The dual of a partial order is a partial order and the dual of the dual of a relation is the original relation. A **total order** is a partial order in which any two elements are comparable, and a set equipped with a total order is a **totally ordered set**.

A linear extension of a partial order is a total order that is compatible with the partial order. It will be convenient to represent a linear extension of a poset $\mathcal{P} = (X, \leq)$ as the mapping $\tau : X \to [\lvert X \rvert]$ such that $\tau(i) < \tau(j)$ if $i < j$ in the linear extension.

A doubly partially ordered pattern (dpop) $P$ is a pair, denoted by $P = (P_v, P_p)$, of posets $P_v = (X, \leq_v)$ and $P_p = (X, \leq_p)$ defined over the same set $X$. We call $P_v$ and $P_p$ the value poset and the position poset, respectively. A dpop $P = (P_v, P_p)$ is symmetric if $P_v = P_p$, dual if $P_v = P_p^\partial$, and semi-total if one of $P_v$ or $P_p$ is a total order. We let $\text{height}(P)$ and $\text{width}(P)$ stand for $\max\{\text{height}(P_v), \text{height}(P_p)\}$ and $\max\{\text{width}(P_v), \text{width}(P_p)\}$, respectively. Finally, the size of $P$ is defined as the cardinality $\lvert X \rvert$ and is denoted by $\lvert P \rvert$.

**Definition 1** (DPOP Matching). Given a permutation $\pi \in \mathfrak{S}(n)$ and a dpop $P = (P_v, P_p)$, an occurrence (or mapping) of $P$ in $\pi$ is an injective function $\varphi : X \to [n]$ such that:

- $\pi \circ \varphi$ is $\leq_v$-non-decreasing, i.e., for all $x, y \in X$, if $x \leq_v y$ then $\pi(\varphi(x)) \leq \pi(\varphi(y))$, and
- $\varphi$ is $\leq_p$-non-decreasing, i.e., for all $x, y \in X$, if $x \leq_p y$ then $\varphi(x) \leq \varphi(y)$.

The **DPOP Matching** problem consists in deciding whether $P$ occurs in $\pi$.

**First Observations**

**Observation 2.** **Permutation Pattern Matching** is the special case of DPOP Matching where both $\leq_v$ and $\leq_p$ are total orders.

We note that applying a vertical and/or horizontal symmetry on both pattern and permutation does not alter the existence of an occurrence.
Observation 3. Let \( P = (P_v, P_p) \) be a dpop and \( \pi \) be a permutation. The following statements are equivalent:

1. \( (P_v, P_p) \) occurs in \( \pi \);
2. \( (P_v, (P_p)^3) \) occurs in \( \pi^2 \);
3. \( ((P_v)^3, P_p) \) occurs in \( \pi^2 \);
4. \( ((P_v)^3, (P_p)^3) \) occurs in \( \pi^4 \);
5. \( (P_p, P_v) \) occurs in \( \pi^{-1} \).

The following reformulation will prove useful.

Observation 4. Let \( P = (P_v, P_p) \) be a dpop with \( P_v = (X, \leq_v) \), \( P_p = (X, \leq_p) \) and \( k = |X| \), and let \( \pi \in S(n) \) be a permutation. The following statements are equivalent:

- \( P \) occurs in \( \pi \).
- There exists a linear extension \( \tau_v : X \to [k] \) of \( P_v \) and a linear extension \( \tau_p : X \to [k] \) of \( P_p \) such that the permutation \( \sigma \in S(k) \) defined by \( \sigma(i) = \tau_v(\tau_p^{-1}(i)) \) for \( 1 \leq i \leq k \) is contained in \( \pi \).

The rationale for the reformulation introduced in Observation 4 stems from the following corollary that sets the general context.

Corollary 5 ([17]). DPOP MATCHING is FPT for the parameter \(|P|\).

Indeed, it is enough to guess two linear extensions \( \tau_v : X \to [k] \) of \( P_v \) and \( \tau_p : X \to [k] \) of \( P_p \), and to check if the permutation \( \sigma \in S(k) \) defined by \( \sigma(i) = \tau_v(\tau_p^{-1}(i)) \) for \( 1 \leq i \leq k \) is contained in \( \pi \). There are \( O(k!) \) pairs of such extensions and, for each of them, one can check whether \( \sigma \) occurs in \( \pi \) in \( n^{2O(k^2 \log k)} \) time [17].

4 Semi-Total Patterns

In this section we focus on semi-total patterns, i.e., without loss of generality, on the case where \( P_p \) is a total order (up to symmetry by Observation 3). This case still contains PERMUTATION PATTERN MATCHING as a special case, and is thus NP-hard. We focus on small-height value partial orders, i.e., on dops with constant height\((P_v)\), and give an XP algorithm for weak orders (Proposition 6) and \( \text{paraNP} \)-hardness in general (Proposition 7).

Proposition 6. DPOP MATCHING is solvable in \( O(n^{\text{height}(P_v)}) \) time if \( P_v \) is a weak order and \( P_p \) is a total order.

Proof. Let \( \pi \in S(n) \) be a permutation and \( P = (P_v, P_p) \) be a dpop on some ground set \( X \), where \( P_v \) is a weak order and \( P_p \) is a total order. Without loss of generality, we assume that \( X \) is the set \([k]\) and that \( P_p \) is the usual order on integers. For every \( x \in X \), we abusively denote by \( \text{height}(x) \) the maximum cardinality of a chain with maximum element \( x \) in \( P_v \).

Finally, set \( \ell = \text{height}(P_v) \).

For any two distinct variables \( x, y \in X \), we have \( x <_v y \) if and only if \( \text{height}(x) < \text{height}(y) \). Thus, \( P \) occurs in \( \pi \) if and only there exists a sequence \( 0 = a_0 < a_1 < a_2 < \ldots < a_\ell = n \) such that \( w_\pi \) is a subsequence of \( w_\pi \), where \( w_\pi \in [\ell]^n \) and \( w_\sigma \in [\ell]^k \) are the two words defined by \( w_\pi[i] = \min \{ j : a_{j-1} < \pi(i) \leq a_j \} \) and \( w_\sigma[i] = \text{height}(i) \).

As for the running time, there exist \( \binom{\ell-1}{\ell-1} \) distinct sequences \( (a_i)_{0 \leq i \leq \ell} \) and deciding whether \( w_\sigma \) occurs in \( w_\pi \) as a subsequence is a linear-time procedure.

Proposition 7. DPOP MATCHING is NP-complete even if \( \text{height}(P_v) = 2 \), \( P_p \) is a total order and \( \pi \) avoids 1234.
which defines the total order $\leq_p$ on $X$, such that $X_1 \leq_p X_2$ if and only if $\exists j \in [m] : \phi(a_i) \leq_j \phi(b_j) \leq_p X_2$. Finally, for each edge $(i, j) \in E$ with $i < j$, we set $a_i \leq_v b_j$; all other elements of $X$ are pairwise incomparable by $\leq_v$. This defines a partial order $\leq_v$ that has height $\leq_v = 2$.

Write now $N = 3n + 3$ and $m = (k + 1)N - 2$, and define a permutation $\pi \in \mathcal{S}(m)$ as follows:

- $\pi(iN + j) = (m + 1) - (iN + j + k)$ whenever $0 \leq i < k$ and $1 \leq j \leq N - 2$;
- $\pi(iN - 1) = (k + 1) - i$ and $\pi(iN) = (m + 1) - i$ whenever $1 \leq i \leq k$.

It is straightforward to check that $\pi$ is 1234-avoiding. It is also easy to see how the construction, illustrated in Figure 4, can be accomplished in polynomial-time.

Let us see under which conditions an injective $\leq_p$-non-decreasing function $\phi : X \to [m]$ maps $P$ into $\pi$. We say that a vertex $i$ belongs to the $j$-th gadget if one of the integers $\phi(a_i)$ or $\phi(b_i)$ is equal to $jN - 1$ or to $jN$, i.e., if $\{\phi(a_i), \phi(b_i)\} \cap \{jN - 1, jN\} \neq \emptyset$. When two elements in the range of $\phi$ are consecutive, either they are integers $\phi(a_i)$ and $\phi(b_i)$ for a given $i$, or one of them is an integer $\phi(c_i)$ for some $i$. Therefore, no two distinct vertices $i$ and $i'$ can belong to the same $j$-th gadget. Consequently, and since there are $k$ gadgets, the set $V'$ of vertices $i$ that belong to some gadget is of size at most $k$.

Then, we define a notion of height as follows: for each element $x$ of $X$, we set $\text{height}(x) = 0$ if $N$ divides $\phi(x) + 1$, $\text{height}(x) = 1$ if $N$ divides $\phi(x)$, and $\text{height}(x) = 2$ otherwise. By construction, for all $x, y \in X$ such that $x \leq_p y$, we have $\text{height}(x) \leq \text{height}(y)$ if and only if $x$
is of smaller height than \( y \). Therefore, if \( \varphi \) maps \( P \) into \( \pi \), and for each relation \( a_i \leq_v b_j \), either \( a_i \) has height 0 or \( b_j \) has height 2. In particular, either \( i \) or \( j \) must belong to \( V' \), and therefore \( V' \) is a vertex cover of size at most \( k \).

Conversely, provided that there exist vertices \( v(1) < v(2) < \ldots < v(k) \) that form a vertex cover \( V' \), we construct an occurrence of \( P \) in \( \pi \) as follows. First, we abusively set \( v(0) = 0 \). Then, for all \( i \in [k] \), we set \( f(i) = jN + 3(i - v(j)) \), where \( j \) is the largest integer such that \( v(j) \leq i \). We set \( \varphi(a_j) = f(j) - 1 \), \( \varphi(b_j) = f(j) \) and \( \varphi(c_j) = f(j) + 1 \).

By construction, we have \( f(i) + 3 \leq f(i + 1) \) for all \( i \), and therefore \( \varphi \) is an injective \( \leq_P \)-non-decreasing function. Moreover, for every \( i \in [n] \), the elements \( a_i \) and \( b_i \) have heights 0 and 2 if \( i \in V' \), and they have height 1 if \( i \notin V' \). It follows that \( \pi(\varphi(a_i)) \leq \pi(\varphi(b_i)) \) whenever \( a_i \leq_v b_j \), i.e., that \( \varphi \) is an occurrence of \( P \) in \( \pi \).

\[ \diamond \]

\section{Symmetric Patterns}

This section is devoted to studying complexity issues of pattern matching for symmetric \( \text{dpop} \) (i.e., those \( \text{dpops} \ P = (P, P) \), whose value and position posets coincide). We further focus on two special cases, first when \( P \) has a bounded width, then when \( P \) is restricted to constrained pattern-avoiding classes of permutations.

\subsection{Symmetric Pattern with Bounded Width}

We first observe that the problem is polynomial for width 1 (Observation 8). We further prove \( \text{W}[1] \)-hardness for the parameter \( k \) when \( P \) is a disjoint union of \( k \) chains (Proposition 10). We complement this result with an \( \text{XP} \) algorithm for the slightly more general case where \( P \) is a disjoint union of weak orders (Proposition 11). Note that the existence of an \( \text{XP} \) algorithm for the width parameter remains open, and we conjecture that the problem is \( \text{NP} \)-hard even for constant width.

\begin{observation}
\textbf{Observation 8.} \textbf{DPOP Matching} is solvable in \( O(n \log \log |P|) \) time for a symmetric \( \text{dpop} \ P \) of width 1 (i.e., a total symmetric \( \text{dpop} \ P \)).
\end{observation}

\begin{proof}
If \( P \) has width 1, then \( P = (P, P) \) for some total order \( P = (X, \lessdot) \). In particular, we can write \( X = \{x_1, \ldots, x_{|X|}\} \) with \( x_i \lessdot x_j \) for \( i < j \), and in any mapping \( \phi : X \to [n] \), the elements \( \pi_{\phi(x_i)}, \ldots, \pi_{\phi(x_{|X|})} \) must form an increasing subsequence of \( \pi \). Conversely, any size-\( |X| \) increasing subsequence of \( \pi \) can be used as an image for \( \phi \), so in this setting \textbf{DPOP Matching} corresponds to the longest increasing subsequence problem, which can be solved in \( O(n \log \log |X|) \) time [12].

To simplify the exposition of our next result, we introduce a new problem that may be of independent interest. Given a positive integer \( k \) and a permutation \( \pi \in \mathcal{S}(kn) \), \textbf{Balanced} \( k \)-\textbf{Increasing Coloring} is the problem of deciding whether there exists a balanced \( k \)-coloring of \( \pi \) (i.e., a partition of \( [kn] \) into \( k \) subsets of size exactly \( n \)) such that each color induces an increasing subsequence of \( \pi \).

\begin{proposition}
\textbf{Balanced} \( k \)-\textbf{Increasing Coloring} for \( 312 \)-avoiding permutations is \text{W}[1]-hard for the parameter \( k \).
\end{proposition}

\begin{proof}
We perform a reduction from \textbf{Unary Bin Packing} parameterized by the number of bins, which is known to be \text{W}[1]-hard [20]. In this version of \textbf{Bin Packing}, we are given a list of integers \( s_1, s_2, \ldots, s_n \) encoded in unary, and two integers \( B \) and \( k \). These integers are interpreted as item sizes, and the task is to decide whether the items can be partitioned
into \( k \) subsets, each of total size \( B \). We show that there is a reduction from \textsc{Unary Bin Packing}, parameterized by the number of bins, to \textsc{Balanced \( k \)-Increasing Coloring}, parameterized by the number of colors.

Consider an arbitrary instance of \textsc{Unary Bin Packing} containing \( n \) items with item sizes \( S = \{s_1, s_2, \ldots, s_n\} \), and two integers \( B \) and \( k \). Define \( \pi \in \Theta(kB + kn) \) by

\[
\pi = \bigoplus_{i=1}^{n} (ip(s_i + 1) \odot dp(k - 1)).
\]

Each pattern \( ip(s_i + 1) \odot dp(k - 1) \) is called the \( i \)-th block of \( \pi \). See Figure 5 for an illustration. It is straightforward to check that \( \pi \) is 312-avoiding.

We claim that the \( n \) items \( s_1, s_2, \ldots, s_n \) can be partitioned into \( k \) subsets, each of total size \( B \), if and only if there exists a \( k \)-coloring of \( \pi \) such that each color induces an increasing pattern of length \( B + n \).

Suppose first that the \( n \) items \( s_1, s_2, \ldots, s_n \) can be partitioned into \( k \) subsets, each of total size \( B \). Write \( S = S_1 \cup S_2 \cup \cdots \cup S_k \) such a partition. Define a \( k \)-coloring of \( \pi \) as follows. Consider the \( i \)-th block \( ip(s_i + 1) \odot dp(k - 1) \) of \( \pi \), and suppose that \( s_i \in S_j \). Color the whole ascending pattern \( ip(s_i + 1) \) with color \( c_j \) and arbitrarily color the elements of the descending pattern \( dp(k - 1) \) with the remaining \( k - 1 \) colors (each element of \( dp(k - 1) \) is assigned to a distinct color). We claim that every color \( c_j \) induces an increasing pattern of length \( B + n \) in \( \pi \). First, it is clear that the above \( k \)-coloring induces increasing patterns only. As for the length of each induced increasing pattern, focus on any color \( c_j \). We note that,
in every block $\text{ip}(s_i + 1) \triangleq \text{dp}(k - 1)$ of $\pi$, either the whole subpattern $\text{ip}(s_i + 1)$ is colored with color $c_j$ (if $s_i \in S_j$) or exactly one element of the subpattern $\text{dp}(k - 1)$ is colored with color $c_j$ (if $s_i \notin S_j$). It follows that the increasing pattern induced by color $c_j$ in $\pi$ has length $\sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j| = B + n$.

For the reverse direction, suppose now that there exists a $k$-coloring of $\pi$ such that each color induces an increasing pattern of length $B + n$. Every block $\text{ip}(s_i + 1) \triangleq \text{dp}(k - 1)$ requires at least $k$ colors, as it contains a decreasing subpattern of length $k$. Therefore, the whole subpattern $\text{ip}(s_i + 1)$ is colored with the same color. For every $j \leq k$, let $S_j$ be the set of all $s_i$ such that, in the $i$-th block $\text{ip}(s_i + 1) \triangleq \text{dp}(k - 1)$, the subpattern $\text{ip}(s_i + 1)$ is colored with color $c_j$. We have $B + n = \sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j|$, and hence $\sum_{s_i \in S_j} s_i = B$. Therefore, the $n$ items $s_1, s_2, \ldots, s_n$ can be packed into $k$ bins, each of capacity $B$.

Most of the interest in Proposition 9 stems from the following proposition.

**Proposition 10.** DPOP Matching for symmetric dpop and 312-avoiding permutations is W[1]-hard for the parameter width($P$).

**Proof.** We perform a reduction from Balanced $k$-Increasing Coloring, which is W[1]-hard for the parameter $k$. Let $\pi \in \mathcal{O}(kn)$ for some positive integers $k$ and $n$. We construct a symmetric dpop $P = (P, P)$, where $P = (X, \lessdot)$, as follows: $X = [k] \times [n]$ and $(i, j) \lessdot (i', j')$ if and only if $i = i'$ and $j \leq j'$. We claim that $P$ occurs in $\pi$ if and only $\pi$ admits a $k$-coloring for which every color induces an increasing pattern of length $n$.

If $\pi$ admits such a $k$-coloring into colors $c_1, c_2, \ldots, c_k$, the function $\varphi: X \rightarrow [kn]$ that maps each pair $(i, j)$ to the $j$-th smallest position with color $c_i$ is an occurrence of $P$ in $\pi$.

Conversely, suppose that some injective function $\varphi: X \rightarrow [kn]$ is an occurrence of $P$ in $\pi$. For each $i \leq k$, the set $\{i\} \times [n]$ forms a chain of $\lessdot$, and therefore it is mapped to an increasing pattern of size $n$. Coloring this pattern in color $c_i$ produces the desired $k$-coloring.

We show now that the problem where $P$ consists of $k$ independent chains is XP for the parameter $k$. In fact, we generalize this result to $k$-weak partial orders (i.e., if $P$ consists of $k$ independent weak orders).

**Proposition 11.** DPOP Matching for $k$-weak symmetric dpop is XP with parameter $k$.

**Proof.** Let $P$ be a disjoint union of $k$ weak symmetric dpops $P_1, P_2, \ldots, P_k$. For each dpop $P_i$, let $\lessdot_i$ be a linear extension of $P_i$, and let $P_{1,1}, P_{1,2}, \ldots, P_{1,P_i}$ be the maximal antichains of $P_i$, ordered by $\lessdot_i$. Finally, for each $k$-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ of integers such that $a_i \lessdot |P_i|$, we denote by $P_\mathbf{a}$ the dpop obtained from $P$ by removing the $a_i \lessdot_i$-least elements of each dpop $P_i$, and by $P^\text{min}_{i, \mathbf{a}}$ the set of $\lessdot_i$-minimal elements of $P_\mathbf{a}$.

Then, given a permutation $\pi \in \mathcal{O}(n)$, a $k$-tuple $\mathbf{I} = (I_1, \ldots, I_k)$ of intervals of $[n]$, a $k$-tuple $\mathbf{a}$ and an integer $\ell$, a function $\varphi: P_\mathbf{a} \rightarrow \{\ell, \ell + 1, \ldots, n\}$ is called a partial matching for $(\pi, \mathbf{I}, \mathbf{a}, \ell)$ if:

- $\varphi \circ \varphi \lessdot_i$-non-decreasing for each $i$, and
- for each $i$, and each element $x$ of $P_\mathbf{a}$, $\pi(x) \in I_i$ if and only if $x \in P^\text{min}_{i, \mathbf{a}}$.

Before going further, we denote by $\mathbf{I}_i$ the $k$-tuple with one element 1 (in position $i$) and $k - 1$ elements 0. We also denote by $\lessdot_i$ the partial order on tuples $\mathbf{I}$ of intervals, where $\mathbf{I}_i \lessdot \mathbf{I}_j$ if $I_j = I'_j$ whenever $j \neq i$ and $x < x'$ whenever $x \in I_i$ and $x' \in I'_i$. 


When \(a_i = |P_i|\) for all \(i\), such a partial matching exists for all permutations \(\pi\), tuples of intervals \(I\) and integers \(\ell\). When \(a_i = 0\) for all \(i\) and \(\ell = 1\), and once \(\pi\) is fixed, such partial matchings coincide with (standard) matchings, and thus we are interested in checking whether a partial matching exists. Finally, for all tuples \(I\) and \(a\) and for all \(\ell \leq n\), a partial matching \(\varphi\) for \((\pi, I, a, \ell)\) exists precisely when one of the following cases occur:

1. \(\varphi\) is a partial matching for \((\pi, I, a, \ell + 1)\), i.e., \(\ell \notin \varphi(P_a)\);
2. there exists an integer \(i \leq k\) for which the \(\leq\)-least element of \(P_{i,a}^{\min}\), say \(x\), is such that \(\varphi(x) = \ell\) and \(\pi(\ell) \in I_i\), and either
   - \(x\) is not the only element of \(P_{i,a}^{\min}\). and \(\varphi\) is a partial matching for \((\pi, I, a + 1, \ell + 1)\), or
   - \(x\) is the only element of \(P_{i,a}^{\min}\) and there exists a tuple \(I' > I\) such that \(\varphi\) is a partial matching for \((\pi, I', a + 1, \ell + 1)\).

Consequently, we can compute by dynamic programming the list of triples \((I, a, \ell)\) such that there exists a partial matching for \((\pi, I, a, \ell)\): deciding whether adding a triple \((I, a, \ell)\) to the list simply requires to check which triples of the form \((I', a', \ell + 1)\) already belong to the list. Since there are less than \(n^{3k+1}\) triples, this provides us with an \(\tilde{O}(n^{6k+2})\) algorithm.

### 5.2 Symmetric Pattern and Pattern-Avoiding \(\pi\)

In this final section, we consider restrictions on the shape of \(\pi\), via pattern-avoiding restrictions. Our goal here is to identify tractable cases among classes of permutations avoiding one or more size-3 patterns. We give an almost complete dichotomy of polynomial/NP-hard cases, as shown in Table 1. Hardness results are proven in Proposition 12, and also apply to height-2 dpos. Polynomial cases are proven in Proposition 13 and apply to dpos of any height.

**Proposition 12.** DPOP Matching for height-2 symmetric dpop \(P\) and permutation \(\pi\) is NP-hard even if \(\pi\) is separable (it avoids 2413 and 3142) and one of the following restrictions occurs:

1. \(\pi\) is 123-avoiding;
2. \(\pi\) is (132, 213)-avoiding;
3. \(\pi\) is (132, 231)-avoiding;
4. \(\pi\) is (231, 312)-avoiding;
5. \(\pi\) is (132, 321)-avoiding;
6. \(\pi\) is (231, 321)-avoiding;
7. \(\pi\) is (213, 312)-avoiding;
8. \(\pi\) is (132, 231)-avoiding;
9. \(\pi\) is (213, 231)-avoiding.

**Proof.** In each of the cases presented below, we define a symmetric dpop \(P = (\mathcal{P}, \mathcal{P})\) for some partially ordered set \(\mathcal{P} = (X, \preceq)\). Each time, we identify \(P\) with the partial order \(\preceq\).

<table>
<thead>
<tr>
<th>(\pi) avoids</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
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<tbody>
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</tr>
<tr>
<td>123</td>
<td>12.1</td>
<td>3P</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>132</td>
<td>13.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>213</td>
<td>13.5</td>
<td>12.2</td>
<td>bcn</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>231</td>
<td>13.1</td>
<td>12.8</td>
<td>bcn</td>
<td>12.9</td>
<td>bcn</td>
<td></td>
</tr>
<tr>
<td>312</td>
<td>13.4</td>
<td>12.5</td>
<td>bcn</td>
<td>12.7</td>
<td>bcn</td>
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</tr>
<tr>
<td>321</td>
<td>13.3</td>
<td>12.3</td>
<td>bcn</td>
<td>12.6</td>
<td>bcn</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 Polynomial (green/light) and NP-hard (red/dark) cases for DPOP Matching with symmetric dpop and pattern-avoiding permutation \(\pi\), for combinations of size-3 avoided patterns. For each case, see the referenced proposition and case for more details. Diagonal cases follow from any other hard case in the same row or column. For hard cases, the problem used for reduction is indicated as follows: bic: Biclique, 3P: 3-Partition, bin: Unary Bin Packing, bis: Bisection.
Case 1: $\pi$ is 123-avoiding and separable. We use a reduction from 3-PARTITION, as illustrated in Figure 6 with permutation $\pi_1$. Let $(A, B)$ be an instance of 3-PARTITION, where $A$ is a list of integers $a_1, a_2, \ldots, a_{3n}$ with sum $nB$, all being larger than 1.

For all $p \leq n$, we define a bin gadget $b_p$ as the permutation $dp(3) \oplus dp(B - 3)$: we see this gadget as consisting of two parts. Our permutation $\pi$ is now defined by $\pi = \bigoplus_{p=1}^{n} b_p$. Then, our partial order $\leq$ is defined on a set $X$ of $nB$ elements, noted $x_i, y_{i,2}, \ldots, y_{i,a_i}$ for each $i \leq 3n$, so that $x_i \preceq y_{i,j}$ for all $i$ and $j$.

If $P$ has an occurrence $\varphi: X \rightarrow [nB]$ in $\pi$, this occurrence is bijective. Moreover, each element $x_i$ is sent to the bottom-left of $y_{i,2}$, and thus it must be mapped to the left part of some gadget, say $b_{f(i)}$. Each element $y_{i,j}$ must then be mapped to the right part of the same gadget. Now, for each $p \leq n$, the set $S_p = \{i: f(i) = p\}$ is of size 3, and exactly $B$ elements of $X$ are mapped to the gadget $b_p$, which means that $\sum_{i \in S_p} a_i = B$. Moreover, $S_1 \cup \ldots \cup S_p$ forms a partition of $[3n]$, hence it yields a 3-partition of $A$.

Conversely, given a partition $S_1 \cup \ldots \cup S_n$ of $[3n]$ such that $|S_p| = 3$ and $\sum_{i \in S_p} a_i = B$ for each $p$, we build an occurrence of $P$ in $\pi$ by mapping the three elements $x_i$ (for $i \in S_p$) to the left part of $b_p$, and the $B - 3$ elements $y_{i,j}$ (for $i \in X_p$) to the right part of $b_p$.

Case 2: $\pi$ is (132,213)-avoiding. We use a reduction from Unary Bin Packing, as illustrated in Figure 6 with permutation $\pi_2$. Given an instance $(A, B, k)$ of Unary Bin Packing, where $A$ is a list of integers $a_1, a_2, \ldots, a_n$ larger than 1, we use the same dpop as in Case 1: our partial order $\preceq$ is defined on a set $X$ of $nB$ elements, noted $x_i, y_{i,2}, \ldots, y_{i,a_i}$ for each $i \leq n$, so that $x_i \preceq y_{i,j}$ for all $i$ and $j$. However, this time, our gadget $b_p$ is the permutation $ip(B)$, and our permutation $\pi$ is again defined by $\pi = \bigoplus_{p=1}^{n} b_p$.

If $P$ has an occurrence $\varphi: X \rightarrow [nB]$ in $\pi$, this occurrence is bijective. Each element $x_i$ is sent to some gadget, say $b_{f(i)}$, and the elements $y_{i,j}$ must then be mapped to the same gadget. Now, for each $p \leq k$, let $S_p = \{i: f(i) = p\}$. Exactly $B$ elements of $X$ are mapped to the gadget $b_p$, which means that $\sum_{i \in S_p} a_i = B$. This means that $(S, B, k)$ is a positive instance of the Unary Bin Packing problem.

Conversely, given a partition $S_1 \cup \ldots \cup S_k$ of $[n]$ such that $\sum_{i \in S_p} a_i = B$ for each $i$, we build an occurrence of $P$ in $\pi$ by mapping the $B$ elements $x_i$ and $y_{i,j}$ (for $i \in S_p$) to $b_p$.

Case 3: $\pi$ is (132,321)-avoiding. We use a reduction from Bisection, as illustrated in Figure 7. Given a graph $G = (V, E)$ and an integer $k$, the Bisection problem consists in deciding whether $V$ admits a partition $V_1 \cup V_2$ such that $|V_1| = |V_2|$ and that splits at most $k$ edges (i.e., at most $k$ edges have one endpoint in $V_1$ and one endpoint in $V_2$).
Our reduction is as follows. Let \( n = |V|/2, m = |E|, W = m + k + 1, \) and \( L = nW + m. \) Our permutation is defined by \( \pi = (\text{ip}(L) \odot \text{ip}(L)) \odot \text{ip}(k). \) These three parts of \( \pi \) are noted \( A, B \) and \( C, \) from left to right. Then, our partial order \( \preccurlyeq \) is defined on a set \( X \) of \( 2nW + m \) elements: \( 2nW \) elements, noted \( v_{v,i} \) for each \( v \in V \) and \( i \leq W, \) and \( m \) elements, noted \( y_e \) for each \( e \in E. \) This order contains the relations \( x_{v,i} \preccurlyeq y_e \) for which \( v \) is an endpoint of \( e. \)

Assume that there exists a mapping of \( P \) into \( \pi. \) For each \( v \in V, \) and since \( C \) has size \( k < W, \) at least one of the elements \( x_{v,i} \) is mapped to \( A, \) in which case we say that \( v \) has type \( A, \) or to \( B, \) in which case \( v \) has type \( B. \) Then, each vertex has at least one type, and possibly both. We partition \( V \) into three sets \( V_A, V_B, V_{AB} \) containing the vertices of type \( A, B \) and both \( A \) and \( B, \) respectively. Moreover, for each \( v \in V_A, \) each element \( x_{v,i} \) must be mapped either to \( A \) or to \( C: \) these two parts together contain \( L + k \) elements, so \( |V_A| \leq (L + k)/W = n + 1 - 1/W, \) and \( |V_A| \leq n. \) Similarly, \( |V_B| \leq n. \)

We build a set \( V_1 \) as the union of \( V_A \) with \( n - |V_A| \) vertices of \( V_{AB}, \) and \( V_2 \) as \( V \setminus V_1, \) so that \( |V_1| = |V_2| = n. \) Moreover, for every \( v \in V_1 \) (resp., \( v \in V_2 \)), some element \( x_{v,i}, \) say \( x_{v,1}, \) is mapped to \( A \) (resp., to \( B). \) Then, each edge \( e = (u,v) \) that is split by \( (V_1, V_2) \) must be mapped to a point above some point of \( A \) and to the right of some point of \( B. \) This means that \( y_e \) is mapped to \( C, \) and that \( (V_1, V_2) \) splits at most \( k \) edges, i.e., is a valid bisection.

Conversely, given a bisection \( (V_1, V_2) \) splitting at most \( k \) edges, we map \( P \) into \( \pi \) as follows: map elements \( x_{v,i} \) for \( v \in V_1 \) (resp., \( V_2 \)) to the first \( nW \) elements of \( A \) (resp., \( B), \) map elements \( y_e \) for which \( e \) is induced by \( V_1 \) (resp., \( V_2 \)) to the following elements of \( A \) (resp., \( B), \) and finally map all elements \( y_e \) such that \( e \) is split by \( (V_1, V_2) \) into \( C. \) This mapping is an occurrence of \( P \) in \( \pi. \)

**Case 4:** \( \pi \) is \((231,312)-avoiding.** We use a reduction from **BICLIQUE**, as illustrated in Figure 8 with permutation \( \pi_1. \) Given a bipartite graph \( G = (V, E) \) and an integer \( k, \) the **BICLIQUE** problem consists in deciding whether \( V \) admits a complete bipartite subgraph \( K_{k,k}. \)

If \( V = A \cup B \) is a partition of \( V \) into two independent sets, adding independent vertices if needed allows us to assume that \( A \) and \( B \) have the same size \( n, \) and that no vertex in either side is fully connected to the other side.

Our permutation \( \pi \) is defined by \( \pi = \text{dp}(n - k) \oplus \text{dp}(2k) \oplus \text{dp}(n - k). \) These three parts of \( \pi \) are noted \( b_1, b_2 \) and \( b_3. \) Our partial order \( \preccurlyeq \) is the order on \( V \) such that \( x \preccurlyeq y \) whenever \( x \in A, y \in B \) and \( \{x, y\} \notin E. \)
Consider a mapping of $P$ into $\pi$. For each element $x \in A$, there exists $y \in B$ such that $x \not\leq y$, and therefore $x$ cannot be mapped into $b_3$. Symmetrically, no element $y \in B$ may be mapped into $b_1$. Overall, since $|\pi| = |V| = 2n$, $b_1$ contains $n - k$ elements from $A$, $b_3$ contains $n - k$ elements from $B$, and $b_2$ contains a size-$k$ subset $A'$ of $A$ and a size-$k$ subset $B'$ of $B$. No two elements $x \in A'$ and $y \in B'$ that are mapped into $b_2$ are comparable for $\leq$, which means that $\{x, y\} \in E$ for each such pair, i.e., that $(A', B')$ is a biclique.

Conversely, if $G$ has a biclique $(A', B')$, we map all elements of $A \setminus A'$ into $b_1$, all elements of $A' \cup B'$ into $b_2$, and all elements of $B \setminus B'$ into $b_3$. This mapping satisfies all relations $x \not\leq y$ with $x \in A$ and $y \in B$, except for $x \in A'$ and $y \in B'$, but indeed there is no such relation since $(A', B')$ is a biclique.

**Case 5:** $\pi$ is $(132,312)$-avoiding. We also use a reduction from BICLIQUE, as illustrated in Figure 6 with permutation $\pi_2$. Our partial order $\leq$ is the same as in Case 4, and our permutation $\pi$ is defined by $\pi = ((\text{dp}(n-k) \oplus \text{ip}(k)) \ominus \text{dp}(k)) \ominus \text{ip}(n-k)$. These three parts of $\pi$ are noted $b_1, b_2, b_3$ and $b_4$.

Consider a mapping of $P$ into $\pi$. For each element $y \in B$, there exists $x \in A$ such that $x \not\leq y$, and therefore $y$ cannot be mapped into $b_1$ or $b_3$. Thus, and since $|B| = n$, the elements of $B$ are mapped to $b_2$ or $b_4$, and the elements of $A$ are mapped to $b_1$ or $b_3$. Hence, $b_2$ contains a size-$k$ subset $B'$ of $B$ and $b_3$ contains a size-$k$ subset $A'$ of $A$. No element of $b_2$ is comparable to any element of $b_3$, and therefore $(A', B')$ is a biclique.

Conversely, if $G$ has a biclique $(A', B')$, we map all elements of $A \setminus A'$ into $b_1$, all elements of $A'$ into $b_3$, all elements of $B'$ into $b_2$ and all elements of $B \setminus B'$ into $b_4$. This mapping satisfies all relations $x \not\leq y$ with $x \in A$ and $y \in B$, except for $x \in A'$ and $y \in B'$, but indeed there is no such relation since $(A', B')$ is a biclique.

**Cases 6–9:** These cases are symmetric to Cases 3, 5, 5 and 7, respectively. Indeed, if $(P, \pi)$ is an instance of DPOP MATCHING with $P$ a height-2 symmetric dpop, $(P^0, \pi^{cr})$ and $(P, \pi^{-1})$ are equivalent instances of DPOP MATCHING with height-2 symmetric dpops, and

- **Case 6:** if $\pi$ avoids $132$ and $321$ (Case 3), $\pi^{cr}$ avoids $132^{cr} = 213$ and $321^{cr} = 321$;
- **Case 7:** if $\pi$ avoids $132$ and $312$ (Case 5), $\pi^{cr}$ avoids $132^{cr} = 213$ and $312^{cr} = 312$;
- **Case 8:** if $\pi$ avoids $132$ and $312$ (Case 5), $\pi^{-1}$ avoids $321^{-1} = 132$ and $312^{-1} = 231$;
- **Case 9:** if $\pi$ avoids $213$ and $312$ (Case 7), $\pi^{-1}$ avoids $213^{-1} = 213$ and $312^{-1} = 231$. 

![Figure 8](image-url) Reduction from BICLIQUE to DPOP MATCHING on (231,312)-avoiding and (213,312)-avoiding permutations. Left: a bipartite graph $G$ with a $(2, 2)$ biclique and the corresponding height-2 dpop $P$, built from the complement of $G$. Right: permutations $\pi_1$ and $\pi_2$ with a mapping of the vertices 1 to 8, including the biclique vertices mapped into the central 2$k$ positions.
Proposition 13. DPOP Matching is in P for symmetric dpop $P$ if one of the following restrictions on $\pi$ occurs:

1. $\pi$ is $(123,231)$-avoiding;
2. $\pi$ is $(123,132)$-avoiding;
3. $\pi$ is $(123,321)$-avoiding;
4. $\pi$ is $(123,312)$-avoiding;
5. $\pi$ is $(123,213)$-avoiding.

Proof. In each of the cases presented below, we are given a permutation $\pi$ and a symmetric dpop $P = (P, \pi)$ for some partially ordered set $P = (X, \prec)$. Each time, we identify $P$ with the partial order $\prec$.

Case 1: $\pi$ is $(123,231)$-avoiding. There exist integers $k$, $\ell$ and $m$, with sum $n$, such that $\pi = dp(k) \cup (dp(\ell) \cup dp(m))$. These three parts of $\pi$ are noted $b_1$, $b_2$ and $b_3$. Then, for every pair $(u, v)$ such that $u < v$, we must map $u$ into $b_2$ and $v$ into $b_3$. Such values can be mapped greedily, since elements in $b_2$ are pairwise incompatible, as well as those in $b_3$. Thus, $P$ can be mapped into $\pi$ if and only if it has height at most 2, there are at most $a$ elements that are lower bounds, and at most $b$ elements that are upper bounds.

Note that, if $P$ is not symmetric, the problem becomes NP-hard, since reversing the horizontal order of $P_0$ and $\pi$ transforms $\pi$ into the $(132,321)$-avoiding permutation of the NP-hard Case 3 in Proposition 12.

Case 2: $\pi$ is $(123,132)$-avoiding. The permutation $\pi$ is a skew sum $\pi = \bigoplus_{p=1}^k d_p$ of patterns of the form $d_p = dp(a_p) \cup dp(1)$ for some integer $a_p \geq 0$. Then, no two elements in $X$ can share a strict lower bound, i.e., if $u < v$ and $u < w$ then $v = w$. Thus, $P$ is of height at most 2, and there exists a partition $S_1 \cup \ldots \cup S_k$ of $X$ in which each set $S_i$ contains a distinguished element $s_i$, such that $x \preceq s_i$ if and only if $x \in S_i$. Up to reordering the patterns $d_p$ and the sets $S_i$ which are pairwise incomparable, we assumed that $a_1 \geq a_2 \geq \ldots \geq a_k$ and that $|S_1| \geq |S_2| \geq \ldots \geq |S_k|$. Let also $m$ be the number of sets $S_i$ with size at least 2.

Each set $S_i$ must be mapped into a single pattern, say $d_{p(i)}$, and if $i \leq m$, i.e., if $|S_i| \geq 2$, the element $s_i$ must be mapped to the unique top-right element of $d_{p(i)}$. Such a mapping exists if and only if $k \geq m$ and $a_i \geq |S_i| - 1$ for all $i \leq m$: we shall choose $p(i) = i$ and map greedily the elements of $S_i \setminus \{s_i\}$ to the bottom-left part of $d_i$. Finally, the elements of singleton sets $S_i$ can be mapped to the remaining places in $\pi$.

Case 3: $\pi$ is $(123,321)$-avoiding. Erdős-Szekeres theorem [14] proves that $n \leq 4$.

Cases 4–5: These cases are symmetric to Cases 1 and 2, respectively. Indeed, if $(P, \pi)$ is an instance of DPOP Matching with $P$ a height-2 symmetric dpop, $(P^0, \pi^0)$ and $(P, \pi^1)$ are equivalent instances of DPOP Matching with height-2 symmetric dops, and

Case 4: if $\pi$ avoids 123 and 231 (Case 1), $\pi^1$ avoids $123^1 = 123$ and $231^1 = 312$;
Case 5: if $\pi$ avoids 123 and 132 (Case 2), $\pi^{c_2}$ avoids $123^{c_2} = 123$ and $132^{c_2} = 213$. ◀

6 Concluding Remarks

Some open complexity questions remain among the parameters we identified for DPOP Matching. For semi-total dops, the complexity is open for constant width, and for most classes of pattern-avoiding permutations (although, according to Propositions 7 and 10, the problem is NP-hard when $\pi$ avoids 1234 or 312, respectively). For symmetric dops, it would be interesting to settle the complexity status of deciding whether a dpop occurs in a $(231,321)$-avoiding or $(312,321)$-avoiding permutation. In particular, for these cases, we conjecture that the problem becomes polynomial when height$(P)$ is constant.

Regarding the original puzzle formulation of the problem, an interesting question is to generate instances that yield a unique solution, i.e., given a permutation $\pi$, find a dpop with a unique occurrence in $\pi$. This can be done by using a semi-total dpop (e.g., take $X$
with $|X| = |\pi|$, let $P_\emptyset$ be a total order and $P_\emptyset$ be an empty order, but one could try to minimize $|X|$ or the number of pairs of comparable elements in $P$ (i.e., the number of clues) in order to have a unique solution.

References


